# How to Multiply

integers, matrices, and polynomials

COS 423
Spring 2007

slides by Kevin Wayne

## Complex Multiplication

Complex multiplication. $(a + bi)\,(c + di) = x + yi.$

Grade-school. $x = ac - bd,\ y = bc + ad.$

4 multiplications, 2 additions

Q. Is it possible to do with fewer multiplications?



## Complex Multiplication

Complex multiplication. $(a + bi)\,(c + di) = x + yi.$

Grade-school. $x = ac - bd,\ y = bc + ad.$

4 multiplications, 2 additions

Q. Is it possible to do with fewer multiplications?

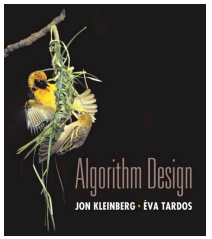
A. Yes. [Gauss] $x = ac - bd,\ y = (a + b)\,(c + d) - ac - bd.$

3 multiplications, 5 additions

Remark. Improvement if no hardware multiply.



# Integer Multiplication




*Section 5.5*

## Integer Addition

Addition. Given two $n$-bit integers $a$ and $b$, compute $a + b$.
Grade-school. $\Theta(n)$ bit operations.

```
  1  1  1  1  1  1  0  1
     1  1  0  1  0  1  0  1
+    0  1  1  1  1  1  0  1
-----------------------------
  1  0  1  0  1  0  0  1  0
```

Remark. Grade-school addition algorithm is optimal.



## Integer Multiplication

Multiplication. Given two $n$-bit integers $a$ and $b$, compute $a \times b$.
Grade-school. $\Theta(n^2)$ bit operations.

```
                1  1  0  1  0  1  0  1
             ×  0  1  1  1  1  1  0  1
-----------------------------------------
                1  1  0  1  0  1  0  1
             0  0  0  0  0  0  0  0
          1  1  0  1  0  1  0  1
       1  1  0  1  0  1  0  1
    1  1  0  1  0  1  0  1
 1  1  0  1  0  1  0  1
 1  1  0  1  0  1  0  1
 0  0  0  0  0  0  0  0
-----------------------------------------
 0  1  1  0  1  0  0  0  0  0  0  0  0  0  1
```

Q. Is grade-school multiplication algorithm optimal?



## Divide-and-Conquer Multiplication: Warmup

To multiply two $n$-bit integers $a$ and $b$:

- Multiply four $\frac{1}{2}n$-bit integers, recursively.
- Add and shift to obtain result.

$$\begin{aligned} a &= 2^{n/2} \cdot a_1 + a_0 \\ b &= 2^{n/2} \cdot b_1 + b_0 \\ ab &= \left(2^{n/2} \cdot a_1 + a_0\right)\left(2^{n/2} \cdot b_1 + b_0\right) = 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \end{aligned}$$

Ex. $a$ = 10001101 ($a_1$ = 1000, $a_0$ = 1101) $\quad b$ = 11100001 ($b_1$ = 1110, $b_0$ = 0001)

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

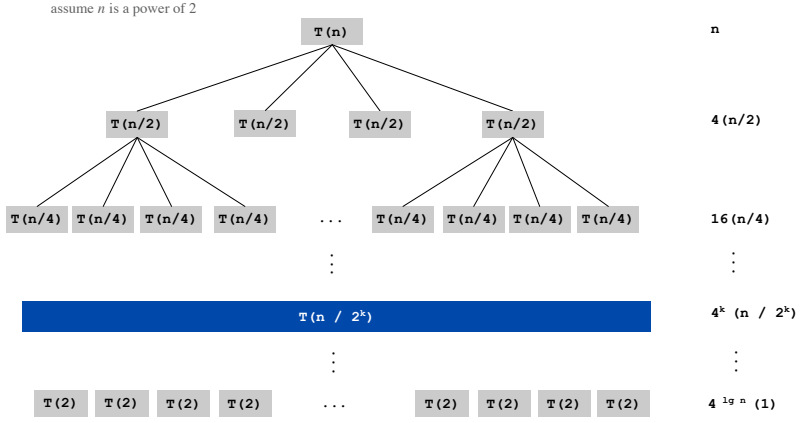


## Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4T(n/2) + n & \text{otherwise} \end{cases}$$

assume $n$ is a power of 2

$$T(n) = \sum_{k=0}^{\lg n} n\,2^k = n\left(\frac{2^{1+\lg n} - 1}{2 - 1}\right) = 2n^2 - n$$



## Karatsuba Multiplication

To multiply two $n$-bit integers $a$ and $b$:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$\begin{aligned} a &= 2^{n/2} \cdot a_1 + a_0 \\ b &= 2^{n/2} \cdot b_1 + b_0 \\ ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot (a_1 b_0 + a_0 b_1) + a_0 b_0 \\ &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \big( (a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0 \big) + a_0 b_0 \end{aligned}$$

(1) (2) (1) (3) (3)



## Karatsuba Multiplication

To multiply two $n$-bit integers $a$ and $b$:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$\begin{aligned} a &= 2^{n/2} \cdot a_1 + a_0 \\ b &= 2^{n/2} \cdot b_1 + b_0 \\ ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot (a_1 b_0 + a_0 b_1) + a_0 b_0 \\ &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \big( (a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0 \big) + a_0 b_0 \end{aligned}$$

(1) (2) (1) (3) (3)

Theorem. [Karatsuba-Ofman 1962] Can multiply two $n$-bit integers in $O(n^{1.585})$ bit operations.

$$T(n) \le \underbrace{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1 + \lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\lg 3}) = O(n^{1.585})$$

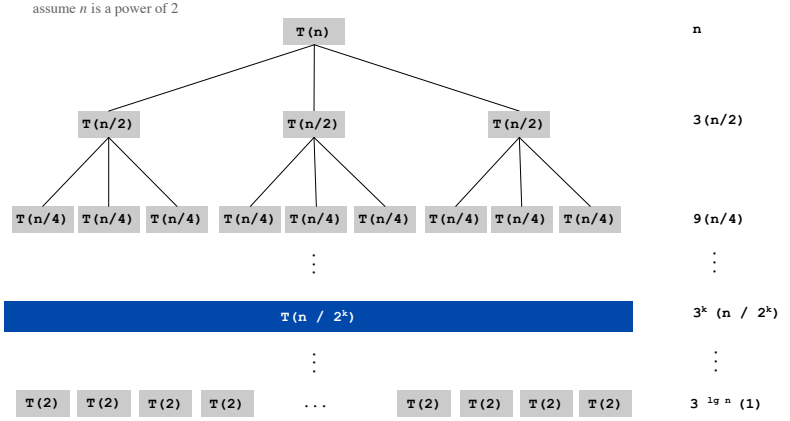


## Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

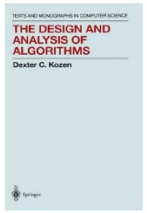
assume $n$ is a power of 2

$$T(n) = \sum_{k=0}^{\lg n} n \left(\tfrac{3}{2}\right)^k = n \left( \frac{\left(\frac{3}{2}\right)^{1+\lg n} - 1}{\frac{3}{2} - 1} \right) = 3n^{\lg 3} - 2n$$






# Integer Division




*Section 30.3*

## Integer Division

Integer division. Given two integer $s$ and $t$ of at most $n$ bits each, compute the quotient and remainder: $q = \lfloor s / t \rfloor$ , $r = s \bmod t$.

Ex.

- $s = 1{,}000,\ t = 110 \Rightarrow q = 9, r = 10.$
- $s = 4{,}905{,}648{,}605{,}986{,}590{,}685,\ t = 100 \Rightarrow r = 85.$

Long division. $\Theta(n^2)$.

Q. Is grade-school long division algorithm optimal?



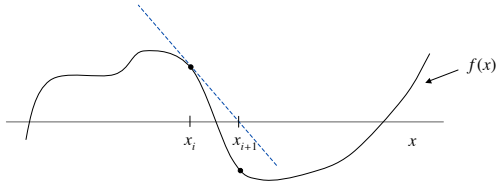
## Newton's Method

Goal. Given a function $f(x)$, find a value $x^*$ such that $f(x^*) = 0$.

sufficiently smooth

Newton's method.

- Start with initial guess $x_0$.
- Compute a sequence of approximations: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$.




Convergence. No guarantees in general.



## Integer Division: Newton's Method

Goal. Given two integer $s$ and $t$ compute $q = \lfloor s / t \rfloor$.

Our approach: Newton's method.

- Approximately compute $x = 1 / t$ using exact arithmetic.

$$\begin{aligned} f(x) &= t - \frac{1}{x} \\ x_{i+1} &= 2x_i - tx_i^2 \end{aligned}$$

- After not too many iterations, quotient $q$ is either $\lfloor s\, x_k \rfloor$ or $\lceil s\, x_k \rceil$.



## Integer Division: Newton's Method Example

Ex. `t = 7`

- $x_0$ = 0.1
- $x_1$ = 0.13
- $x_2$ = 0.1417
- $x_3$ = 0.142847770
- $x_4$ = 0.14285714224218970
- $x_5$ = 0.142857142857142854495685444449737
- $x_6$ = 0.142857142857142857142857142857142808090231138678393076316441581 70

number of digits of accuracy doubles after each iteration

$$\begin{aligned} f(x) &= t - \frac{1}{x} \\ x_{i+1} &= 2x_i - tx_i^2 \end{aligned}$$

Ex. `s/t = 123456/7`

- $s\, x_5$ = 17636.571428571428244619342235867310720000
- Correct answer is either 17636 or 17637.

## Integer Division: Newton's Method

$(q, r)$ = NewtonDivision($s$, $t$)

---

Choose $x$ to be unique fractional power of 2 in interval $(1 / (2t),\ \ 1 / t\ ]$

**repeat** lg $n$ times

$x \leftarrow 2x - t x^2$

**set** $q = \lfloor s\, x \rfloor$ **or** $q = \lceil s\, x \rceil$

**set** $r = s - q\, t$

---



## Analysis

L1. Iterates converge monotonically.

$$\frac{1}{2t} < x_0 \le x_1 \le x_2 \le \cdots \le \frac{1}{t}.$$

Pf. [by induction on $i$]

- Base case: by construction, $\frac{1}{2t} < x_0 \le \frac{1}{t}$

- Inductive hypothesis: $\frac{1}{2t} < x_0 \le x_1 \le \cdots \le x_i \le \frac{1}{t}$.

$$\begin{aligned} x_{i+1} &= 2x_i - t\, x_i^2 \\ &= x_i(2 - t\, x_i) \\ &\ge x_i(2 - t\,(1/t)) \\ &= x_i \end{aligned}$$

inductive hypothesis

*(monotonic)*

$$\begin{aligned} x_{i+1} &= 2x_i - t\, x_i^2 \\ &= (2x_i - t\, x_i^2 - 1/t) + 1/t \\ &= -t(x_i - 1/t)^2 + 1/t \\ &\le 1/t \end{aligned}$$

*(bounded)*



## Analysis

L2. Iterates converge quadratically to $1 / t$: $1 - t x_i < \frac{1}{2^{2^i}}$.

$x_i$ is approximates $1 / t$ to $2^i$ significant bits of accuracy

Pf. [by induction on $i$]

- Base case: by construction, $\frac{1}{2t} < x_0 \ \Rightarrow\ 1 - t\, x_0 < \frac{1}{2}$

- Inductive hypothesis: $1 - t\, x_i < \frac{1}{2^{2^i}}$

$$\begin{aligned} 1 - t\, x_{i+1} &= 1 - t\,(2x_i - t\, x_i^2) \\ &= (1 - t\, x_i)^2 \\ &< \left(\frac{1}{2^{2^i}}\right)^2 \\ &= \frac{1}{2^{2^{i+1}}} \end{aligned}$$

inductive hypothesis



## Analysis

L3. Algorithm returns correct answer.

Pf.

- By L2, after $k = \lceil \lg \lg (s / t) \rceil$ steps, we have: $1 - t x_k < \frac{1}{2^{2^k}} \le \frac{t}{s}$.

L2 choice of $k$

- Thus, $0 \le \frac{s}{t} - s x_k < 1$

$x_k \le 1 / t$ by L1 rearranging expression above

- This implies, $q = \lfloor s / t \rfloor$ is either $\lfloor s\, x_k \rfloor$ or $\lceil s\, x_k \rceil$.

- Note: $k \le \lg n$.

## Analysis

Theorem. Algorithm computes quotient and remainder in $O(M(n))$ time, where $M(n)$ is the time to multiply two $n$-bit integers.

Pf.

- The number of iterations is $k = \lg n$.
- By L2, the algorithm returns the correct answer.
- Each iterate involves $O(1)$ multiplications and additions.

$$\begin{aligned} f(x) &= t - \frac{1}{x} \\ x_{i+1} &= 2x_i - tx_i^2 \end{aligned}$$

- Note: algorithm still works if we only keep track of $2^i$ significant digits in iteration $i$.

- Overall running time: $M(1) + M(2) + M(4) + \ldots + M(2^k) = O(M(n))$.



## Analysis

Theorem. Algorithm computes quotient and remainder in $O(M(n))$ time, where $M(n)$ is the time to multiply two $n$-bit integers.

Corollary. Can do integer division in $O(n^{1.585})$ bit operations.



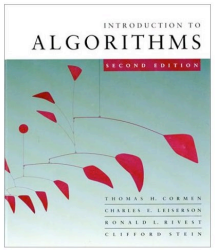
## Integer Arithmetic

Theorem. The following have the same asymptotic bit complexity.

- Multiplication.
- Squaring.
- Quotient.
- Remainder.



# Matrix Multiplication




*Chapter 28.2*

## Dot Product

Dot product. Given two length $n$ vectors $a$ and $b$, compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$

$$\begin{aligned} a &= \begin{bmatrix} .70 & .20 & .10 \end{bmatrix} \\ b &= \begin{bmatrix} .30 & .40 & .30 \end{bmatrix} \\ a \cdot b &= (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32 \end{aligned}$$

Remark. Grade-school dot product algorithm is optimal.



## Matrix Multiplication

Matrix multiplication. Given two $n$-by-$n$ matrices $A$ and $B$, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm optimal?



## Block Matrix Multiplication

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$C_{11}$ $A_{11}$ $A_{12}$ $B_{11}$ $B_{11}$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$



## Matrix Multiplication: Warmup

To multiply two $n$-by-$n$ matrices $A$ and $B$:

- Divide: partition $A$ and $B$ into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

## Fast Matrix Multiplication

Key idea. multiply 2-by-2 blocks with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

- 7 multiplications.
- $18 = 8 + 10$ additions and subtractions.



## Fast Matrix Multiplication

To multiply two $n$-by-$n$ matrices $A$ and $B$: [Strassen 1969]

- Divide: partition $A$ and $B$ into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Compute: 14 $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices via 10 matrix additions.
- Conquer: multiply 7 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume $n$ is a power of 2.
- $T(n)$ = # arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$



## Fast Matrix Multiplication: Practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm around $n = 128$.

Common misperception. *"Strassen is only a theoretical curiosity."*

- Apple reports 8x speedup on G4 Velocity Engine when $n \approx 2{,}500$.
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize" $Ax = b$, determinant, eigenvalues, SVD, ....



## Fast Matrix Multiplication: Theory

Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?
A. Yes! [Strassen 1969] $\Theta(n^{\log_2 7}) = O(n^{2.807})$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?
A. Impossible. [Hopcroft and Kerr 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Two 3-by-3 matrices with 21 scalar multiplications?
A. Also impossible. $\Theta(n^{\log_3 21}) = O(n^{2.77})$

Begun, the decimal wars have. [Pan, Bini et al, Schönhage, ...]

- Two 20-by-20 matrices with 4,460 scalar multiplications. $O(n^{2.805})$
- Two 48-by-48 matrices with 47,217 scalar multiplications. $O(n^{2.7801})$
- A year later. $O(n^{2.7799})$
- December, 1979. $O(n^{2.521813})$
- January, 1980. $O(n^{2.521801})$

# Fast Matrix Multiplication: Theory

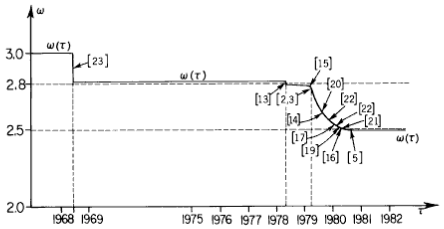



FIG. 1. *$\omega(t)$ is the best exponent announced by time $\tau$.*

Best known. $O(n^{2.376})$ [Coppersmith-Winograd 1987]

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. Theoretical improvements to Strassen are progressively less practical.