COS 445 - PSet 4

Due online Monday, April 4th at 11:59 pm.

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- The cheatsheet gives problem solving tips, and tips for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp22.pdf`.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Combinatorial Auctions (20 points, no collaboration)

In a combinatorial auction there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set (so $v_i(S) = 5$ means that bidder i gets value 5 for receiving exactly the set S). These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$. But you should make no other assumptions on $v_i(\cdot)$. Each item can be awarded to at most one bidder.

The designer's goal is to distribute the items to the bidders (in a way such that each item is awarded to at most one bidder), and to do so in a way that maximizes the welfare (the sum over all i of the value that bidder i has for the set they receive). Design an auction that is incentive compatible and maximizes welfare. A complete solution should describe:

- On bids $v_1(\cdot), \dots, v_n(\cdot)$, what set S_i does bidder i get?
- On bids $v_1(\cdot), \dots, v_n(\cdot)$, what price does bidder i pay?

You should give both answers as an explicit formula. For example, "bidder i should pay 5" is an explicit formula. So is "bidder i should pay $\sum_{j \neq i} v_j(\{1\})$." "Bidder i should pay the harm they cause to bidder $i - 1$" is not an explicit formula. Similarly, "Bidder i should get set $\{1, 2\}$" is an explicit formula. So is "Bidder i should get the set S_i which maximizes $v_1(S_i) + v_i(S_i)$." "Bidder i should get the set which they are awarded in the welfare-maximizing allocation" is not an explicit formula.

Hint: The intention of this problem is for you to figure out how to mechanically instantiate the VCG mechanism for this setting. You are allowed to use the VCG auction for guidance and provide a complete proof that your auction is incentive compatible and maximizes welfare. You are also allowed to design an auction and prove that your auction is a special case of VCG (this proof might be pretty short).

Problem 2: Some issues with Greedy (40 points)

In a combinatorial auction there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set (so $v_i(S) = 5$ means that bidder i gets value 5 for receiving set S). These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$. Unless otherwise specified, you should not make any other assumptions on $v_i(\cdot)$.

Consider the following mechanism for allocating items:

- Initialize $S_i = \emptyset$ (each bidder i initially gets no items). Initialize $p_i = 0$ (each bidder initially pays 0).
- For $j = 1$ to m (for each item in order)
  - Ask each bidder their marginal value for item j: $b_{ij}(S_i) = v_i(S_i \cup \{j\}) - v_i(S_i)$ (how much additional value would they get right now by adding j).
  - Reveal all bids to all bidders (that is, for all i, reveal the marginal value that bidder i reported for item j to all bidders).
  - Award item j to the bidder i who reports the largest value (breaking ties lexicographically), add to their payment the second-highest report. That is, if i reports the largest marginal value and i' reports the second-largest, Update S_i to $S_i \cup \{j\}$, and p_i to $p_i + b_{i'j}(S'_i)$.
- Award bidder i the set S_i of items and charge them p_i.

Part a (20 points)

Prove that if all valuation functions are additive (that is, $v_i(S) = \sum_{j \in S} v_i(\{j\})$, for all S), then it is a Nash equilibrium for all bidders to truthfully report $b_{ij}(S_i) := v_i(S_i \cup \{j\}) - v_i(S_i)$ in every round.

Hint: Remember that a list of strategies are a Nash equilibrium if every player i is best responding to the other players.

Part b (10 points)

Prove that, even if all valuation functions are additive, and even if $n = m = 2$, it is not a dominant strategy for bidders to truthfully report $b_{ij}(S_i) := v_i(S_i \cup \{j\}) - v_i(S_i)$ in every round.

Hint: It may help to explicitly think about what strategies a bidder can use in this auction. Recall that a dominant strategy is a best response to every strategy the other player might use. So if you want to show that something is not a dominant strategy,. . . .

Part c (10 points)

Provide one example of valuation functions $v_1(\cdot)$ and $v_2(\cdot)$, such that it is not a Nash equilibrium for both bidders to bid their true marginal valuations. Specifically, prove (in your example) that if bidder 2 tells the truth, then bidder 1 can do strictly better by lying.

Problem 3: Revenue Equivalence (50 points)

Consider a single-item auction with two bidders whose values are drawn from the equal-revenue curve ER, ($F(x) = 1 - 1/x$ for all $x \geq 1$, and $f(x) = 1/x^2$ for all $x \geq 1$). The following parts will guide you through a proof to find a Bayes-Nash equilibrium of the first-price auction using Revenue Equivalence. You should complete all parts and not provide an alternative proof.

Recall that a bidding strategy $b_1(\cdot)$ is a best response to $b_2(\cdot)$ if: for all v_1, in expectation over $v_2 \leftarrow ER$, and bidder two bidding $b_2(v_2)$, bidder 1's payoff is (weakly) maximized by bidding $b_1(v_1)$. Recall also that your payoff from a first price auction is equal to $v - b$ if you win and bid b, and zero otherwise.

Part a (10 points)

What is the expected revenue of the second-price auction when two bidders with values independently drawn from equal-revenue curves bid their true value? You should also prove that you computed your answer correctly.

Part b (10 points)

In the second-price auction, what is the expected payment made by bidder one, conditioned on bidding v_1, and that bidder two truthfully reports $v_2 \leftarrow ER$? You should also prove that you computed your answer correctly.

Note that we are **not** conditioning on bidder 1 winning. To be extra formal, let $P_1^{SPA}(v_1)$ denote the random variable that is equal to v_2 if $v_1 > v_2$, and 0 otherwise (that is, Player One pays v_2 when they win, and 0 if they do not). What is $\mathbb{E}_{v_2 \leftarrow ER}[P_1^{SPA}(v_1)]$?

Part c (10 points)

For a given bidding strategy $b(\cdot)$, define $P_1^{FPA}(v_1, b)$ to be the random variable that is equal to $b(v_1)$ if $v_1 > v_2$, and 0 otherwise. Find a bidding strategy $b(\cdot)$ such that:

- $b(\cdot)$ is strictly monotone increasing on $[1, \infty)$ ($b(v) > b(v') \Leftrightarrow v > v'$). That is, bidder 1 will win the first price auction exactly when $v_1 > v_2$ if both bidders use strategy $b(\cdot)$.
- For all $v_1 \in [1, \infty)$, $\mathbb{E}_{v_2 \leftarrow ER}[P_1^{FPA}(v_1, b)] = \mathbb{E}_{v_2 \leftarrow ER}[P_1^{SPA}(v_1)]$. That is, the expected payment made by bidder 1, conditioned on v_1, and $v_2 \leftarrow ER$, is the same in both the first-price auction (when both bidders use $b(\cdot)$) and second-price auction (when both bidders tell the truth).

You should also prove that your answer has these properties.

Part d (20 points)

Prove that the strategy you found in Part c is a Bayes-Nash Equilibrium of the first-price auction for two bidders with values drawn from the equal-revenue curve. You will receive partial credit for correctly setting up the necessary equations and verifying them with an online solver. For full credit, you should also solve the necessary equations.

Hint: Proving this inevitably will require taking derivatives, but there is a clever trick that avoids overly painful calculations. If you are dreading the calculus you're about to do, try to be creative with other ways you can work through the math.

Extra Credit: Walrasian Equilibria

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation grade. Some extra credits are **quite** challenging and will contribute significantly.

For this problem, you may collaborate with any students and office hours. You may not consult course resources or external resources, as this is a proof of a well-known result.[2]

In a combinatorial auction there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set (so $v_i(S) = 5$ means that bidder i gets value 5 for receiving set S). These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$. A Walrasian Equilibrium is a non-negative price for each item $\vec{p}$ such that:

- Each buyer i selects to purchase a set $B_i \in \arg\max_S\{v_i(S) - \sum_{j \in S} p_j\}$.
- The sets B_i are disjoint, and $\cup_i B_i = [m]$.

Prove that a Walrasian equilibrium exists for $v_1, \ldots, v_n$ if and only if the optimum of the LP relaxation below (called the configuration LP) is achieved at an integral point (i.e. where each $x_{i,S} \in \{0, 1\}$).

$$\max \sum_i \sum_S v_i(S) \cdot x_{i,S}$$
$$\forall i,\ \sum_S x_{i,S} = 1$$
$$\forall j,\ \sum_{S \ni j} \sum_i x_{i,S} \leq 1$$
$$\forall i, S,\ x_{i,S} \geq 0.$$

Finally, provide an example of two valuation functions v_1, v_2 over two items where a Walrasian equilibrium doesn't exist.

[2] You may consult course resources for general refreshers on Linear Programming, but not for anything specific to this problem.