COS 445 - PSet 2

Due online Monday, February 21st at 11:59 pm

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- The cheatsheet gives problem solving tips, and tips for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp22.pdf`.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Vote Cancelling (20 points, no collaboration)

In this problem we define another desirable property of voting rules: that certain configurations of votes cancel out. There are n voters and m candidates. As a reminder, here are the rules we will be interested in for this problem.

- Plurality: pick the candidate with the most first-place votes. Tie-break in favor of the lexicographically-first candidate. For this problem, you must tie-break according to this rule (but your proofs/counterexamples may make use of this tie-breaking rule in order to simplify analysis).
- Borda: each candidate receives $m-1$ points for every first place vote, $m-2$ for every second place vote, and so on. The candidate with the most points wins. Tie-break in favor of the lexicographically-first candidate. For this problem, you must tie-break according to this rule (but your proofs/counterexamples may make use of this tie-breaking rule in order to simplify analysis).

Because both rules are anonymous, for simplicity of notation in this problem we will just refer to a set of votes $\{\succ_1, \ldots, \succ_n\}$, rather than a particular ordering. This problem will use the following definition:

Definition 1 (Cancels Out) *We say a voting rule F cancels out with respect to a set of votes $p^* := \{\succ'_1, \ldots, \succ'_k\}$, if for every set $p = \{\succ_1, \ldots, \succ_n\}$ of votes, the outcome of the election is the same for p and $p \cup p^*$. That is to say, for all p: $F(p) = F(p \cup p^*)$.*

For example, if $m = 2$, and F is the Majority voting rule (tie-breaking for a over b), Majority cancels out with respect to the set of votes $\{a \succ'_1 b, b \succ'_2 a\}$. This is because no matter what p you start with, if you add the two votes $a \succ b$, and $b \succ a$, each candidate gets one additional vote, and therefore the majority is preserved. Similarly, Majority cancels out with respect to the set of votes $\{a \succ'_1 b, a \succ'_2 b, b \succ'_3 a, b \succ'_4 a\}$.

Majority does not cancel out with respect to the set of votes $p^* := \{b \succ'_1 a, b \succ'_2 a, b \succ'_3 a\}$. This is because if you start from a p where currently a has one more vote than b (for example, $p = \{a \succ_1 b\}$), then adding $\{b \succ'_1 a, b \succ'_2 a, b \succ'_3 a\}$ causes the majority to switch from a to b. This is an explicit choice of p such that $F(p) \neq F(p \cup p^*)$, and therefore Majority does not cancel out with respect to p^*.

Part a (definitions): Opposite votes

Define a pair of votes to be opposing if they rank the candidates in exactly the opposite order. Specifically, the preferences $\succ_i$ and $\succ_j$ are opposing if: $c_1 \succ_i c_2 \succ_i \ldots \succ_i c_m$ and $c_m \succ_j c_{m-1} \succ_j \ldots \succ_j c_1$.

We say a voting rule is opposite-cancelling if it cancels out with respect to **every** pair of opposite votes. That is to say a rule is opposite-cancelling if for every profile, adding/removing any pair of opposing votes preserves the outcome of the election.

Part a (question): Opposite votes (10 points)

For each of the two rules defined above (Plurality and Borda), prove that it is opposite-cancelling, or provide a counterexample. If you provide a counterexample, include a brief explanation of why it is a counterexample.

Part b (definition): Cyclic votes

Define a set of (exactly) m votes to be cyclic if they rank the candidates in cyclic order. Specifically, preferences $\succ_1, \ldots, \succ_m$ are cyclic if:

- Voter 1 votes like this: $c_1 \succ_1 c_2 \succ_{v_1} \ldots \succ_1 c_m$,
- Voter 2 votes like this: $c_2 \succ_2 c_3 \succ_{v_2} \ldots \succ_2 c_1$,
- ...
- Voter j votes like this: $c_j \succ_j c_{j+1} \succ_{v_j} \ldots \succ_j c_{j-1}$,
- ...
- Voter m votes like this: $c_m \succ_m c_1 \succ_m \ldots \succ_m c_{m-1}$.

For example, if there were 4 candidates (say, a, b, c, d) then the set of voters Alice, Bob, Charlie, Danielle would be cyclic if, for instance,

- $a \succ_{Alice} b \succ_{Alice} c \succ_{Alice} d$,
- $b \succ_{Bob} c \succ_{Bob} d \succ_{Bob} a$,
- $c \succ_{Charlie} d \succ_{Charlie} a \succ_{Charlie} b$,
- $d \succ_{Danielle} a \succ_{Danielle} b \succ_{Danielle} c$.

We say a voting rule is cycle-cancelling if it cancels out with respect to **every** set of cyclic votes. That is to say a rule is cycle-cancelling if for every set of votes, adding/removing any set of cyclic votes preserves the outcome of the election.

Part b (question): Cyclic votes (10 points)

For each of the two rules defined above (Plurality and Borda), prove that it is cycle-cancelling, or provide a counterexample. If you provide a counterexample, include a brief explanation of why it is a counterexample.

Problem 2: Two Candidates, One Rule (40 points)

For this problem, there are n voters and $m = 2$ candidates, and $n \geq 3$ is odd. Therefore, for a voting rule to have property X, it only needs to have property X when $m = 2$ and $n \geq 3$ is odd. For a voting rule to not have property X, there must exist a counterexample with $m = 2$ and $n \geq 3$ odd. Recall also the following definitions:

Definition 2 (Unanimous) *A voting rule F is unanimous if for all candidates a, whenever all voters select a as their favorite candidate, F outputs a. Formally, F is unanimous if whenever there exists a candidate a such that $a \succ_i b$ for all i and $b \neq a$, then $F(\succ_1, \ldots, \succ_n) = a$.*

Definition 3 (Anonymous) *A voting rule F is anonymous if the identities of the voters does not matter. Put another way, F is anonymous if the output of F is invariant under relabeling the voters. Formally, F is anonymous if for all $\succ_1, \ldots, \succ_n$, and all permutations of voters σ, $F(\succ_1, \ldots, \succ_n) = F(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)})$.*

Definition 4 (Neutral) *A voting rule F is neutral if the identity of the candidates does not matter. Put another way, F is neutral if relabeling the candidates causes the output of F to be similarly relabeled. Formally, F is neutral if for all $\succ_1, \ldots, \succ_n$, and all permutations of candidates, τ, $F(\tau(\succ_1), \ldots, \tau(\succ_n)) = \tau(F(\succ_1, \ldots, \succ_n))$. Here, we have abused notation and let $\tau(\succ_i)$ denote the ordering which places candidate $\tau(a)$ over candidate $\tau(b)$ if and only if $a \succ_i b$.*

Definition 5 (Strategyproof) *A voting rule F is strategyproof if no voter can ever be strictly happier by lying. Formally, F is strategyproof if for all voters i, all true preferences $\succ_i$, and all possible lies $\succ_i'$, and all possible votes of other voters $\vec{\succ}_{-i}$, $F(\succ_i; \vec{\succ}_{-i}) \succeq_i F(\succ_i'; \vec{\succ}_{-i})$.*

Part a (5 points)

Design a voting rule which **is not** unanimous, **is not** neutral, and **is** strategyproof (and briefly prove that it **is not** unanimous, **is not** neutral, and **is** strategyproof).

Part b (5 points)

Design a voting rule which **is** unanimous, **is** neutral, and **is not** strategyproof (and briefly prove that it **is** unanimous, **is** neutral, and **is not** strategyproof).

Part c (5 points)

Design a voting rule which **is** unanimous, **is not** neutral, and **is** strategyproof (and briefly prove that it **is** unanimous, **is not** neutral, and **is** strategyproof).

Part d (5 points)

Design a voting rule which **is** neutral, **is** anonymous, and **is** strategyproof (and briefly prove that it **is** neutral, **is** anonymous, and **is** strategyproof).

Part e (20 points)

Prove that every voting rule which is neutral and strategyproof is also unanimous.

Problem 3: "Never Worst" Preference Sets (40 points)

In this problem, there are $m \geq 3$ candidates. Any claim you prove must hold for all $m \geq 3$. Any counterexample you provide can pick a particular $m \geq 3$ of your choice.

In Lecture 6, we saw that if we know voter preferences are single-peaked, then we can design voting rules with interesting properties.

Definition 6 *A set $V \neq \emptyset$ of voter preferences are single-peaked if there exists an ordering of the candidates $c_1 \dots, c_m$ such that for all $\succ \in V$, the following holds. Let c_i denote the favorite candidaate of $\succ$. Then for all $j < k \leq i$, $c_k \succ c_j$. Also, for all $j > k \geq i$, $c_k \succ c_j$.*

One of the properties we saw in Lecture 6 concerned the concept of a Condorcet winner:

Definition 7 (Condorcet Winner) *A candidate a is a Condorcet winner if for every other candidate b, a wins a strict majority of votes in a head-to-head against b. Note that a Condorcet winner does not necessarily exist.*

This problem will explore a generalization of single-peaked preferences, called Never-Worst.

Definition 8 *A set $V \neq \emptyset$ of voter preferences are Never-Worst if for all sets of three candidates $\{a, b, c\}$, there exists an $x \in \{a, b, c\}$ such that every single $\succ \in V$ has $x \succ y$ for some $y \in \{a, b, c\}$. That is, for all sets of three candidates, there is some candidate x in that set, such that all voters agree that x is not the worst in that set.*

Part a (10 points)

Prove that if V is single-peaked, then V is Never-Worst.

Part b (10 points)

Prove that being single-peaked is not the only way to be Never-Worst. That is, find a set V of preferences that is not single-peaked, but is Never-Worst, and (briefly) prove that it is Never-Worst, and (briefly) prove that it is not single-peaked.

Observe that to prove a set of preferences is not single-peaked, you must show that no ordering on the candidates results in single-peaks.

You may not use code to find your example. When you find your example, your proof should be readable by a human, and should not simply exhaust every possible ordering to confirm it is not single-peaked. There exists an example with four candidates and two preferences, and a short non-exhaustive proof (but you are allowed to use another comparably-sized example with a comparably-short proof).

Part c (20 points)

Prove that if V is Never-Worst, and there are an odd number of voters with preferences in V, then there is a Condorcet winner.

Hint: You may want to first try to prove the claim when there are only three candidates.

Extra Credit: Why is nothing Strategyproof?

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation grade. Some extra credits are **quite** challenging and will contribute significantly.

For this problem, you may collaborate with any students and office hours. You may not consult course resources or external resources. In this problem we will guide you through the proof of a well-known result, so you should not copy the proof from one of the course texts (nor should you try to find a proof from external sources). You must follow the guide below (and not provide an alternative proof).

A Full-Ranking-Function (FRF) F is given a set of alternatives A and a profile of preferences over n voters, p, (just as with voting rules) except that now it must output a full ranking over A instead of a single winner.

Here are some desirable properties of FRFs:

- Unanimous: a FRF F is unanimous if whenever $a \succ_i b$ for all i, $\succ = F(\succ_1, \ldots, \succ_n)$ has $a \succ b$ (whenever everyone likes a better than b, the final ranking has a above b).
- Independence of Irrelevant Alternatives: consider two profiles $p = (\succ_1, \succ_2, ..., \succ_n)$, $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ and let $\succ = F(p), \succ' = F(p')$.

  A FRF F satisfies Independence of Irrelevant Alternatives if for any two alternatives $a, b \in A$, if $a \succ_i b \iff a \succ'_i b, \forall i$ (every voter has the same preference between a and b) then $a \succ b \iff a \succ' b$ (the ordering output by F ranks a vs. b the same). Intuitively this property suggests that our preferences for c should not interfere with the ranking of a and b, and is related to strategyproof-ness.

Here is an undesirable property of FRFs:

- Dictatorship: voter i is a dictator in a in FRF F if for all $p = (\succ_1, \succ_2, ..., \succ_n)$, $\succ_i = F(p)$. That is to say, no matter what everyone else submits, the FRF chooses the ordering of the dictator. F is a dictatorship if some voter is a dictator.

It would be nice to produce FRFs that are unanimous with Independence of Irrelevant Alternatives, and are not dictatorships. Unfortunately, the following theorem says that for $|A| \geq 3$, this is not possible:

Theorem 9 *Every unanimous FRF F satisfying Independence of Irrelevant Alternatives over a set of more than 2 alternatives is a dictatorship.*

Part a

Prove the following lemma:

Lemma 10 *Let $p = (\succ_1, \succ_2, ..., \succ_n)$, $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ be two profiles such that for every player i, $a \succ_i b \iff c \succ'_i d$. Then if F is unanimous with Independence of Irrelevant Alternatives, $a \succ b \iff c \succ' d$, where $\succ = F(p), \succ' = F(p')$ (when there are > 2 alternatives).*

Note that a complete proof needs to consider all of the following cases:

1. $a = c, b = d$.

2. $a = c, b \neq d$.

3. $a \notin \{c, d\}, b = c$.

4. $a \notin \{c, d\}, b = d$.

5. $a \notin \{c, d\}, b \notin \{c, d\}$.

6. $a = d, b = c$.

7. $a = d, b \neq c$.

You do not need to provide a proof for all 7 cases, as many are similar. Provide a proof for cases One, Two, and Five.

Part b

Take any $a \neq b \in A$, and for every $0 \leq i \leq n$ let π^i be some preference profile in which exactly the first i voters rank a above b, and the remaining voters rank b above a.

Prove that, if F is unanimous with Independence of Irrelevant Alternatives, there must be some $i^* \in [1, n]$ such that in $F(\pi^{i^*-1})$ we have $b \succ a$, but in $F(\pi^{i^*})$ we have $a \succ b$ (at this point, i^* might not be unique).

Part c

Use the lemma from part a to show the following lemma and corollary.

Lemma 11 *Let a and b be any two candidates in A. For all i, let π^i be some preference profile in which exactly the first i voters rank a above b, and the remaining voters rank b above a. Let F be unanimous with Independence of Irrelevant Alternatives, and let i^* be such that in $F(\pi^{i^*-1})$ we have $b \succ a$, but in $F(\pi^{i^*})$ we have $a \succ b$.*

Then, for any $c \neq d \in A$, and any preference profile: if $c \succ_{i^*} d$ then F ranks c above d. Conclude that i^* is a dictator for F.