On the Consistency under Expectation of Tournament Rules

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Abstract

We define the notion of a tournament rule being *consistent under expectation* as a metric of its fairness. In particular, a rule is said to be consistent under expectation if the expectation (over the randomness of the rule) of the rank of a team equals n minus the number of teams this team defeats. We show that Nested Randomized King of the Hill, a tournament rule introduced in [2], is consistent under expectation and use this result to conclude that no set of teams of any size can collude to manipulate a tournament and gain any advantage in expectation given a uniform prize vector. We further bound the advantage that a set of players can gain through manipulation when the prize vector is close to uniform. Finally, we provide examples of tournament rules that are not consistent under expectation.

1 Introduction

Given a set of n teams, we form a tournament by having teams play matches against each other in order to form a full ranking of these n teams. Generating a ranking of the teams based on the results of the pairwise matches requires care, as the results of the matches may not be transitive (eg. we can have that Team A beats Team B, and that Team B beats Team C, but Team C beats Team A). Therefore, a topological sort where a team is always ranked above every team that they beat may not be possible. The most intuitive thing to do here is to output that ranking that minimizes the number of times that a team is ranked below a team that it defeats. However, computing such a ranking is known to be NP-Hard [1]. To tackle this, we study tournament rules which algorithmically assign rankings to teams based on the results of the matches in the tournament.

Generating a fair ranking is especially important when prizes are awarded to teams based on their rank. Many sports competitions offer prizes and seek to reward better teams as defined by the results of the tournament. However, the introduction of prizes may also incentivize collusion between teams in order to improve their combined winnings. This motivates the question of how susceptible a particular tournament ranking rule is to manipulation by the participants.

Prior work [2] proposes a tournament rule, known as Nested Randomized King of the Hill (NRKotH), which has certain desirable properties — Condorcet–Consistency and Cover–Consistency. Condercet–Consistency guarantees that if there exists a team that defeats all other teams, then this team will be ranked first. Cover–Consistency guarantees that if a team u defeats a team v as well as all teams that v defeats, then u will get a better ranking than v. Furthermore, NRKotH bounds the total expected gain from two team colluding by $1/3$. This bound applies to any prize vector (which is a non-decreasing assignment of a reward between 0 and 1 to a team based on their rank in the tournament).

In this paper, we improve this bound for the NRKotH Tournament Rule for a specific class of prize vectors. We show that for a uniform prize vector, collusion between a set of teams of any size yields 0 expected gain. To do this, we define the metric of a tournament rule being consistent under expectation as when the expected rank of any specific team is equal to the total number of teams in the tournament minus the number of teams that this specific team beats. We then prove that NRKotH satisfies this property, which gives us our bound. Additionally, we provide another bound for prize vectors that are sufficiently close to the uniform prize vector.

We end with examples of tournament ranking rules that are not consistent under expectation.

2 Preliminaries

We use notation consistent with [2]. Throughout the rest of this section, we consider some tournament T on n teams and some Tournament Ranking Rule r.

Definition 2.1 ***(Tournament):*** *A tournament on n is a set of matches between these teams and the results of these matches. This can be thought of as a directed complete graph on n vertices (representing the teams), with the edge between two teams pointing to the winner of the match between the two teams.*

Definition 2.2 ***(Tournament Rule):*** *For a tournament on n teams, a tournament rule is an algorithm that produces an assignment of the teams to ranks $\{1, 2, \ldots, n\}$ based on the results of the tournament, where each team gets a unique rank. In general, we think of lower ranks as being given to teams that perform "better."*

Definition 2.3 *For any individual team t and set of teams S, we define $w_S(t)$ to be the set of teams in S that t defeats.*

Definition 2.4 *For any team t, we define $\sigma_S(t)$ to be the ranking of t under some tournament rule, where S is the set of teams in the tournament. Note that each $\sigma_S(t)$ must be a unique integer from 1 through $|S|$.*

Definition 2.5 ***(Prize Vector):*** *We define the prize vector for a tournament T on n participants as the vector $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ where the participant with rank i receives reward p_i. We consider only prize vectors in $[0, 1]^n$ that satisfy the following condition: $p_i \geq p_{i+1}$. That is to say that it is never worse to receive a better rank.*

Definition 2.6 ***(Uniform Prize Vector):*** *We define a uniform prize vector for a tournament T on n participants as the vector $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ where $p_i = 1 - (i-1)/(n-1)$.*

Definition 2.7 *($S-$**Adjacent Tournament**):* *Consider any two tournaments T and T' on the same set of teams. Additionally, consider any subset of these teams S. We define T and T' to be $S-$Adjacent if and only if the results of the matches in T and T' differ only in matches between teams in S.*

Definition 2.8 ***(Manipulation):*** *When a set of teams collude to change the result of matches between teams in the set in order to receive greater total winnings, we call this manipulating a tournament. We define $\alpha_k^{\vec{p}}(r)$ to be the maximum expected gain that any set of k teams can get by manipulating any tournament they are in under the tournament rule r.*

Definition 2.9 ***(NRKotH):*** *The Nested Randomized King of the Hill (NRKotH) tournament rule presented in [2] operates as follows: First choose a uniformly random team i. Then, rank all the teams that i beat below i and all the teams that beat i above i, fixing the rank of i. Then, recursively apply the same algorithm to the set of teams above and below i (with a base case of doing nothing when the set is of size 0).*

3 Main Results

Definition 3.1 *We say a tournament rule on a set of n teams, S, is said to be "consistent under expectation" if it satisfies the following constraint for all teams t :*

$$\mathbb{E}\left[\sigma_S(t)\right] = |S| - |w_S(t)|$$

Theorem 3.1 *NRKotH is consistent under expectation.*

Proof. We prove the result by strong induction on $n = |S|$.

The base case of $n = 1$ is trivial. Assume, as a part of the inductive hypothesis, that for all sets A with $|A| < n$, we have that $\mathbb{E}\left[\sigma_A(t)\right] = |A| - |w_A(t)|$. Consider some team $u \in S$. During the first iteration of NRKotH, let v be the prince. We compute the expectation of $\sigma_S(u)$ by conditioning on whether $v = u$, v defeats u, or v loses to u. Note that since the prince is selected uniformly at random, the probability that a given team is the prince is $1/|S| = 1/n$. If $v = u$, the rank of u equals $n - |w_S(u)|$. Now if u defeats v, the rank of u equals the rank of u among those teams that defeat v. On the other hand, if v defeats u, the rank of u equals the rank of v plus the rank of u among those teams that v defeats. Since the rank of v must be $n - |w_S(v)|$, we have that

$$\mathbb{E}\left[\sigma(u)\right] = \frac{1}{n}\left(n - |w_S(u)| + \sum_{v \in w_S(u)} \mathbb{E}\left[\sigma_{S \setminus w_S(v)}(u)\right] + \sum_{\substack{v \in S \setminus w_S(u) \\ v \neq u}} \left(n - |w_S(v)| + \mathbb{E}\left[\sigma_{w_S(v)}(u)\right]\right)\right).$$

By the inductive hypothesis, we have that

$$\mathbb{E}\left[\sigma_{S\backslash w_S(v)}(u)\right] = |S\backslash w_S(v)| - |w_{S\backslash w_S(v)}(u)| = n - |w_S(v)| - |w_S(u)\backslash w_S(v)| = n - |w_S(u)| - |w_S(v)\backslash w_S(u)|$$

since $w_{S\backslash w_S(v)}(u)$ is the set of teams in S that lose to u but defeat v. In addition, we used the fact that $|w_S(v)| + |w_S(u)\backslash w_S(v)| = |w_S(u)| + |w_S(v)\backslash w_S(u)|$ since they both equal $|w_S(v) \cup w_S(u)|$. Very similarly, we have

$$\mathbb{E}\left[\sigma_{w_S(v)}(u)\right] = |w_S(v)| - |w_{w_S(v)}(u)| = |w_S(v)| - |w_S(u) \cap w_S(v)| = |w_S(v)| - |w_S(u)| + |w_S(u)\backslash w_S(v)|$$

since $w_{w_S(v)}(u)$ is the set of teams in S that lose to u and v. Plugging these expressions into the original equation gives us

$$\mathbb{E}\left[\sigma(u)\right] = \frac{1}{n}\Bigg(n - |w_S(u)| + \sum_{v \in w_S(u)} n - |w_S(u)| - |w_S(v)\backslash w_S(u)| + \sum_{\substack{v \in S\backslash w_S(u) \\ v \neq u}} n - |w_S(u)| + |w_S(u)\backslash w_S(v)|\Bigg)$$

Taking the $n - |w_S(u)|$ term out of each of the sums (there are n in total) gives us

$$\mathbb{E}\left[\sigma(u)\right] = n - |w_S(u)| + \frac{1}{n}\left(\sum_{\substack{v \in S\backslash w_S(u) \\ v \neq u}} |w_S(u)\backslash w_S(v)| - \sum_{v \in w_S(u)} |w_S(v)\backslash w_S(u)|\right)$$

It suffices to prove that these two sums are equal. We now write $|w_S(u)\backslash w_S(v)|$ as a sum of indicator variables:

$$|w_S(u)\backslash w_S(v)| = \sum_{\substack{x \in S \\ x \neq u,v}} \mathbb{I}[x \in w_S(u), x \notin w_S(v)].$$

Note that we have the constraint $x \neq u$ since $u \notin w_S(u)$ and we have the constraint $x \neq v$ since if $x = v$, we have that $v \in S\backslash w_S(v)$ and $v \in w_S(v)$, which is a contradiction. Thus, we can write the entire sum as

$$\sum_{\substack{v \in S\backslash w_S(u) \\ v \neq u}} |w_S(u)\backslash w_S(v)| = \sum_{\substack{v \neq u \\ x \in S \\ x \neq u,v}} \mathbb{I}[u \in w_S(v), x \in w_S(u), x \notin w_S(v)]$$

Very similarly, we can write

$$\sum_{v \in w_S(u)} |w_S(v)\backslash w_S(u)| = \sum_{\substack{v \neq u \\ x \in S \\ x \neq u,v}} \mathbb{I}[v \in w_S(u), x \in w_S(v), x \notin w_S(u)]$$

where we have $v \neq u$ since $v \notin w_S(u)$. Notice that interchanging v and x in this sum gives us the previous sum (since $v \in w_S(x) \iff x \notin w_S(v)$ for $x \neq v$). Consequently, we have that

$$\mathbb{E}\left[\sigma(u)\right] = n - |w_S(u)|$$

and we are done. ∎

Theorem 3.2 *For a uniform prize vector $\vec{p} \in \mathbb{R}^n$, we have that $\alpha_k^{\vec{p}}(r) = 0$ for all natural numbers $k \leq n$ and all tournament rules r.*

Proof. For a uniform prize vector $\vec{p}$, we have $p_i = 1 - (i-1)/(n-1) = (n-i)/(n-1)$. Now consider a set of k teams $\mathcal{A} = \{a_1, a_2, \cdots, a_k\}$ that decide to collude.

Let W be the total winnings of teams in $\mathcal{A}$. Then we have:

$$W = \sum_{a \in A} p_{\sigma(a)} = \sum_{a \in A} \frac{n}{n-1} - \frac{\sigma(a)}{n-1}.$$

So, their expected total winnings is

$$\mathbb{E}\left[W\right] = \sum_{a \in A} \mathbb{E}\left[\frac{n}{n-1} - \frac{\sigma(a)}{n-1}\right] = k\frac{n}{n-1} - \frac{1}{n-1}\sum_{a \in \mathcal{A}} \mathbb{E}[\sigma(a)]$$

By Theorem 3.1, we know $\mathbb{E}\left[\sigma(a)\right] = n - |w(a)|$ Additionally, we can write the number of wins of a player a as the sum of the number of wins against players in $\mathcal{A}$ and the number of wins against players outside of $\mathcal{A}$. Therefore, $\mathbb{E}\left[\sigma(a)\right] = n - (|w_{\mathcal{A}}(a)| + |w_{S \setminus \mathcal{A}}(a)|)$. Plugging this into our previous equation yields

$$\begin{aligned}
\mathbb{E}\left[W\right] &= \frac{kn}{n-1} - \frac{1}{n-1}\sum_{a \in \mathcal{A}} \big(n - (|w_{\mathcal{A}}(a)| + |w_{S \setminus \mathcal{A}}(a)|)\big) \\
&= \frac{1}{n-1}\sum_{a \in \mathcal{A}} \big(|w_{\mathcal{A}}(a)| + |w_{S \setminus \mathcal{A}}(a)|\big) \\
&= \frac{1}{n-1}\left(\sum_{a \in \mathcal{A}} |w_{\mathcal{A}}(a)| + \sum_{a \in \mathcal{A}} |w_{S \setminus \mathcal{A}}(a)|\right)
\end{aligned}$$

Regardless of the results of the matches between teams in $\mathcal{A}$, the total number of wins between players in $\mathcal{A}$ is $\binom{k}{2}$, as there are $\binom{k}{2}$ matches between teams in $\mathcal{A}$, and each has exactly one winner. Thus our expression becomes

$$\mathbb{E}\left[W\right] = \frac{1}{n-1}\left(\binom{k}{2} + \sum_{a \in \mathcal{A}} |w_{S \setminus \mathcal{A}}(a)|\right)$$

Now, we note that the expected total winnings of teams in $\mathcal{A}$ is only dependent on the results of matches between teams in this set and teams outside of this set. Therefore, for any $\mathcal{A}-$adjacent tournaments T and T', the expected total winnings of teams in $\mathcal{A}$ will be the same (as the results in these two tournaments only differ for matches between teams in $\mathcal{A}$). Thus, $\alpha_k^{\vec{p}}(\text{NRKotH}) = 0$, and we are done. ■

Definition 3.2 *($\epsilon-$Uniform Prize Vectors) We define the class of $\epsilon-$uniform prize vectors to be the set of prize vectors $\vec{p^\epsilon} = (p_1^\epsilon, p_2^\epsilon, \ldots, p_n^\epsilon) \in \mathbb{R}^n$ such that $p_i^\epsilon \in \left[\frac{n-i}{n-1} - \epsilon, \frac{n-i}{n-1} + \epsilon\right]$ (ie. every element is within ϵ of the corresponding element in the uniform prize vector).*

Theorem 3.3 *For any $\epsilon-$uniform prize vector $\vec{p^\epsilon} \in \mathbb{R}^n$, we have that $\alpha_k^{\vec{p^\epsilon}}(r) \leq 2k\epsilon$ for all natural numbers $k \leq n$ and all tournament rules r.*

Proof. Again consider a set of teams $\mathcal{A}$ with $|\mathcal{A}| = k$. For a given tournament T, also consider a $\mathcal{A}-$adjacent tournament T'. We claim that the expected benefit for teams in $\mathcal{A}$ to manipulate results to create tournament T' instead of T is at most $2k\epsilon$.

Let W_T be the total winnings of teams in $\mathcal{A}$ from tournament T, and $W_{T'}$ be the same from T'. Further, let $\sigma(i)$ be the rank of team i given by NRKotH on T, and let $\sigma'(i)$ be the rank of team i given by NRKotH on T'. Now, we want to bound $\mathbb{E}\left[W_{T'} - W_T\right] = \mathbb{E}\left[W_{T'}\right] - \mathbb{E}\left[W_T\right]$.

Note that, by definition,

$$W_T = \sum_{a \in \mathcal{A}} p_{\sigma(a)}^\epsilon \geq \sum_{a \in \mathcal{A}} \frac{n}{n-1} - \frac{\sigma(a)}{n-1} - \epsilon.$$

Where the inequality comes from the fact that $\vec{p^\epsilon}$ is $\epsilon-$uniform. From this, we can bound the expectation of W_T by

$$\mathbb{E}\left[W_T\right] \geq \sum_{a \in \mathcal{A}} \mathbb{E}\left[\frac{n}{n-1} - \frac{\sigma(a)}{n-1} - \epsilon\right] = \frac{1}{n-1}\left(\binom{k}{2} + \sum_{a \in \mathcal{A}} |w_{S \setminus \mathcal{A}}(a)|\right) - k\epsilon,$$

where the last equality follows from removing the additive $-k\epsilon$ from the expectation and then following the same calculations as in the proof of Theorem 3.2. Similarly, we know:

$$W_{T'} = \sum_{a \in \mathcal{A}} p^{\epsilon}_{\sigma'(a)} \leq \sum_{a \in A} \frac{n}{n-1} - \frac{\sigma'(a)}{n-1} + \epsilon,$$

which then gives us

$$\mathbb{E}\left[W_{T'}\right] \leq \sum_{a \in \mathcal{A}} \mathbb{E}\left[\frac{n}{n-1} - \frac{\sigma'(a)}{n-1} + \epsilon\right] = \frac{1}{n-1}\left(\binom{k}{2} + \sum_{a \in \mathcal{A}} |w_{S \setminus \mathcal{A}}(a)|\right) + k\epsilon$$

We note that the results of matches between teams in $\mathcal{A}$ and teams outside of $\mathcal{A}$ are the same in both T and T', as they are $\mathcal{A}-$adjacent. Thus, the $\sum_{a \in \mathcal{A}} |w_{S \setminus \mathcal{A}}(a)|$ terms in each expression are equivalent. Thus, combining these two inequalities gives us that $\mathbb{E}\left[W_{T'} - W_T\right] = \mathbb{E}\left[W_{T'}\right] - \mathbb{E}\left[W_T\right] \leq 2k\epsilon$. ■

4 Non-Examples

Definition 4.1 ***(Single Elimination-Based Tournament Rule)*** *Assume that the number of teams is a power of* 2. *First we randomly pair all teams in the tournament and have each pair play a match. The winners of each match are ranked in the top half, and the losers are ranked in the bottom half. We then recursively perform this on each half (doing nothing if the size of the half is 0 or 1) to arrive at our final ranking.*

Theorem 4.1 *The single elimination based tournament ranking algorithm defined above is not consistent under expectation.*

Proof. Consider a tournament with 4 teams (A, B, C, and D) with the following results: C beats everyone, B beats everyone except C, D only beats A, and A loses to everyone.

There are 2 teams that B beats. We now analyze the expected rank for B outputted by our single elimination based tournament ranking algorithm. With probability $1/3$, B plays C and D plays A in the first round, which would result in B getting rank 3. With probability $1/3$, B plays D and C plays A in the first round, which would result in B getting rank 2. Finally, with probability $1/3$, B plays A and D plays C in the first round, which would result in B getting rank 2. Therefore, the expected rank of B is $7/3$, which is not equal to $4 - 2$, so this algorithm is not consistent under expectation. ■

Definition 4.2 ***(Bubble Sort-Based Tournament Rule)*** *Assume that the number of teams is a power of* 2. *First we randomly shuffle the* n *participating teams. Then, we do the following operation* n *times: starting from* $i = 1$ *and ending at* $i = n - 1$, *serially swap the positions of the* i^{th} *and* $(i+1)^{st}$ *teams if and only if the* i^{th} *team defeated the* $(i+1)^{st}$ *team.*

Theorem 4.2 *The bubble sort based ranking algorithm is not consistent under expectation.*

Proof. Consider a tournament with 4 teams $(A, B, C,$ and $D)$ with the following results: A loses to B and C but wins against D. B wins against C but loses to D and C loses to D. Similar to our approach in the proof of Theorem 3.1, we can show that, over the randomness of the initial shuffling, the expected rank of B is $2.75 \neq 4 - 2 = 2$. ■

References

[1] Noga Alon. Ranking tournaments. *SIAM J. Discret. Math.*, 20(1):137–142, Jan 2006.

[2] Authors. Approximately strategyproof tournament rules with multiple prizes (not yet submitted).