COS 445 - PSet 5

Due online Thursday, April 21st at 11:59 pm.

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- The cheatsheet gives problem solving tips, and tips for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp22.pdf`.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Scoring Rules (20 points, no collaboration)

Recall that we use the notation $S(\vec{x}, i)$ to denote the payoff that a scoring rule awards to the predictor when she reports probability distribution $\vec{x}$ and event i occurs. Prove that, for all $\alpha > 1$, the scoring rule $S(\vec{x}, i) = x_i^{\alpha-1} - \frac{\alpha-1}{\alpha} \cdot \left(\sum_j x_j^\alpha\right)$ is proper.

Hint: Observe that when $\alpha = 2$, this is Brier's scoring rule.

Problem 2: Fair Division (70 points)

Consider a cake-cutting setup as in class: there's a single cake, represented as the interval $[0, 1]$. There are n players, each with valuation function $V_i(\cdot)$ which takes as input a subset of $[0, 1]$. These valuations are additive, normalized, and divisible (as in lecture).

In all parts below, $\mathcal{A}$ denotes the set of all potential allocations of cake. That is, elements of $\mathcal{A}$ partition the cake into $S_1, \ldots, S_n$. Below, we will also use $\mathcal{P}$ to denote the set of **proportional** allocations of the cake (where each player gets value at least $1/n$). That is, an allocation $S_1, \ldots, S_n$ is in $\mathcal{P}$ if and only if $V_i(S_i) \geq 1/n$ for all i.

Finally, to simplify notation in the subsequent problem statements, we will let $\vec{S}$ refer to $(S_1, \ldots, S_n)$, and denote by $V(\vec{S}) := \sum_i V_i(S_i)$. Therefore, $V(\vec{S})$ denotes the total value of all players for the allocation $\vec{S}$.

Part a (10 points)

When $n = 2$, prove that **for all** V_1, V_2, $\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\min_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} \leq 2$. That is, for all V_1, V_2, the ratio of the total value of the best allocation to the total value of the worst proportional allocation is at most 2.

Part b (10 points)

Design two valuation functions V_1, V_2 such that $\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\min_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} = 2$. That is, design two valuation function where the ratio of the total value of the best allocation to the total value of the worst proportional allocation is exactly 2. Prove that your example is correct.

Part c (10 points)

Prove that for all $V_1, \ldots, V_n$, $\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\min_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} \leq n$. That is, for all $V_1, V_2, \ldots, V_n$, the ratio of the total value of the best allocation to the total value of the worst proportional allocation is at most n.

Part d (10 points)

For all n, design n valuation functions such that $\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\min_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} = n$. Prove that your example is correct. That is, design n valuation functions where the ratio of the total value of the best allocation to the total value of the worst proportional allocation is exactly n. Prove that your example is correct.

Part e (15 points)

When $n = 2$, prove that for all V_1, V_2:

$$\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\max_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} \leq 3/2$$

That is, the ratio of the total value of the best allocation to the **best proportional allocation** is at most $3/2$.[2]

[2] If you are able to prove that $\max_{\vec{S} \in \mathcal{A}}\{V(\vec{S})\}/\max_{\vec{S} \in \mathcal{P}}\{V(\vec{S})\} \leq x$ for an $x \in (3/2, 2)$, you will get significant partial credit.

Part f (15 points)

Design two valuation functions, V_1, V_2 such that $\max_{\vec{S}\in\mathcal{A}}\{V(\vec{S})\}/\max_{\vec{S}\in\mathcal{P}}\{V(\vec{S})\} > 1$. That is, design two valuation functions such that no proportional allocation is value-maximizing. Prove that your example is correct.

Problem 3: Price of Stability (60 points)

Consider the following game: there are n players who are each nodes in a graph. For each player v, and each other node u, player v may decide to build an edge to u, for cost α (v can build as many edges as they want). If neither u nor v select to build, there is no edge. If either u or v select to build, then there is an edge (and the edge does not get "better" if both build).

Once the edges are built, we have an undirected graph G, and each player v incurs cost equal to the total distance to all other nodes: $\sum_{u \neq v} d_G(v, u)$. Here, $d_G(v, u)$ denotes the length of the shortest path from v to u in G (and is infinite if u and v are disconnected in G).

So the total cost incurred by player v is $\alpha n_v + \sum_{u \neq v} d_G(v, u)$, where n_v is the number of edges purchased by v. The total social cost is then $\sum_v \alpha n_v + \sum_{(u,v), u \neq v} 2d_G(v, u)$. Observe that when no edge is built twice, this is exactly: $\alpha |E(G)| + \sum_{(u,v), u \neq v} 2d_G(v, u)$

Note: For the following problem, you may want to recall that a Nash equilibrium is a profile of strategies (one for each player), such that all players are best responding. You may also want to remember that the optimal solution is a profile of strategies (one for each player) which minimizes the social cost. Finally, you may want to remember that the Price of Stability is the ratio of the best Nash equilibrium (i.e. the Nash equilibrium which minimizes cost) over the optimum.

Part a (20 points)

Prove that when $\alpha \leq 1$, the Price of Stability is 1.

Hint: Try finding the optimum first.

Part b (20 points)

Prove that when $\alpha \geq 2$, the Price of Stability is 1.

Hint: Again, try finding the optimum first.

Part c (20 points)

Prove that when $\alpha \in (1, 2)$, the Price of Stability is at most $4/3$.

Extra Credit: Proportionality for large n

Let there be n players with normalized, additive values for a cake $[0, 1]$. Let also $\mathcal{A}$ denote the set of all partitions of cake to the n players. Let $\mathcal{P}$ denote the set of all proportional partitions of cake to the n players (that is, each player has value at least $1/n$ for their allocated cake).

For notation below, for an allocation $S := S_1 \sqcup \ldots \sqcup S_n$, let $V(S) := \sum_i V_i(S_i)$. Prove that for all n, and all valuations $V_1, \ldots, V_n$, $\max_{S \in \mathcal{A}}\{V(S)\}/\max_{S \in \mathcal{P}}\{V(S)\} = O(\sqrt{n})$. That is, the welfare of the best proportional allocation is at least an $O(\sqrt{n})$ factor of the best welfare without proportional constraints.

For all n, provide a list of valuations $V_1, \ldots, V_n$ such that $\max_{S \in \mathcal{A}}\{V(S)\}/\max_{S \in \mathcal{P}}\{V(S)\} = \Omega(\sqrt{n})$ (that is, prove that the previous bound is tight up to constant factors).

Hint: You will for sure want to use ideas from the algorithm we saw in class to find a proportional allocation.

Hint: Try to break it down into cases where not-that-many players contribute more than $1/\sqrt{n}$ to the total value, and those where many players contribute more than $1/\sqrt{n}$.