A Short Proof of the Kahn-Kalai-Linial Inequality

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Abstract

In this note, we show a simple proof of Talagrand's strengthening of the Kahn-Kalai-Linial (KKL) inequality, which states that for any boolean function $f : \{-1,1\}^n \to \{-1,1\}$,

$$\mathbf{Var}(f) \leq 2 \cdot \sum_{i=1}^{n} \frac{\mathbf{Inf}_i[f]}{\ln\left(\frac{1}{\mathbf{Inf}_i[f]}\right)}.$$

As a corollary, we obtain the KKL inequality with the best known constant[1]: For every boolean function $f : \{-1,1\}^n \to \{-1,1\}$, there exists a coordinate $i \in [n]$ with influence

$$\mathbf{Inf}_i[f] \geq \left(\frac{1}{2} - o_n(1)\right) \cdot \mathbf{Var}[f] \cdot \frac{\ln n}{n}.$$

The proof uses the so called semigroup-interpolation method.

1 Talagrand's Inequality

The reader is referred to the book by O'Donnell for an excellent introduction to the subject [O'D14]. We note that the proof we present is already known in the literature, see for example [Led19].

Let $f : \{-1,1\}^n \to \mathbb{R}$, and let $\rho \in [0,1]$. We recall some basic definitions and facts:

1. $\mathbf{Stab}_\rho[f] := \mathbb{E}[f(x)f(y)]$, where the expectation is over ρ-correlated (x,y).
2. For $i \in [n]$, $D_i f(x) := \frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2}$.
3. For $i \in [n]$, $\mathbf{Inf}_i^{(\rho)}[f] := \mathbf{Stab}_\rho[D_i f] = \sum_{S \ni i} \rho^{|S|-1} \widehat{f}(S)^2$.
4. $\mathbf{I}^{(\rho)}[f] = \sum_{i=1}^{n} \mathbf{Inf}_i^{(\rho)}[f] = \sum_{k=1}^{n} k\rho^{k-1}\mathbf{W}^k[f]$, where $\mathbf{W}^k[f] = \sum_{|S|=k} \widehat{f}(S)^2$.
5. $\mathbf{Var}[f] = \sum_{k=1}^{n} \mathbf{W}^k[f]$.

Integrating the above expression for $\mathbf{I}^{(\rho)}[f]$, we get:

Lemma 1. *Let* $f : \{-1,1\}^n \to \mathbb{R}$, $\rho \in [0,1]$. *Then,*

$$\mathbf{Var}[f] = \int_0^1 \mathbf{I}^{(\rho)}[f]\ d\rho = \sum_{i=1}^{n} \int_0^1 \mathbf{Inf}_i^{(\rho)}[f]\ d\rho.$$

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[1] See Exercise 9.30 in [O'D14].

Another ingredient we shall need is the well-known $(p,2)$-hypercontractivity inequality:

Theorem 2. *Let* $f : \{-1,1\}^n \to \mathbb{R}$ *and* $\rho \in [0,1]$. *Then,* $\mathbf{Stab}_\rho[f] \le \|f\|_{1+\rho}^2$.

We are now ready to prove Talagrand's inequality:

Theorem 3. (Talagrand) *For any boolean function* $f : \{-1,1\}^n \to \{-1,1\}$,

$$\mathbf{Var}(f) \le 2 \cdot \sum_{i=1}^{n} \frac{\mathbf{Inf}_i[f]}{\ln\left(\frac{1}{\mathbf{Inf}_i[f]}\right)}.$$

Proof. For any $i \in [n]$, applying Theorem 2 on $D_i f$, which has range $\{-1,0,1\}$, we get

$$\mathbf{Inf}_i^{(\rho)}[f] = \mathbf{Stab}_\rho[D_i f] \le \|D_i f\|_{1+\rho}^2 = \mathbb{E}\left[|D_i f|^{1+\rho}\right]^{\frac{2}{1+\rho}} = \mathbb{E}\left[|D_i f|\right]^{\frac{2}{1+\rho}} = \mathbf{Inf}_i[f]^{\frac{2}{1+\rho}}.$$

Now, by Lemma 1, we have

$$\mathbf{Var}[f] = \sum_{i=1}^{n} \int_0^1 \mathbf{Inf}_i^{(\rho)}[f]\ d\rho \le \sum_{i=1}^{n} \int_0^1 \mathbf{Inf}_i[f]^{1+\frac{1-\rho}{1+\rho}}\ d\rho \le 2 \cdot \sum_{i=1}^{n} \frac{\mathbf{Inf}_i[f]}{\ln\left(\frac{1}{\mathbf{Inf}_i[f]}\right)}.$$

In the last step, we use that for every real $x \in [0,1]$,

$$\int_0^1 x^{\frac{1-\rho}{1+\rho}}\ d\rho = \int_0^1 \frac{2x^s}{(1+s)^2}\ ds \le \int_0^1 2x^s\ ds = \frac{2(1-x)}{\ln\left(\frac{1}{x}\right)} \le \frac{2}{\ln\left(\frac{1}{x}\right)}.$$ □

Remarks.

1. As a corollary, we obtain the KKL inequality stated in the abstract:

   Let $f : \{-1,1\}^n \to \{-1,1\}$. Suppose for the sake of contradiction, that for each $i \in [n]$, $\mathbf{Inf}_i[f] \le \left(\frac{1}{2} - \epsilon\right) \cdot \mathbf{Var}[f] \cdot \frac{\ln n}{n}$, where $\epsilon > 0$ is some constant. Then, for large n, the above gives us

   $$\mathbf{Var}[f] \le \sum_{i=1}^{n} \frac{2 \cdot \mathbf{Inf}_i[f]}{\ln\left(\frac{1}{\mathbf{Inf}_i[f]}\right)} \le \sum_{i=1}^{n} \frac{2 \cdot \left(\frac{1}{2} - \epsilon\right) \mathbf{Var}[f] \cdot \frac{\ln n}{n}}{\ln\left(\frac{n}{\ln n}\right)} < \mathbf{Var}[f],$$

   which is a contradiction.

2. In the above proof, for bounded functions $f : \{-1,1\}^n \to [-1,1]$, one can apply the inequality $|D_i f|^{1+\rho} \le |D_i f|$, and obtain

   $$\mathbf{Var}[f] \le 2 \cdot \sum_{i=1}^{n} \frac{\mathbb{E}\,|D_i f|}{\ln\left(\frac{1}{\mathbb{E}|D_i f|}\right)}.$$

3. In the above proof, for real valued functions $f : \{-1,1\}^n \to \mathbb{R}$, one can apply Hölder's inequality to bound $\|D_i f\|_{1+\rho}^2 \le \|D_i f\|_2^{\frac{4\rho}{1+\rho}} \cdot \|D_i f\|_1^{\frac{2(1-\rho)}{1+\rho}} = \|D_i f\|_2^2 \cdot \left(\frac{\|D_i f\|_1}{\|D_i f\|_2}\right)^{\frac{2(1-\rho)}{1+\rho}}$, and get Talagrand's $L^1 - L^2$ influence inequality:

   $$\mathbf{Var}[f] \le \sum_{i=1}^{n} \frac{\|D_i f\|_2^2}{\ln\left(\frac{\|D_i f\|_2}{\|D_i f\|_1}\right)}.$$

References

[Led19] Michel Ledoux. Four talagrand inequalities under the same umbrella. *Preprint*, 2019. https://arxiv.org/pdf/1909.00363. 1

[O'D14] Ryan O'Donnell. *Analysis of Boolean functions*. Cambridge University Press, New York, 2014. 1