Modern Expander-Based Error-Correcting Codes

Pedro Paredes

Rutgers/DIMACS Theory of Computing Seminar

December 4, 2024

Part I: Coding Theory

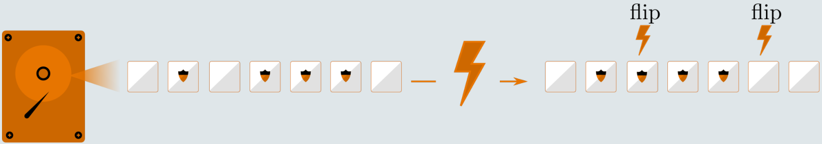
• Why codes? •

Sending a message through a noisy channel

• Why codes? •

Sending a message through a noisy channel

Storing data on a noisy medium



• What is a Code? •

Definition: Error-Correcting Code

A (binary) code $C \subseteq \{0,1\}^n$ is a subset of length-n strings

• What is a Code? •

Definition: Error-Correcting Code

A (binary) code $C \subseteq \{0,1\}^n$ is a subset of length-n strings

- **Blocklength:** n (the size of encoded messages)
- **Dimension/Rate:** $\dim(C)$ (the efficiency of the code)
- **Distance:** $\Delta(C)$ (the amount of errors we can correct)

• What is a Code? •

Definition: Error-Correcting Code

A (binary) code $C \subseteq \{0,1\}^n$ is a subset of length-n strings

- **Blocklength:** n (the size of encoded messages)
- **Dimension/Rate:** $\dim(C)$ (the efficiency of the code)
- **Distance:** $\Delta(C)$ (the amount of errors we can correct)

Definition: Dimension/Rate

$$\dim(C) = \log_2 |C| \qquad R = \frac{\dim(C)}{n}$$

• What is a Code? •

Definition: Error-Correcting Code

A (binary) code $C \subseteq \{0,1\}^n$ is a subset of length-n strings

- **Blocklength:** n (the size of encoded messages)
- **Dimension/Rate:** $\dim(C)$ (the efficiency of the code)
- **Distance:** $\Delta(C)$ (the amount of errors we can correct)

Definition: Dimension/Rate

$$\dim(C) = \log_2 |C| \qquad R = \frac{\dim(C)}{n}$$

Definition: Distance

$$\Delta(x,y) = \frac{1}{n}\sum_i 1[x_i \neq y_i] \qquad \Delta(C) = \min_{x,y \in C, x \neq y} \Delta(x,y)$$

• Linear Codes •

Definition: Linear Code

A code $C \subseteq \mathbb{F}_2^n$ is **linear** if it forms a subspace of $\mathbb{F}_2^n$.

• Linear Codes •

Definition: Linear Code

A code $C \subseteq \mathbb{F}_2^n$ is **linear** if it forms a subspace of $\mathbb{F}_2^n$.

Definition: Hamming Weight

$$|x| = \frac{1}{n}\sum_i 1[x_i = 1]$$

• Linear Codes •

Definition: Linear Code

A code $C \subseteq \mathbb{F}_2^n$ is **linear** if it forms a subspace of $\mathbb{F}_2^n$.

Definition: Hamming Weight

$$|x| = \frac{1}{n} \sum_i 1[x_i = 1]$$

Proposition: Linear Code Distance

If C is a linear code then $\Delta(C) = \min_{\substack{c \in C \\ c \neq 0}} |c|$

Pf. Say $\Delta(x, y) = \Delta(C)$, then $x + y \in C$ and $\Delta(x, y) = |x + y|$.

• Linear Codes •

Definition: Linear Code

A code $C \subseteq \mathbb{F}_2^n$ is **linear** if it forms a subspace of $\mathbb{F}_2^n$.

Definition: Hamming Weight

$$|x| = \frac{1}{n} \sum_i 1[x_i = 1]$$

Proposition: Linear Code Distance

If C is a linear code then $\Delta(C) = \min_{\substack{c \in C \\ c \neq 0}} |c|$

Pf. Say $\Delta(x, y) = \Delta(C)$, then $x + y \in C$ and $\Delta(x, y) = |x + y|$.

Definition: Parity Check Matrix

Matrix $H \in \mathbb{F}_2^{(n-k)\times n}$ such that $H \cdot c^\top = 0$ for all $c \in C$.

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma|L| \quad |\mathrm{N}(S)| \geqslant \epsilon|S|.$$

2-sided expansion means the same for R

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma |L| \quad |\mathrm{N}(S)| \geqslant \epsilon |S|.$$

2-sided expansion means the same for R

Definition: Lossless Expanders

A (d_1, d_2)-regular graphs is a *lossless expander* if it is a $(\gamma, (1-\epsilon)d_1)$-vertex expanders, for some constant γ and ϵ (which depends on γ).

So lossless expanders are optimal vertex expanders.

Definition: Unique-Neighborhood

Given a graph $G = (V, E)$ and a set $S \subseteq V$, $\mathrm{U}(S)$ is the set of neighbors of S adjacent to exactly one vertex in S

Definition: Unique-Neighborhood

Given a graph $G = (V, E)$ and a set $S \subseteq V$, $\mathrm{U}(S)$ is the set of neighbors of S adjacent to exactly one vertex in S

Definition: Unique-Neighbor Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-unique-neighbor expander if:

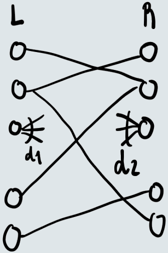
$$\forall S \subseteq L, |S| \leqslant \gamma |L| \quad |\mathrm{U}(S)| \geqslant \epsilon |S|.$$

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma|L| \quad |\mathrm{N}(S)| \geqslant \epsilon|S|.$$

2-sided expansion means the same for R

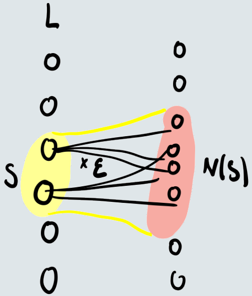


Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma|L| \quad |\mathrm{N}(S)| \geqslant \epsilon|S|.$$

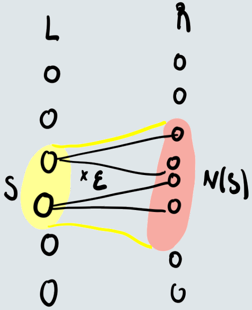
2-sided expansion means the same for R



Definition: Unique-Neighbor Expander

A (d_1, d_2)-regular graph (L, R, E) is a (1-sided) (γ, ϵ)-unique-neighbor expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma|L| \quad |\mathrm{U}(S)| \geqslant \epsilon|S|.$$



• Expander Codes [SS'96] •

Definition: Graph Code

Given a bipartite graph $G = (L, R, E)$, we can define a linear code $C_G \subseteq \mathbb{F}_2^{|L|}$ as:

$$c \in C_G : \forall w \in R, \bigoplus_{v \sim w} c_v = 0.$$

In other words, the parity check matrix $H \in \mathbb{F}_2^{|L| \times |R|}$ is such that $H_{u,v} = 1$ is $u \sim v$.

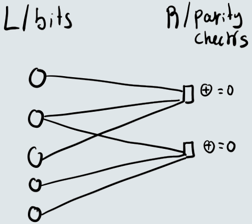
• Expander Codes [SS'96] •

Definition: Graph Code

Given a bipartite graph $G = (L, R, E)$, we can define a linear code $C_G \subseteq \mathbb{F}_2^{|L|}$ as:

$$c \in C_G : \forall w \in R, \bigoplus_{v \sim w} c_v = 0.$$

In other words, the parity check matrix $H \in \mathbb{F}_2^{|L| \times |R|}$ is such that $H_{u,v} = 1$ is $u \sim v$.



• Expander Codes [SS'96] •

Proposition: Graph Code Dimension

$$\dim(C_G) \geqslant |L| - |R|.$$

Pf. $|R|$ linear constraints on a space of dimension $|L|$.

• Expander Codes [SS'96] •

Proposition: Graph Code Dimension

$$\dim(C_G) \geqslant |L| - |R|.$$

Pf. $|R|$ linear constraints on a space of dimension $|L|$.

Theorem: Expander Codes

Suppose G is a (γ, ϵ)-unique-neighbor expander (where $\epsilon > 0$), then $\Delta(C_G) \geqslant \gamma$.

• Expander Codes [SS'96] •

Proposition: Graph Code Dimension

$$\dim(C_G) \geqslant |L| - |R|.$$

Pf. $|R|$ linear constraints on a space of dimension $|L|$.

Theorem: Expander Codes

Suppose G is a (γ, ϵ)-unique-neighbor expander (where $\epsilon > 0$), then $\Delta(C_G) \geqslant \gamma$.

Pf. Suppose $c \in C_G$ is such that $|c| < \gamma$. Let $S \subseteq L$ be the set of indices corresponding to 1s in c. Then $\mathrm{U}(S) \geqslant \epsilon|S| > 0$, which means there is some parity check (in R) that is adjacent to exactly one bit (in L) from c. But then this parity check isn't satisfied, which is a contradiction.

• Expander Codes [SS'96] •

Theorem: Expander Codes

Suppose G is a (γ, ϵ)-unique-neighbor expander (where $\epsilon > 0$), then $\Delta(C_G) \geqslant \gamma$.

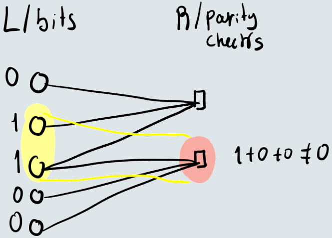
Pf. Suppose $c \in C_G$ is such that $|c| < \gamma$. Let $S \subseteq L$ be the set of indices corresponding to 1s in c. Then $\mathrm{U}(S) \geqslant \epsilon|S| > 0$, which means there is some parity check (in R) that is adjacent to exactly one bit (in L) from c. But then this parity check isn't satisfied, which is a contradiction.

• Expander Codes [SS'96] •

Theorem: Expander Codes

Suppose G is a (γ, ϵ)-unique-neighbor expander (where $\epsilon > 0$), then $\Delta(C_G) \geqslant \gamma$.

Pf. Suppose $c \in C_G$ is such that $|c| < \gamma$. Let $S \subseteq L$ be the set of indices corresponding to 1s in c. Then $\mathrm{U}(S) \geqslant \epsilon|S| > 0$, which means there is some parity check (in R) that is adjacent to exactly one bit (in L) from c. But then this parity check isn't satisfied, which is a contradiction.



A non-exhaustive list of modern problems:

- Local Testability and Decodability

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms
- (Efficient) List Decoding

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms
- (Efficient) List Decoding

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms
- (Efficient) List Decoding

Definition: Local Testability

Given a code $C \subseteq \mathbb{F}_2^n$, a local tester is an oracle tester T that given some word $x \in \mathbb{F}_2^n$ as input:

- It makes a constant number of queries to indices of x

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms
- (Efficient) List Decoding

Definition: Local Testability

Given a code $C \subseteq \mathbb{F}_2^n$, a local tester is an oracle tester T that given some word $x \in \mathbb{F}_2^n$ as input:

- It makes a constant number of queries to indices of x
- If $x \in C$, then $\Pr[T(x) = 1] = 1$

• Modern Problems in Coding Theory •

A non-exhaustive list of modern problems:

- Local Testability and Decodability
- Quantum Codes
- Efficient Decoding Algorithms
- (Efficient) List Decoding

Definition: Local Testability

Given a code $C \subseteq \mathbb{F}_2^n$, a local tester is an oracle tester T that given some word $x \in \mathbb{F}_2^n$ as input:

- It makes a constant number of queries to indices of x
- If $x \in C$, then $\Pr[T(x) = 1] = 1$
- If $x \notin C$, then $\Pr[T(x) \neq 1] = \Omega(\min_{c \in C} \Delta(x, c))$

• Locally Testable Codes •

The Problem

Are there explicit codes with constant distance, constant rate and locally testable? (aka LTCs)

• Locally Testable Codes •

The Problem

Are there explicit codes with constant distance, constant rate and locally testable? (aka LTCs)

Theorem: LTCs exist [DELLM'22]

LTCs exist!

• Locally Testable Codes •

The Problem

Are there explicit codes with constant distance, constant rate and locally testable? (aka LTCs)

Theorem: LTCs exist [DELLM'22]

LTCs exist!

Theorem: LTCs exist [HH'22]

Given a 1-sided lossless expander, LTCs exist.

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

Theorem: Expander Codes Aren't LTC

There exist expander graphs G for which the above natural tester fails, i.e. given some $x \notin C(G)$ but close to $C(G)$, $\Pr[T(x) \neq 1] = O(1/n)$.

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

Theorem: Expander Codes Aren't LTC

There exist expander graphs G for which the above natural tester fails, i.e. given some $x \notin C(G)$ but close to $C(G)$, $\Pr[T(x) \neq 1] = O(1/n)$.

Pf sketch. This is true for a random graph G.

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

Theorem: Expander Codes Aren't LTC

There exist expander graphs G for which the above natural tester fails, i.e. given some $x \notin C(G)$ but close to $C(G)$, $\Pr[T(x) \neq 1] = O(1/n)$.

Pf sketch. This is true for a random graph G.

- Consider G' by removing a random parity check from G

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

Theorem: Expander Codes Aren't LTC

There exist expander graphs G for which the above natural tester fails, i.e. given some $x \notin C(G)$ but close to $C(G)$, $\Pr[T(x) \neq 1] = O(1/n)$.

Pf sketch. This is true for a random graph G.

- Consider G' by removing a random parity check from G
- There are $x \in C(G') \setminus C(G)$ which are far from $C(G)$

• Expander Codes Aren't Locally Testable •

Definition: A Natural Tester

Given a (γ, ϵ)-unique-neighbor expander G consider its expander code C_G. A natural local tester would:

- Pick $O(1)$ parity checks uniformly at random
- Accept is all of the parity checks are satisfied

Theorem: Expander Codes Aren't LTC

There exist expander graphs G for which the above natural tester fails, i.e. given some $x \notin C(G)$ but close to $C(G)$, $\Pr[T(x) \neq 1] = O(1/n)$.

Pf sketch. This is true for a random graph G.

- Consider G' by removing a random parity check from G
- There are $x \in C(G') \setminus C(G)$ which are far from $C(G)$
- These words only fail a single parity check of $C(G)$, so probability of the tester picking it is low

• Quantum Codes •

The Problem

Are there explicit quantum codes with constant distance and constant rate?

• Quantum Codes •

The Problem

Are there explicit quantum codes with constant distance and constant rate?

Theorem: Good Quantum Codes [PK'22]

Good Quantum Codes exist!

• Quantum Codes •

The Problem

Are there explicit quantum codes with constant distance and constant rate?

Theorem: Good Quantum Codes [PK'22]

Good Quantum Codes exist!

Theorem: Good Quantum Codes [HH'22]

Given a 2-sided lossless expander, good quantum codes exist.

Part II: Expanders

• Recall: Expander Graphs •

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma |L| \quad : \quad |\mathrm{N}(S)| \geqslant \epsilon |S|.$$

• Recall: Expander Graphs •

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma |L| \quad : \quad |\mathrm{N}(S)| \geqslant \epsilon |S|.$$

Definition: Spectral Expander

A graph G is a λ-**spectral expander** if $\lambda_2(G) \leqslant \lambda$, where $\lambda_2(G)$ is the second largest eigenvalue of the adjacency matrix of G.

• Recall: Expander Graphs •

Definition: Vertex Expander

A (d_1, d_2)-regular graph (L, R, E) is a (γ, ϵ)-vertex expander if:

$$\forall S \subseteq L, |S| \leqslant \gamma|L| \quad : \quad |\mathsf{N}(S)| \geqslant \epsilon|S|.$$

Definition: Spectral Expander

A graph G is a λ-**spectral expander** if $\lambda_2(G) \leqslant \lambda$, where $\lambda_2(G)$ is the second largest eigenvalue of the adjacency matrix of G.

Definition: Ramanujan Graph (Optimal Spectral Expander)

A (d_1, d_2)-regular graph is a **Ramanujan graph** if it is a $(\sqrt{d_1 - 1} + \sqrt{d_2 - 2})$-spectral expander.

• Properties of Expanders •

Proposition: Vertex to Unique-Neighbor Expanders

Suppose G is a (d_1, d_2)-regular graph. If G is a $(\gamma, d_1\epsilon)$-vertex expander then it is a $(\gamma, d_1(2\epsilon - 1))$-unique-neighbor expander.

• Properties of Expanders •

Proposition: Vertex to Unique-Neighbor Expanders

Suppose G is a (d_1, d_2)-regular graph. If G is a $(\gamma, d_1\epsilon)$-vertex expander then it is a $(\gamma, d_1(2\epsilon - 1))$-unique-neighbor expander.

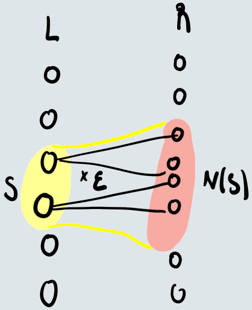
Proposition: Random Graphs

Random (d_1, d_2)-regular graphs are 2-sided *lossless expanders* with constant probability, i.e. $(\gamma, (1-\epsilon)d_1)$-vertex expanders, for any constant γ and ϵ (which depends on γ).

Pf. Union bound over all subsets of vertices.

• The Problem •

Can we construct explicit (d_1, d_2)-regular $(O(1), O(1)d_1)$-unique-neighbor expanders for any constant d_1, d_2?



• Some Answers •

Theorem: Ramanujan to Vertex Expander [Kahale'95,HMMP'24]

A (d_1, d_2)-regular Ramanujan graph is a 2-sided $(\gamma, d_1/2(1 - O(1/\log(1/\gamma)))$-vertex expander for any γ.

• Some Answers •

Theorem: Ramanujan to Vertex Expander [Kahale'95,HMMP'24]

A (d_1, d_2)-regular Ramanujan graph is a 2-sided $(\gamma, d_1/2(1 - O(1/\log(1/\gamma)))$-vertex expander for any γ.

Theorem: 1-sided Expanders [CRVW'02,CRTS'23,Golowich'23]

For any constant d_1, d_2, there exist 1-sided explicit (d_1, d_2)-regular lossless expanders (i.e. $(\gamma, (1 - \epsilon)d_1)$-vertex expanders).

• Some Answers •

Theorem: Ramanujan to Vertex Expander [Kahale'95,HMMP'24]

A (d_1, d_2)-regular Ramanujan graph is a 2-sided $(\gamma, d_1/2(1 - O(1/\log(1/\gamma)))$-vertex expander for any γ.

Theorem: 1-sided Expanders [CRVW'02,CRTS'23,Golowich'23]

For any constant d_1, d_2, there exist 1-sided explicit (d_1, d_2)-regular lossless expanders (i.e. $(\gamma, (1-\epsilon)d_1)$-vertex expanders).

Theorem: 2-sided Expanders [HMMP'24] $\leftarrow$ today

For any constant d_1, d_2, there exist 2-sided explicit (d_1, d_2)-regular $(O(1), O(1)d_1)$-unique-neighbor expanders.

• Some Answers •

Theorem: Ramanujan to Vertex Expander [Kahale'95,HMMP'24]

A (d_1, d_2)-regular Ramanujan graph is a 2-sided $(\gamma, d_1/2(1 - O(1/\log(1/\gamma)))$-vertex expander for any γ.

Theorem: 1-sided Expanders [CRVW'02,CRTS'23,Golowich'23]

For any constant d_1, d_2, there exist 1-sided explicit (d_1, d_2)-regular lossless expanders (i.e. $(\gamma, (1-\epsilon)d_1)$-vertex expanders).

Theorem: 2-sided Expanders [HMMP'24] $\leftarrow$ today

For any constant d_1, d_2, there exist 2-sided explicit (d_1, d_2)-regular $(O(1), O(1)d_1)$-unique-neighbor expanders.

Theorem: 2-sided Expanders [HLMOZ'24] (recent follow up)

For any constant d_1, d_2, there exist 2-sided explicit (d_1, d_2)-regular $(O(1), 3/5d_1)$-unique-neighbor expanders.

• Tripartite Line Product •

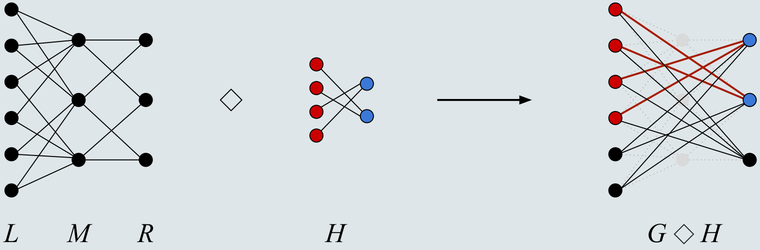
Definition: Tripartite Line Product

Let $G = (L, M, R, E_1 \cup E_2)$ be a tripartite graph consisting of a (k_1, d_1)-regular graph (L, M, E_1), and a (d_2, k_2)-regular graph (M, R, E_2)-regular graph. Let $H = (L_H, R_H)$ be a bipartite graph with $|L_H| = d_1$ and $|R_H| = d_2$. The **tripartite line product** $G \diamond H$ is the bipartite graph on $L \cup R$ and edges obtained by placing a copy of H on the neighbors of v for each $v \in M$.

• Tripartite Line Product •

Definition: Tripartite Line Product

Let $G = (L, M, R, E_1 \cup E_2)$ be a tripartite graph consisting of a (k_1, d_1)-regular graph (L, M, E_1), and a (d_2, k_2)-regular graph (M, R, E_2)-regular graph. Let $H = (L_H, R_H)$ be a bipartite graph with $|L_H| = d_1$ and $|R_H| = d_2$. The **tripartite line product** $G \diamond H$ is the bipartite graph on $L \cup R$ and edges obtained by placing a copy of H on the neighbors of v for each $v \in M$.



• Tripartite Line Product •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ and $G_2 = (M, R, E_2)$ be bipartite Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(O(1), O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

• Tripartite Line Product •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ and $G_2 = (M, R, E_2)$ be bipartite Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(O(1), O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

We use known constructions of Ramanujan graphs for G_1, G_2.

H is a constant-sized gadget, so we use the fact that random graphs are good unique-neighbor expanders and find one by brute force.

• Proof Overview: Dream Scenario •

Proof Overview

Let $S \subseteq L(G \diamond H) = L$.

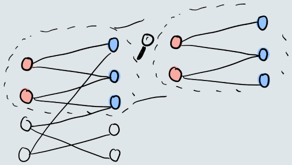
It would be ideal if each $v \in S$ has many unique-neighbors given by each gadget it belongs to.

• Proof Overview: Dream Scenario •

Proof Overview

Let $S \subseteq L(G \diamond H) = L$.

It would be ideal if each $v \in S$ has many unique-neighbors given by each gadget it belongs to.

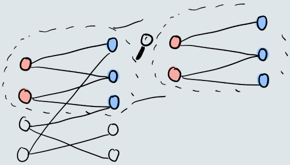


• Proof Overview: Dream Scenario •

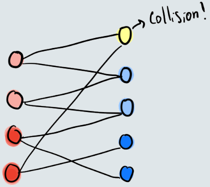
Proof Overview

Let $S \subseteq L(G \diamond H) = L$.

It would be ideal if each $v \in S$ has many unique-neighbors given by each gadget it belongs to.



Alas we might have collisions between gadgets...



• Proof Overview Detour: Subgraph Density •

Theorem: Subgraph Density of Ramanujan Graphs

Let $G = (L, R, E)$ be a (d_1, d_2)-regular $(\sqrt{d_1 - 1} + \sqrt{d_2 - 2})(1 + O(1/d))$-spectral expander. Then, for any $S_1 \subseteq L$ and $S_2 \subseteq R$ such that $|S_1| + |S_2| = O(|L| + |R|)$, the left $\overline{d_L}$ and right $\overline{d_R}$ average degrees of the induced subgraph $G[S_1 \cup S_2]$ satisfy:

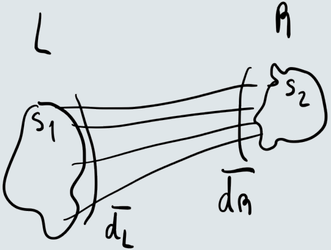
$$(\overline{d_L} - 1)(\overline{d_R} - 1) \leqslant O\left(\sqrt{(d_1 - 1)(d_2 - 1)}\right)$$

• Proof Overview Detour: Subgraph Density •

Theorem: Subgraph Density of Ramanujan Graphs

Let $G = (L, R, E)$ be a (d_1, d_2)-regular $(\sqrt{d_1 - 1} + \sqrt{d_2 - 2})(1 + O(1/d))$-spectral expander. Then, for any $S_1 \subseteq L$ and $S_2 \subseteq R$ such that $|S_1| + |S_2| = O(|L| + |R|)$, the left $\overline{d_L}$ and right $\overline{d_R}$ average degrees of the induced subgraph $G[S_1 \cup S_2]$ satisfy:

$$(\overline{d_L} - 1)(\overline{d_R} - 1) \leqslant O\left(\sqrt{(d_1 - 1)(d_2 - 1)}\right)$$



• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)

• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)
- Right average degree of $G_1[S \cup N_h]$ is at least $O(\sqrt{d_1 + d_2})$

• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)
- Right average degree of $G_1[S \cup N_h]$ is at least $O(\sqrt{d_1 + d_2})$
- Use the Subgraph Density Theorem to show left average degree of $G_1[S \cup N_h]$ is small ($O(\sqrt{k_1 + k_2})$)

• Proof Overview •

Theorem: Main [HMMP'24]

Let $G_1 = (L, M, E_1)$ be (k, d_1)-regular and $G_2 = (M, R, E_2)$ be (d_2, k)-regular Ramanujan graphs, and form the tripartite graph G from them. Let H be a $(1/\sqrt{d_1 + d_2}, O(1)\deg(H))$-unique-neighbor expander. Then $G \diamond H$ is a $(O(1), O(1)d_1)$-unique-neighbor expander.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)
- Right average degree of $G_1[S \cup N_h]$ is at least $O(\sqrt{d_1 + d_2})$
- Use the Subgraph Density Theorem to show left average degree of $G_1[S \cup N_h]$ is small $(O(\sqrt{k_1 + k_2}))$
- Conclude $|N_h|$ is small $(\leqslant .2|N|)$

Proof Overview

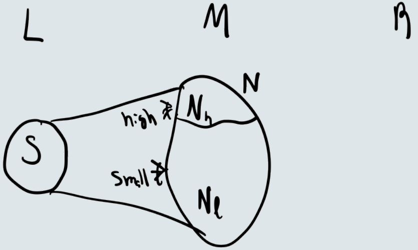
Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)
- Right average degree of $G_1[S \cup N_h]$ is at least $O(\sqrt{d_1 + d_2})$
- Use the Subgraph Density Theorem to show left average degree of $G_1[S \cup N_h]$ is small $(O(\sqrt{k_1 + k_2}))$
- Conclude $|N_h|$ is small ($\leqslant .2|N|$)

Proof Overview

Step 1: "Most edges from S go into low-degree vertices in M"

- Partition N into low-degree vertices N_ℓ (degree in $G_1 \leqslant \sqrt{d_1 + d_2}$) and high-degree vertices N_h (degree in $G_1 \geqslant \sqrt{d_1 + d_2}$)
- Right average degree of $G_1[S \cup N_h]$ is at least $O(\sqrt{d_1 + d_2})$
- Use the Subgraph Density Theorem to show left average degree of $G_1[S \cup N_h]$ is small $(O(\sqrt{k_1 + k_2}))$
- Conclude $|N_h|$ is small ($\leqslant .2|N|$)



Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$.

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$. Also let T be the set of unique-neighbors of S within some gadget

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$. Also let T be the set of unique-neighbors of S within some gadget

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$. Also let T be the set of unique-neighbors of S within some gadget

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree (a constant that depends on the expansion of H, d_1 and k)

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$. Also let T be the set of unique-neighbors of S within some gadget

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree (a constant that depends on the expansion of H, d_1 and k)
- Use the Subgraph Density Theorem to show right average degree of $G_2[N \cup T]$ is small ($\leqslant 1.1$)

Proof Overview

Let $S \subseteq L(G \diamond H) = L$ and $N = N_G(S) \subseteq M$. Also let T be the set of unique-neighbors of S within some gadget

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree (a constant that depends on the expansion of H, d_1 and k)
- Use the Subgraph Density Theorem to show right average degree of $G_2[N \cup T]$ is small ($\leqslant 1.1$)
- Conclude that the number of collisions is small, i.e. large number of unique-neighbors

Proof Overview

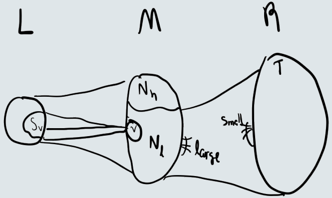
Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree
- Use the Subgraph Density Theorem to show right average degree of $G_2[N \cup T]$ is small ($\leqslant 1.1$)
- Conclude that the number of collisions is small

Proof Overview

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree
- Use the Subgraph Density Theorem to show right average degree of $G_2[N \cup T]$ is small ($\leqslant 1.1$)
- Conclude that the number of collisions is small

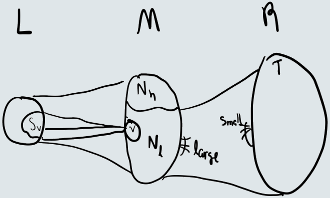


• Proof Overview •

Proof Overview

Step 2: "Low-degree vertices don't cause collisions"

- If $v \in N_\ell$ is low-degree, then $S_v = S \cap N(v)$ is small ($|S_v| \leqslant \sqrt{d_1 + d_2}$)
- So we can use the unique-neighbor property of H to conclude that $G_2[N \cap T]$ has high average left degree
- Use the Subgraph Density Theorem to show right average degree of $G_2[N \cup T]$ is small ($\leqslant 1.1$)
- Conclude that the number of collisions is small



□

Thanks!