the previous example is

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} d & 5 & 0 & 0 & 0 \\ 4 & d & 3 & 0 & 0 \\ 0 & 8 & d & 1 & 0 \\ 0 & 0 & 6 & d & 2 \\ 0 & 0 & 0 & 7 & d \end{bmatrix} \end{array}.$$

Then it is clear that the oriented communication net corresponding to this branch capacity matrix B has the terminal capacity matrix which is equal to T. For example the oriented communication net corresponding to B in the previous example is shown in Fig. 5. Notice that the communication net whose terminal capacity matrix satisfies Theorem 5 contains at most $2(v - 1)$ different terminal capacities, where v is the number of vertices in the net.

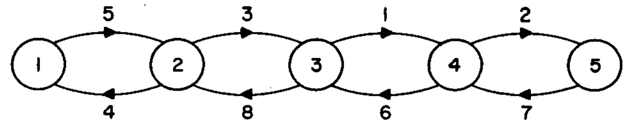

Fig. 5—Communication net corresponding to B.

One of the future problems in this field is to find a sufficient condition for realizability of a terminal capacity matrix in which there are k different elements for a fixed k. Another problem is to define the optimum oriented communication net and to obtain a method of synthesizing such a net whose terminal capacity matrix satisfies Theorem 5.

Sufficient Conditions on Pole and Zero Locations for Rational Positive-Real Functions*

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Summary—**The problem of finding sufficient conditions on the pole and zero locations to insure that a rational function $W(s)$ is positive-real has been an outstanding one in network theory. Several solutions to this problem are presented in this paper. In particular, assuming that $W(s)$ has n poles and n zeros, certain regions in the left-half s plane are constructed which have the following property: If these poles and zeros are placed in one of these regions in any arbitrary manner (with the restriction, of course, that complex elements appear in complex-conjugate pairs), the resulting $W(s)$ will be positive-real. These results are then extended to the case where the number of poles and the number of zeros differ by one.**

In addition certain paths in these regions are derived which allow one to place any number of poles and zeros into any of these regions. That is, if the poles and zeros alternate in groups of n elements on any such path, $W(s)$ will again be positive-real. The simple alternation of poles and zeros on the real negative axis and on a vertical line or circle in the closed left-half s plane, which is a known result, is a special case of these considerably more general conclusions.

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I. Introduction

THE RATIONAL, positive-real functions are those rational functions $W(s)$ of the complex variable $s = \sigma + j\omega$ that are real-valued for real s and have a non-negative real part whenever the real part of s is positive. Well-known necessary conditions on the pole and zero locations for such functions are that there can be no poles or zeros in the open right-half s plane and that any imaginary poles and zeros must be simple. Furthermore complex poles and zeros must occur in complex-conjugate pairs. The converse problem of determining sufficient conditions on the pole and zero locations, which insure that $W(s)$ is positive-real, has been an outstanding one in circuit theory. A number of specialized results have been known for some time. For instance if the poles and zeros are simple and alternate on the real, negative axis or on a vertical line in the closed left-half s plane, $W(s)$ is certainly positive-real.[1] A more recent result is given by F. M. Reza,[2] who shows

[1] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N. Y.; 1957.

[2] F. M. Reza, "RLC canonic forms," *J. Appl. Phys.*, vol. 25, pp. 297–301; March, 1954.

among other things that the same conclusion holds if the poles and zeros are simple and alternate on a circle that lies in the left-half s plane and has its center on the real axis. This result can be obtained by making a bilinear transformation from a vertical line to a circle and using the fact that a positive-real function of a positive-real function is again positive-real.

In this paper some new conditions on the pole and zero locations are developed which insure the positive-reality of $W(s)$. For instance, assuming that $W(s)$ has exactly n poles and n zeros, certain regions of the left-half s plane are constructed which have the following property: If these poles and zeros are placed in one of these regions in any arbitrary manner (with the restriction, of course, that complex elements appear in complex-conjugate pairs), the resulting $W(s)$ will be positive-real. Furthermore other regions are developed for which the magnitude of the phase angle of $W(j\omega)$ never exceeds a given quantity θ for $-\infty < \omega < \infty$. Moreover these results are extended to the cases where the number of poles and the number of zeros of $W(s)$ differ by one.

In addition certain paths in the left-half s plane are derived which allow one to place any number of poles and zeros into any of the aforementioned regions. In particular, if the poles and zeros alternate in groups of n elements on any such path in a given region, $W(s)$ will again be positive-real. The simple alternations of poles and zeros on the real, negative axis and on a vertical line or circle in the closed left-half s plane are special cases of these considerably more general conclusions.

Throughout this paper it is assumed that the constant multiplier of the rational function is positive and that the complex poles and complex zeros appear in complex-conjugate pairs. *These conditions will not be explicitly stated in the forthcoming theorems.*

If $W(s)$ is a rational function having all its poles and zeros in the closed left-half s plane, the phase function $\Phi_W(\omega)$ of $W(j\omega)$ will be measured in the conventional way. That is, the phase function of any factor $j\omega + \eta$, where $Re\ \eta \geq 0$, is restricted to its principal branch:

$$|\arg (j\omega + \eta)| \leq \pi/2.$$

In particular, if $Re\ \eta = 0$, $\arg (j\omega + \eta)$ is taken to be zero at the point $j\omega = -\eta$. Because of this at any imaginary pole or zero (say at $s = j\omega_1$) of $W(s)$ we always have

$$\Phi_W(\omega_1) = \tfrac{1}{2}[\Phi_W(\omega_1 + 0) + \Phi_W(\omega_1 - 0)]. \tag{1}$$

Under the assumptions of the previous paragraph $W(s)$ will be real and positive for real positive values of s, and $\Phi_W(\omega)$ will be an odd function of ω. In addition we may state

Lemma 1: Let $W(s)$ be a rational function having all its poles and zeros in the closed left-half s plane. If $|\Phi_W(\omega)| \leq \pi/2$ for $\omega > 0$, then $W(s)$ is positive-real.

Proof: The fact that $\Phi_W(\omega)$ is bounded by $\pm\pi/2$ implies that $Re\ W(j\omega) \geq 0$ and that each pole on the $j\omega$ axis is simple and has a positive residue. This insures the positive-reality of $W(s)$.[3]

Actually, under our hypothesis on the pole and zero locations of $W(s)$, the converse of this lemma is also true.

We shall refer to the number of poles and zeros in a region and to their multiplicities in the following way. The number of such elements in a given region is obtained by counting the elements according to their multiplicities. As an example, if a certain region contains only a triple pole at one point and a double pole at another point, we shall say that it contains five poles. In referring to the number of poles or the number of zeros of a rational function, we shall be referring to the number of finite poles or zeros; the poles or zeros at $s = \infty$ will not be counted.

II. P Paths and n-Fold Alternation on P Paths

In this section we shall define a certain class of paths which reside entirely in the closed left-half s plane and have the following property: If the zeros of a quadratic polynomial are moved along these paths, the phase function of this quadratic will either increase everywhere on the positive $s = j\omega$ axis or decrease everywhere on this axis, depending on the direction in which the zeros are moved. As will be shown later, these curves enable us to define certain classes of rational functions whose phase function on the $j\omega$ axis is bounded in magnitude by some prescribed angle.

To make these ideas more concrete, consider a quadratic polynomial $Q_1(s)$ whose zeros lie on some circle C which resides entirely within the closed left-half s plane. Let us move the zeros of $Q_1(s)$ some distance to the left on C, as shown in Fig. 1(a), and investigate the behavior of the phase function on the positive $j\omega$ axis. The phase function of the new quadratic $Q_2(s)$ will be equal to that of $Q_1(s)$ plus the phase function of the rational function $Q_2(s)/Q_1(s)$ which has poles at the original location of the zeros and zeros at the final location. Thus the change in the phase function on the $j\omega$ axis due to the shift in zeros will be given by the phase function of $Q_2(s)/Q_1(s)$ on the $j\omega$ axis. This phase function is negative for $\omega > 0$ and positive for $\omega < 0$. To see this consider the root locus defined by

$$Im\,[Q_1(s)/Q_2(s)] = 0.$$

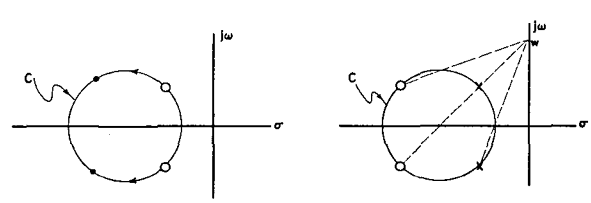

Fig. 1—(a) The zeros of $Q_1(s)$ moved on C. (b) The poles and zeros of $Q_2(s)/Q_1(s)$.

[3] Guillemin, *op. cit.*, see p. 15.

It is well-known[4] that this locus consists of the real axis plus the circle C, which does not intersect the $j\omega$ axis. Therefore arg $[Q_1(j\omega)/Q_2(j\omega)]$ changes sign only at $\omega = 0$ and is of one sign for $\omega > 0$ and the opposite sign for $\omega < 0$. As can be seen from the construction in Fig. 1(b), the phase function of $Q_1(s)/Q_2(s)$ is negative at the point W on the positive $j\omega$ axis. We then have the desired result:

$$\arg\,[Q_1(j\omega)/Q_2(j\omega)]\begin{cases} < 0 & (\omega > 0) \\ = 0 & (\omega = 0) \\ > 0 & (\omega < 0). \end{cases} \quad (2)$$

This means then that as the zeros of $Q_1(s)$ are moved to the left on the circle C, the phase function decreases everywhere on the positive $j\omega$ axis and increases everywhere on the negative $j\omega$ axis. It is clear that moving the zeros to the left on a path consisting of segments of many such circles will have the same effect. We may in fact consider paths with tangent circles in the left-half plane as being made up of an infinite number of such circles. This suggests the following definition:

Definition: Consider two points s_p and $\bar{s}_p$ which need not always be distinct. A *P path* is generated by moving these complex-conjugate points from $s = 0$ to $s = \infty$ in such a way that *Re* s_p is nonincreasing and, wherever *Re* s_p remains constant on any portion of this path, *Im* s_p increases. We also require that a circle whose center is on the negative-real axis and which lies entirely within the closed left-half s plane can be drawn tangent to the P path at all but a finite number of points on any finite portion of this path. By including the limiting cases where the radii of these circles are either zero or infinity, we admit portions of the negative-real axis or vertical lines in the closed left-half s plane as possible parts of these paths.

We shall assign an orientation to every P path. In particular the direction taken by the generating points s_p and $\bar{s}_p$ as they move from $s = 0$ to $s = \infty$ will be taken as the negative direction on the path. Examples of P paths, with the negative orientation indicated, are shown in Fig. 2.

We can now state the following lemma:

Lemma 2: Let $Q(s) = (s + \alpha)^2 + \beta^2$ be a real quadratic, having zeros in the left-half s plane at $s = -\alpha \pm j\beta (\alpha, \beta \geq 0)$. Then as the zeros of $Q(s)$ traverse a P path in the negative direction, the phase function of $Q(s)$ on the $j\omega$ axis, $\Phi_Q(\omega)$, decreases monotonically for every fixed, positive ω.

Proof: The phase function of $Q(j\omega)$ is given by

$$\Phi_Q(\omega) = \tan^{-1} \frac{2\alpha\omega}{\alpha^2 + \beta^2 - \omega^2}. \quad (3)$$

As the zeros of $Q(s)$ move on a P path, α and β change. The total change in phase due to an infinitesimal change in α and β can be computed to be

$$d\Phi_Q = \frac{\partial \Phi_Q}{\partial \alpha}\, d\alpha + \frac{\partial \Phi_Q}{\partial \beta}\, d\beta$$

$$= \frac{-2\omega}{[\alpha^2 + (\omega - \beta)^2][\alpha^2 + (\omega + \beta)^2]} \cdot [(\alpha^2 + \omega^2 - \beta^2)\, d\alpha + 2\alpha\beta\, d\beta]. \quad (4)$$

If $\omega \geq 0$, $d\Phi_Q \leq 0$ if and only if

$$(\alpha^2 + \omega^2 - \beta^2)\, d\alpha + 2\alpha\beta\, d\beta \geq 0 \quad (5)$$

for all ω.

We shall now show that (5) holds at every point $-\alpha \pm j\beta$ on a P path. First assume that α is increasing as we move along the P path. If $\beta = 0$, (5) certainly holds. If $\beta > 0$, a circle can be drawn tangent to the P path with center at $s = -a < 0$ and radius b, where $0 < b \leq a$ (see Fig. 3). The equation of this circle is

$$(-\alpha + a)^2 + \beta^2 = b^2. \quad (6)$$

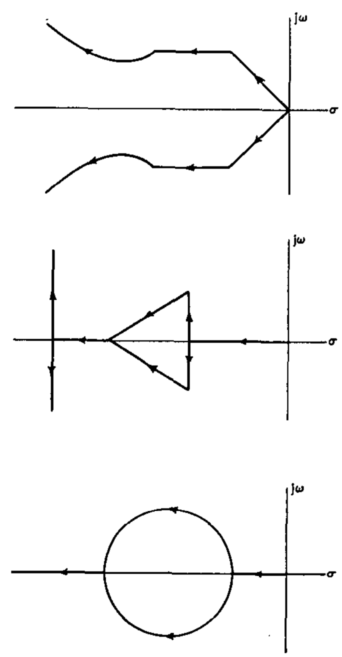

Fig. 2—Examples of P paths with the negative directions indicated.

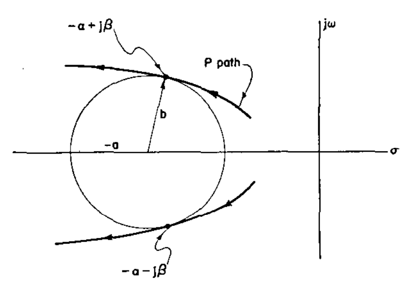

Fig. 3—A circle drawn tangent to a P path at the points $s = -\alpha \pm j\beta$.

[4] K. Steiglitz, "An analytical approach to root loci," IRE Trans. on Automatic Control, vol. AC-6, pp. 326–332; September, 1961.

Differentiating (6), we have

$$d\beta = \frac{-\alpha + a}{\beta} d\alpha. \tag{7}$$

Substituting (7) into (5) we obtain

$$(\omega^2 - \alpha^2 + 2\alpha a - \beta^2)\, d\alpha \geq 0. \tag{8}$$

But from (6)

$$-\alpha^2 + 2\alpha a - \beta^2 = a^2 - b^2 \geq 0. \tag{9}$$

Hence (8) and therefore (5) holds when $d\alpha > 0$.

In the case when α is constant on the P path, β must increase, so that $d\alpha = 0$, $d\beta > 0$, and (5) again holds. Q.E.D.

We shall now define certain ways of placing the poles and zeros of a rational function on a P path to obtain a certain type of alternation. This alternation, plus the property of P paths shown to hold in the previous lemma, will serve to bound the phase angle of this rational function on the $j\omega$ axis. In the following definition we shall traverse a P path and count the number of poles and zeros occurring on it in the following way. For those sections where the P path has two complex-conjugate branches, the complex-conjugate pairs of poles and zeros will be counted simultaneously. In other words we shall always count the poles and zeros for both branches rather than for just one of them.

Definition: We shall say that there is *n-fold alternation* of poles and zeros on a P path if, while traversing it in the negative direction, we encounter first n poles, then n zeros, then n poles, etc., or first n zeros, then n poles, then n zeros, etc., where there are an equal total number of poles and zeros. If poles are encountered first we shall say that the poles are the starting elements, and similarly for zeros. We shall allow some of the poles and zeros to become coincident.

Examples of 3-fold, 4-fold and 6-fold alternation on a P path are shown in Fig. 4.

Lemma 3: Let $W(s)$ have kn poles and kn zeros which possess n-fold alternation on a P path. Let the contribution to the phase function of the first group of n elements be denoted by $\Phi_1(\omega)$, of the second group by $\Phi_2(\omega)$, etc., up to $\Phi_{2k}(\omega)$, and let the total phase function be $\Phi_W(\omega) = \arg W(j\omega)$. Then on the positive $j\omega$ axis

$$0 \leq \Phi_W(\omega) \leq |\Phi_1(\omega)| - |\Phi_{2k}(\omega)| \leq |\Phi_1(\omega)| \leq \frac{n\pi}{2} \tag{10}$$

if zeros are the starting elements, and

$$-\frac{n\pi}{2} \leq -|\Phi_1(\omega)| \leq -|\Phi_1(\omega)| + |\Phi_{2k}(\omega)| \leq \Phi_W(\omega) \leq 0 \tag{11}$$

if poles are the starting elements.

Proof: Assume that zeros are the starting elements.

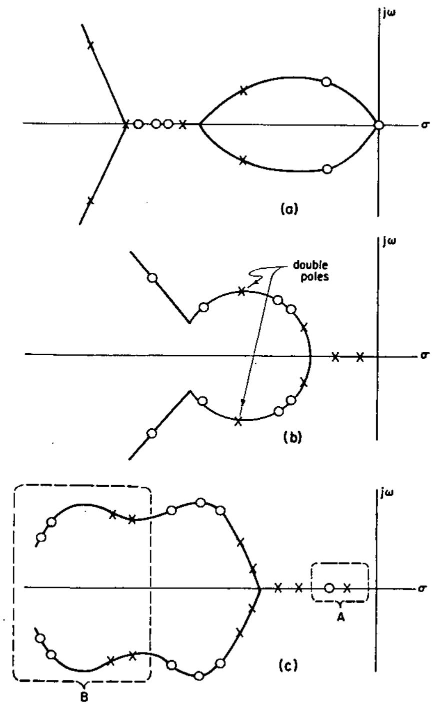

Fig. 4—Examples of 3-fold, 4-fold, and 6-fold alternation on P paths. All poles and zeros are simple unless otherwise noted. In (c), 5 poles and 5 zeros in set A are coincident whereas 2 poles and 2 zeros in set B are coincident. These poles and zeros cancel each other and hence are not shown.

Then for $\omega > 0$

$$\begin{aligned}\Phi_W(\omega) &= \Phi_1(\omega) + \Phi_2(\omega) + \cdots + \Phi_{2k}(\omega)\\ &= |\Phi_1(\omega)| - |\Phi_2(\omega)| + |\Phi_3(\omega)| - \cdots - |\Phi_{2k}(\omega)|\\ &= [|\Phi_1(\omega)| - |\Phi_{2k}(\omega)|] - [|\Phi_2(\omega)|\\ &\quad - |\Phi_{2k-1}(\omega)|] + \cdots .\end{aligned} \tag{12}$$

Because of the property of P paths stated in Lemma 2, each of the bracketed terms is positive and each is less in magnitude than its predecessor. A series of terms which alternate in sign and decrease in magnitude can be no larger in magnitude than its first term and must have the same sign as its first term. Hence for $\omega > 0$

$$0 \leq \Phi_W(\omega) \leq |\Phi_1(\omega)| - |\Phi_{2k}(\omega)|.$$

Again, by Lemma 2, $|\Phi_1(\omega)| \geq |\Phi_{2k}(\omega)|$ for $\omega > 0$. But $|\Phi_1(\omega)| \leq n\pi/2$ because $\Phi_1(\omega)$ is the phase contribution of n zeros in the left-half s plane. Thus for $\omega > 0$ we have arrived at (10). A similar argument yields (11).

In the special case when $n = 1$ and the P path is the real nonpositive axis, we have the familiar simple alternation of an RL or RC driving-point impedance. As we shall show later, if this 1-fold alternation starts with a zero (pole), we may add a real pole (zero) anywhere on the real nonpositive axis and still have a positive-real function.

With 2-fold alternation, Lemma 3 bounds $\Phi(\omega)$ between 0 and π for a starting zero (and between 0 and $-\pi$

for a starting pole). By adding a pole (zero) between the origin and the rightmost starting zero (pole) of the function $W(s)$, the phase function is shifted in the appropriate direction to ensure positive-reality, as we shall now show. This will be our first sufficient condition for a positive-real function.

Theorem 1: Let $W(s) = F(s)\,(s+\alpha)^{\pm 1}$, *where the poles and zeros of* $F(s)$ *have 2-fold alternation on a P path. Furthermore let* α *be real and nonnegative; let the root* $-\alpha$ *be no smaller (i.e., not more negative) than the rightmost element of* $F(s)$*; also, let* $-\alpha$ *be a pole if the starting elements of* $F(s)$ *are zeros, and a zero if the starting elements of* $F(s)$ *are poles. Then* $W(s)$ *is positive-real.*

Proof: Assume without loss of generality that $F(s)$ has starting zeros and that we add a pole. If necessary shift the pole-zero pattern of $W(s)$ to the right so that the added pole is at the origin. The shifted function is

$$W(s-\alpha) = \frac{F(s-\alpha)}{s}.$$

By Lemma 3, for $\omega > 0$

$$0 \le \arg F(j\omega - \alpha) \le \pi.$$

Therefore

$$-\frac{\pi}{2} \le \arg W(j\omega - \alpha) \le \frac{\pi}{2},$$

and by Lemma 1, $W(s-\alpha)$ is positive-real. Shifting a positive-real pole-zero pattern to the left will not disturb the positive-real property. Hence $W(s)$ is also positive-real. Q.E.D.

When the P path of Theorem 1 is a vertical line in the closed left-half s plane and when the added real element is on this vertical line, Theorem 1 yields the familiar LC (or uniformly dissipative LC) case, which can be converted to Reza's result by a bilinear transformation.

III. Regions Generated by P Paths

The results of Section II may be used to generate certain regions of the left-half s plane into which n poles and n zeros may be placed in any fashion to generate positive-real functions. In order to develop some of these regions we shall need to know under what conditions on the real positive quantities γ and ρ the following function is positive-real:

$$F(s) = \left(\frac{s+\gamma}{s+\rho}\right)^n. \tag{13}$$

It is clear that for γ sufficiently close to ρ, $F(s)$ will be positive-real, while for γ sufficiently far from ρ it will not be positive-real. We need merely ascertain under what conditions the phase function of $F(j\omega)$, denoted by $\Phi_F(\omega)$, satisfies

$$|\Phi_F(\omega)| \le \frac{\pi}{2} \qquad (\omega > 0). \tag{14}$$

For (13) we have

$$\Phi_F(\omega) = n\tan^{-1}\frac{\omega}{\gamma} - n\tan^{-1}\frac{\omega}{\rho}. \tag{15}$$

$\Phi_F(\omega)$ is an odd function of ω. Clearly if $\gamma < \rho$, $\Phi_F(\omega) > 0$ for $\omega > 0$, and if $\gamma > \rho$, $\Phi_F(\omega) < 0$ for $\omega > 0$. Computing the maximum value $\Phi_{\max}$ of $\Phi_F(\omega)$ in the standard way, we find

$$|\Phi_F(\omega)| \le \Phi_{\max} = n\tan^{-1}\frac{|\gamma-\rho|}{2\sqrt{\gamma\rho}}, \tag{16}$$

and by Lemma 1 this implies

Lemma 4: Let the function $F(s)$ *be defined by* (13) *where* γ *and* ρ *are real positive quantities.* $F(s)$ *is positive-real if and only if*

$$\tan^{-1}\frac{|\gamma-\rho|}{2\sqrt{\gamma\rho}} \le \frac{\pi}{2n}. \tag{17}$$

Now given an n, consider any interval of the real negative axis in the s plane whose endpoints $-\gamma$ and $-\rho$ satisfy (17). Let us assume for definiteness that $\gamma < \rho$. Also consider any rational function $W(s)$ having n poles and n zeros all of which lie on this interval. This function can be obtained from $F(s)$ by moving the zeros of $F(s)$ to the left and the poles of $F(s)$ to the right. When these poles and zeros are moved the entire length of the interval we shall obtain $1/F(s)$. Since these poles and zeros are moving on a P path, it follows that for any positive ω, the phase function $\Phi_W(\omega)$ of $W(j\omega)$ decreases monotonically from $\Phi_F(\omega)$, which is the phase function of $F(j\omega)$, to $-\Phi_F(\omega)$. Hence for all ω

$$|\Phi_W(\omega)| \le |\Phi_F(\omega)| \le \pi/2,$$

and therefore $W(s)$ is also positive-real. This means that if we place n poles and n zeros on this interval in any fashion we will always generate a positive-real function.

Actually we need not restrict the poles and zeros to the real negative axis. If we move the poles and zeros along P paths that extend from $s = -\gamma$ to $s = -\rho$, the same argument shows that the $W(s)$ is again a positive-real function. In other words any region, each of whose points lies on some P path connecting $s = -\gamma$ and $s = -\rho$, has the property that every rational function obtained by putting n poles and n zeros anywhere in this region in any fashion will be positive-real. Given n and γ, the largest such region is obtained as follows: Choose ρ greater than γ and such that equality is achieved in (17). Construct a circle C with center on the real axis and passing through $s = -\rho$ and $s = 0$. Finally draw a vertical line L through $s = -\gamma$. The closed region whose interior is to the left of L and inside C is the largest region that can be generated by P paths. We shall denote this region by $S(-\gamma, \pi/2n)$. The regions $S(-1, \pi/2n)$ for $n = 2, 3$ and 4 are illustrated in Fig. 5.

Thus we have arrived at

Theorem 2: The rational function $W(s)$*, having n poles*

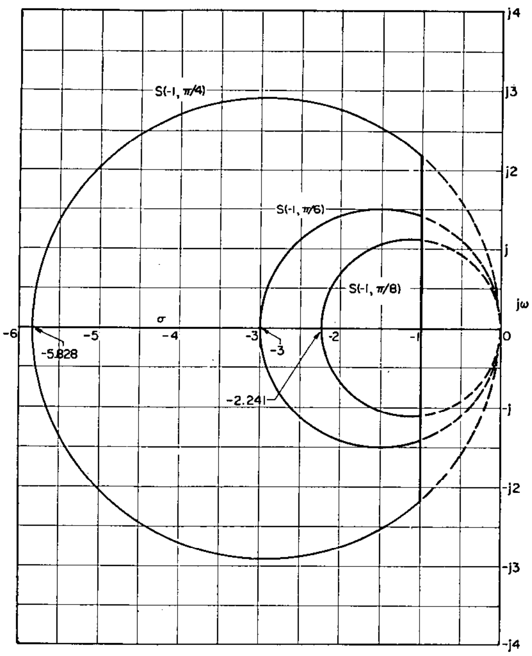

Fig. 5—The regions $S(-1,\ \theta/n)$ for various values of θ/n; n poles and n zeros placed anywhere in $S(-1,\ \theta/n)$ will determine a function for which $|\ \Phi(\omega)\ | < \theta$.

and n zeros all of which lie in a region $S(-\gamma,\ \pi/2n)$, is positive-real.

In Section IV we shall obtain larger regions which generate positive-real functions by employing a somewhat different technique. However these larger regions will not be so easily constructed as the $S(-\gamma,\ \pi/2n)$.

Actually we may place any number of pole-zero pairs into a given $S(-\gamma,\ \pi/2n)$ region to generate a positive-real function if these poles and zeros possess n-fold alternation on P paths. More precisely we may state

Theorem 3: Let $W(s)$ have m poles and m zeros, all occurring in $S(-\gamma,\ \pi/2n)$. Of these m poles and m zeros, let q poles and q zeros $(0 \leq q \leq m)$ possess n-fold alternation on a P path starting with poles and let the rest of them ($m - q$ poles and $m - q$ zeros) possess n-fold alternation on the same or another P path starting with zeros. Then $W(s)$ is positive-real.

Proof: For the q poles and q zeros let $-\xi$ be the pole farthest in the positive direction on its path and let $-\nu$ be the zero farthest in the negative direction. (If ξ is complex take any one of these complex-conjugate elements, and similarly for ν.) Let $\Psi(\omega)$ be the phase function corresponding to

$$\left[\frac{(j\omega + \nu)(j\omega + \bar{\nu})}{(j\omega + \xi)(j\omega + \bar{\xi})}\right]^{n/2}.$$

By Theorem 2, $-\pi/2 \leq \Psi(\omega) \leq 0$ for $\omega > 0$. Now the q poles and q zeros generate a rational function whose phase function $\Phi_q(\omega)$ satisfies $\Psi(\omega) \leq \Phi_q(\omega) \leq 0$ for $\omega > 0$, according to (11) of Lemma 3. Hence $-\pi/2 \leq \Phi_q(\omega) \leq 0$ for $\omega > 0$.

By precisely the same argument, the phase function $\Phi_{m-q}(\omega)$ for the remaining $m - q$ poles and $m - q$ zeros satisfies $0 \leq \Phi_{m-q}(\omega) \leq \pi/2$ for $\omega > 0$. Finally the phase function $\Phi_W(\omega)$ of $W(j\omega)$ equals $\Phi_q(\omega) + \Phi_{m-q}(\omega)$. Therefore $|\ \Phi_W(\omega)\ | \leq \pi/2$ for $\omega > 0$, and by Lemma 1 $W(s)$ is positive-real. Q.E.D.

The number of poles and the number of zeros of a positive-real function may differ by one and this situation has not yet been discussed in this section. For such a case there must be at least one real pole or one real zero. This consideration leads to

Theorem 4: Let $F(s)$ have m poles and m zeros, all occurring in $S(-\gamma,\ \pi/2n)$, and let these poles and zeros possess n-fold alternation on a P path starting with zeros. Let

$$G(s) = \frac{\prod_{i=1}^{q} (s + \eta_i)}{\prod_{i=1}^{q+1} (s + \delta_i)}, \tag{18}$$

where $q \geq 0$, the η_i and δ_i are real, and $0 \leq \delta_1 < \eta_1 < \delta_2 < \eta_2 < \cdots < \eta_q < \delta_{q+1}$. Then $F(s)G(s)$ is positive-real. On the other hand, if the n-fold alternation for $F(s)$ starts with poles, then $F(s)/G(s)$ is positive-real.

Note: If $q = 0$, $G(s) = 1/(s + \delta_1)$.

Proof: Let the n-fold alternation for $F(s)$ start with zeros. As in the proof of Theorem 3, we can show that the phase function $\Phi_F(\omega)$ of $F(j\omega)$ satisfies $0 \leq \Phi_F(\omega) \leq \pi/2$ for $\omega > 0$. We also have that the phase function $\Phi_G(\omega)$ satisfies $-\pi/2 \leq \Phi_G(\omega) \leq 0$ for $\omega > 0$. Hence by Lemma 1, $F(s)G(s)$ is positive-real. The rest of the theorem can be proven in the same way.

In this discussion we have taken the upper bound on $\Phi(\omega)$ to be $\pi/2$ so as to generate positive-real functions. However we may choose other bounds θ on $\Phi(\omega)$ and construct other regions $S(-\gamma,\ \theta/n)$. More precisely let the positive integer n and the positive numbers γ and θ be given. Let ρ be the real number larger than γ satisfying

$$\tan^{-1} \frac{\rho - \gamma}{2\sqrt{\gamma\rho}} = \frac{\theta}{n}. \tag{19}$$

Since $\Phi_F(\omega)$ for (13) can never be greater than $n\pi/2$, (19) provides a solution for ρ only when $\theta/n \leq \pi/2$; therefore, we shall restrict θ/n to such values. Denote by $S(-\gamma,\ \theta/n)$ the region of the s plane that is the intersection of the half-plane given by $Re\ s \leq -\gamma$, with the circular region given by $|\ s + \rho/2\ | \leq \rho/2$. The regions $S(-1,\ \theta/n)$ for $\theta/n = \pi/4$, $\pi/6$ and $\pi/8$ are shown in Fig. 5.

We may now state

Theorem 5: Let $W(s)$ be a rational function having n poles and n zeros all lying in a region $S(-\gamma,\ \theta/n)$. Then the phase function $\Phi_W(\omega)$ of $W(j\omega)$ satisfies

$$|\ \Phi_W(\omega)\ | \leq \theta \qquad (-\infty < \omega < \infty). \tag{20}$$

Proof: Each pole and each zero of $W(s)$ can be placed on a P path that passes through $s = -\gamma$ and $s = -\rho$. If $F(s)$ is given by (13), the phase function $\Phi_F(\omega)$ of

$F(j\omega)$ satisfies

$$|\Phi_F(\omega)| \leq n \tan^{-1} \frac{\rho - \gamma}{2\sqrt{\gamma\rho}} = \theta \qquad (-\infty < \omega < \infty)$$

as can be seen from (16) and (19). Now $W(s)$ can be obtained from $F(s)$ by moving the poles of $F(s)$ in the positive direction on the stated P paths and moving the zeros of $F(s)$ in the negative direction on these P paths. While doing this the phase function of the varying function at any given value of ω will vary monotonically from $\Phi_F(\omega)$ to $-\Phi_F(\omega)$, the latter value being achieved when the poles reach $s = -\gamma$ and the zeros reach $s = -\rho$. Hence

$$|\Phi_W(\omega)| \leq |\Phi_F(\omega)| \leq \theta.$$

Q.E.D.

Obviously if we place m poles and m zeros into $S(-\gamma, \theta/n)$, the bound on $|\Phi(\omega)|$ for all ω will be $m\theta/n$.

A result that is analogous to Theorem 3 also holds for the $S(-\gamma, \theta/n)$ regions and it can be proven in the same way.

Theorem 6: Under the hypothesis obtained by replacing $S(-\gamma, \pi/2n)$ *by* $S(-\gamma, \theta/n)$ *in the hypothesis of Theorem 3, we have that the phase function* $\Phi_W(\omega)$ *of* $W(j\omega)$ *satisfies* $|\Phi_W(\omega)| \leq \theta$ *for all* ω.

The results obtained thus far can be combined into a more general criterion which permits us to ascertain whether certain rational functions are positive-real. This may be accomplished by factoring the given rational function. In short we have

Theorem 7: Let

$$W(s) = \prod_{i=1}^{m} W_i(s) \tag{21}$$

where each $W_i(s)$ *satisfies the hypothesis of either Theorems 5 or 6 and let the corresponding bound on each* $|\arg W_i(j\omega)|$ *be* θ_i. *If*

$$\sum_{i=1}^{m} \theta_i \leq \frac{\pi}{2},$$

then $W(s)$ *is positive-real.*

As an example of the application of Theorem 7 consider the function

$$W(s) = \frac{[(s+1)^2 + 1](s+3)^3}{[(s+2)^2 + 0.16](s+5)^3}.$$

The poles and zeros of this function do not fit inside any $S(-\gamma, \pi/10)$ region no matter what value for γ is tried. However the zeros at $s = -1 \pm j$ and the poles at $s = -2 \pm j\,0.4$ fit inside $S(-1, \pi/8)$ whereas the triple zero at $s = -3$ and the triple pole at $s = -5$ fit inside $S(-3, \pi/12)$. Consequently by Theorem 7 $W(s)$ is positive-real.

We can again include the case where the number of poles and number of zeros differ by one in the following way.

Corollary 7(a): In addition to the hypothesis of Theorem 7, let (21) *be such that the phase functions for all the* $W_i(j\omega)$ *are non-negative for* $\omega > 0$. *Also let* $G(s)$ *be given by* (18) *and its associated restrictions. Then* $W(s)G(s)$ *is positive-real. On the other hand, if the phase functions for all the* $W_i(j\omega)$ *are nonpositive for* $\omega > 0$, *then* $W(s)/G(s)$ *is positive-real.*

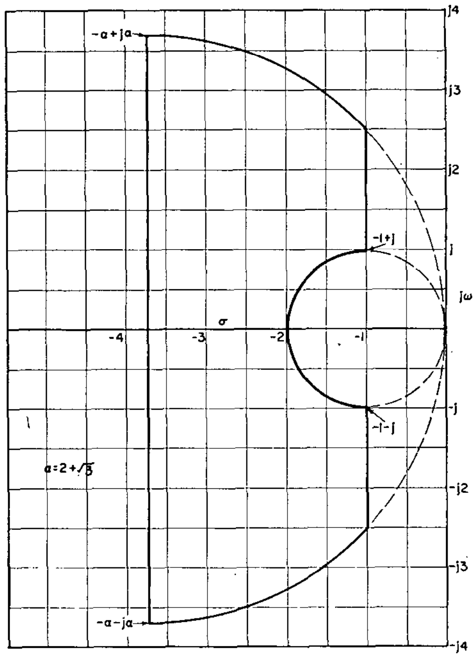

Fig. 6—The region $T(-1+j, -\alpha+j\alpha, \pi/4)$. 2 poles and 2 zeros placed anywhere in this region will generate a positive-real function.

In developing the $S(-\gamma, \theta/n)$ regions we started from (13), where γ and ρ are both real. If we remove the restriction that γ and ρ are real, we may generate other regions which we shall denote by $T(-\gamma, -\rho, \theta/n)$, having the property that n poles and n zeros placed anywhere within them will produce a function whose phase function is bounded in magnitude by θ for all ω. For instance let

$$F(s) = \frac{(s+1)^2 + 1}{(s+\alpha)^2 + \alpha^2} \qquad (\alpha = 2 + \sqrt{3}). \tag{22}$$

This is a positive-real function; moreover, the phase function $\Phi_F(\omega)$ of $F(j\omega)$ achieves $\pi/2$ at some value of ω. Define $T(-1+j, -\alpha+j\alpha, \pi/4)$ as the largest closed region each of whose points lies on a P path that connects either $s = -1+j$ to $s = -\alpha + j\alpha$ or $s = -1-j$ to $s = -\alpha - j\alpha$. The boundary of this region is indicated in Fig. 6. By precisely the same argument that established Theorem 2, we can conclude that if $W(s)$ has 2 poles and 2 zeros appearing anywhere in $T(-1+j, -\alpha+j\alpha, \pi/4)$, then $W(s)$ is positive-real.

More generally the region $T(-\gamma, -\rho, \theta/n)$ will be defined as follows. Let Ψ be the maximum value of the phase function for

$$F(j\omega) = \frac{(j\omega+\gamma)(j\omega+\bar{\gamma})}{(j\omega+\rho)(j\omega+\bar{\rho})} \qquad (Im\,\gamma \geq 0, Im\,\rho \geq 0). \tag{23}$$

The region $T(-\gamma, -\rho, \Psi/2) = T(-\gamma, -\rho, \theta/n)$ consists of all points that lie on a P path connecting either $s = -\gamma$ to $s = -\rho$, or $s = -\bar{\gamma}$ to $s = -\bar{\rho}$. It should be noted that for certain choices of ρ and γ this region will not exist. When the region does exist, its boundary will consist of portions of straight lines and circles, as for example in Fig. 6.

All the results obtained previously for the $S(-\gamma, \theta/n)$ regions may be extended to the $T(-\gamma, -\rho, \theta/n)$ regions where γ and ρ are in general complex. However we shall not pursue this approach any further since all these results can be obtained as special cases of the more general conclusions of Section IV.

IV. Larger Regions

In this section we shall develop regions in the left-half s plane which include both the $S(-\gamma, \theta/n)$ and the $T(-\gamma, -\rho, \theta/n)$ as proper subregions. As a first step consider again the function

$$F(s) = \frac{(s + \gamma)(s + \bar{\gamma})}{(s + \rho)(s + \bar{\rho})} = \frac{s^2 + a_1 s + a_0}{s^2 + b_1 s + b_0}, \qquad (24)$$

where γ and ρ are in general complex but may be real. Also let $Re\ \rho$ and $Re\ \gamma$ be positive. Computing the $Re\ F(j\omega)$ we see that the $Re\ F(j\omega)$ is non-negative if and only if the polynomial

$$P(\omega^2) = \omega^4 + (a_1 b_1 - a_0 - b_0)\omega^2 + a_0 b_0 \qquad (25)$$

is non-negative. The critical case occurs (*i.e.*, $F(s)$ is just on the borderline of being a positive-real function) when

$$a_0 + b_0 - a_1 b_1 \geq 0 \qquad (26)$$

and

$$[a_0 + b_0 - a_1 b_1]^2 - 4a_0 b_0 = 0. \qquad (27)$$

Combining (26) and (27) and using $a_1 = 2\ Re\ \gamma$, $a_0 = |\gamma|^2$, $b_1 = 2\ Re\ \rho$, $b_0 = |\rho|^2$, we obtain

$$Re\ \rho = \frac{(|\gamma| - |\rho|)^2}{4\ Re\ \gamma}. \qquad (28)$$

For a fixed value for γ, (28) defines a locus of values for $-\rho$. For $\gamma = 1$, $1 + j$, and $1 + 2j$, these loci are illustrated in Fig. 7. We shall designate such loci by $L_2(-\gamma, \pi/2)$. They have the property that, for the given value of γ and for $-\rho$ on $L_2(-\gamma, \pi/2)$, the peak value of $|\arg F(j\omega)|$ corresponding to (24) equals $\pi/2$.

Similarly given γ, we can compute the locus of $-\rho$ for which the peak value of $|\arg F(j\omega)|$ equals θ where $\theta \leq \pi/2$. More generally given γ let $L_2(-\gamma, \theta)$ be the locus of $-\rho$ for which the peak value of the phase function $\Phi_F(\omega)$ of $F(j\omega)$ equals θ where $\theta \leq \pi/2$ and $F(s)$ is given by (24). The loci $L_2(-\gamma, \theta)$ were computed in several typical cases by using an analog computer and the results are shown in Figs. 8, 9, and 10.

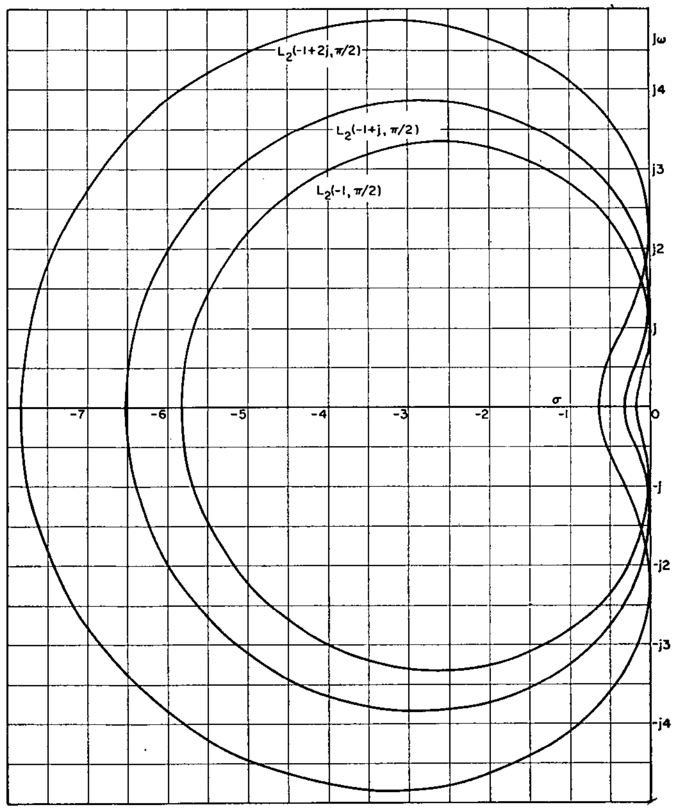

Fig. 7—The loci $L_2(-\gamma, \pi/2)$ for $-\gamma = -1$, $-1 + j$, and $-1 + 2j$.

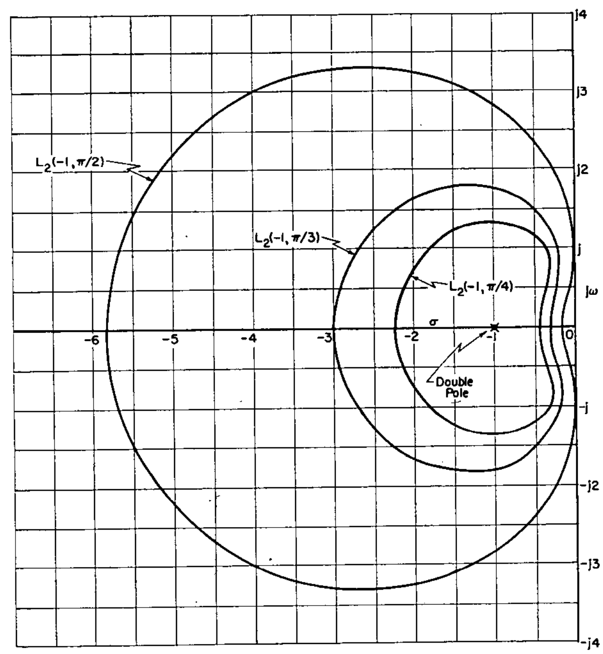

Fig. 8—The loci $L_2(-1, \psi)$ for $\psi = \pi/4$, $\pi/3$, and $\pi/2$.

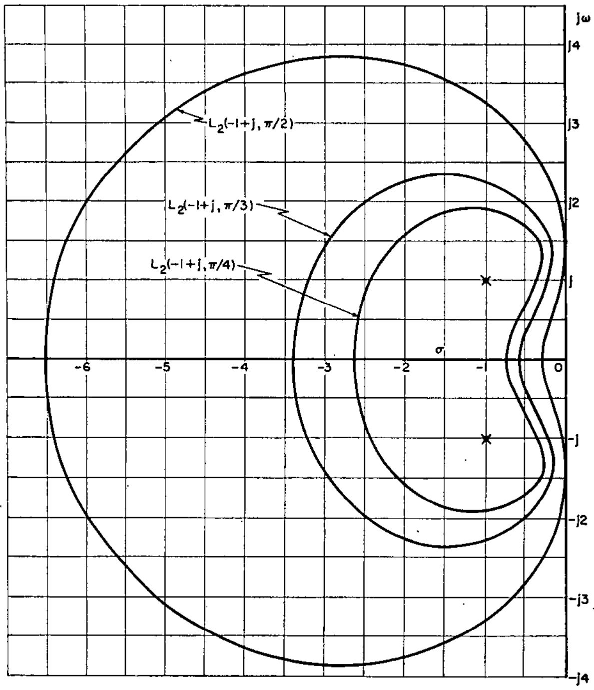

Fig. 9—The loci $L_2(-1+j, \psi)$ for $\psi = \pi/4$, $\pi/3$, and $\pi/2$.

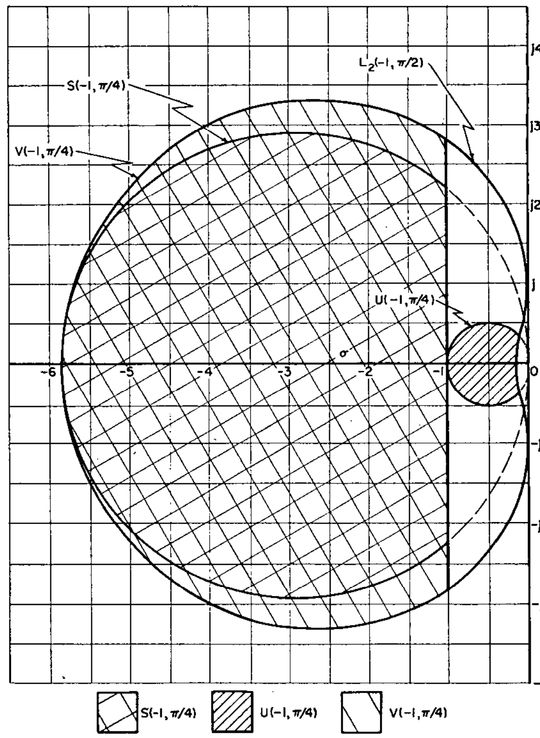

Fig. 11—The regions $S(-1, \pi/4)$, $U(-1, \pi/4)$, and $V(-1, \pi/4)$. Note: $V(-1, \pi/4)$ includes $S(-1, \pi/4)$.

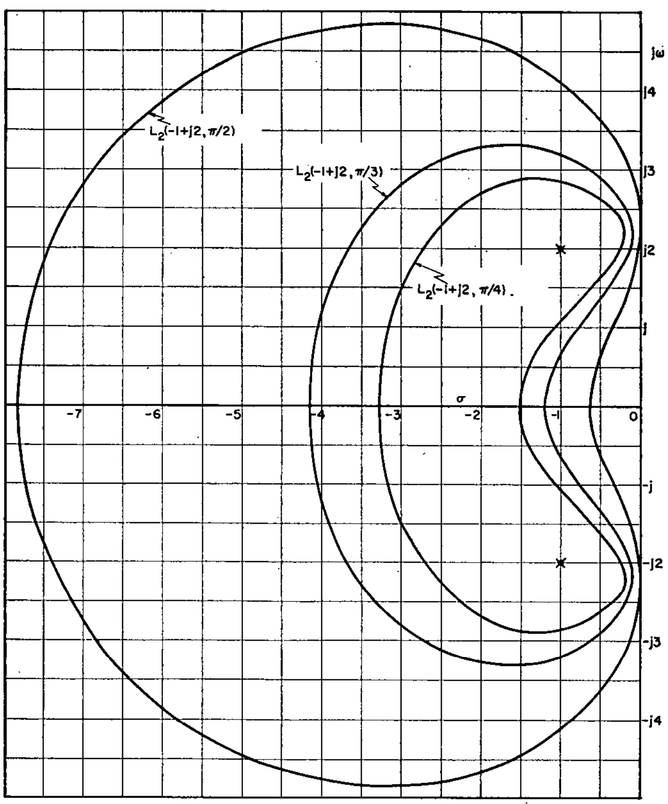

Fig. 10—The loci $L_2(-1+j2, \psi)$ for $\psi = \pi/4$, $\pi/3$, and $\pi/2$.

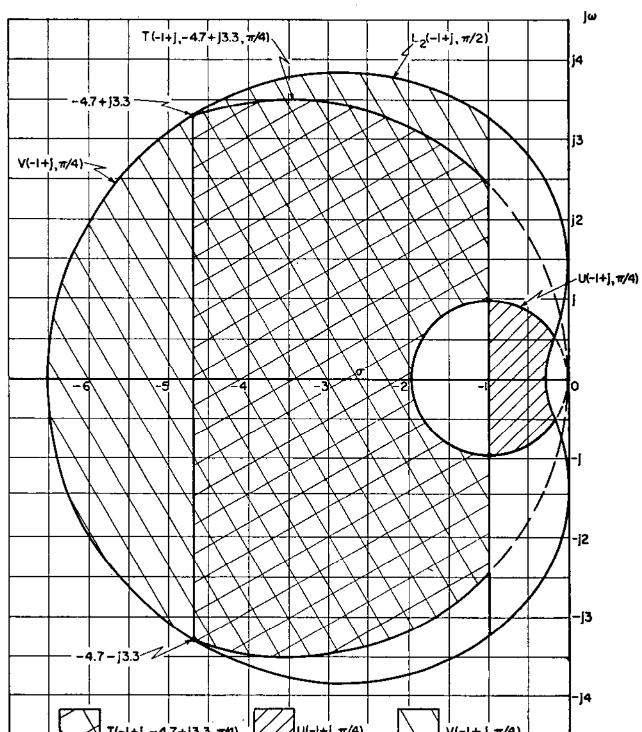

Fig. 12—The regions $T(-1+j, -4.7+j3.3, \pi/4)$, $U(-1+j, \pi/4)$, and $V(-1+j, \pi/4)$. Note: $V(-1+j, \pi/4)$ includes $T(-1+j, -4.7+j3.3, \pi/4)$.

We are now ready to define the regions that we seek. Let n be a positive integer no less than 2 and let γ be given. $U(-\gamma, \theta/n)$ is the largest closed region within $L_2(-\gamma, 2\theta/n)$ having the property that each of its points can be reached by tracing in the positive direction on a P path that starts either at $s = -\gamma$ or $s = -\bar{\gamma}$. Similarly $V(-\gamma, \theta/n)$ is the largest closed region within $L_2(-\gamma, 2\theta/n)$ having the property that each of its points can be reached by tracing in the negative direction on a P path that starts either at $s = -\gamma$ or $s = -\bar{\gamma}$. The regions $U(-\gamma, \pi/4)$ and $V(-\gamma, \pi/4)$ for $\gamma = 1$ and $\gamma = 1 + j$ are shown in Figs. 11 and 12.

Remembering that $S(-\gamma, \theta/n)$ and $T(-\gamma, -\rho, \theta/n)$ are the largest regions that can be covered by tracing on P paths in the appropriate directions from $s = -\gamma$ and $s = -\rho$, it is clear that $S(-\gamma, \theta/n)$ is a subregion of $V(-\gamma, \theta/n)$, where γ is real, and that $T(-\gamma, -\rho, \theta/n)$ is a subregion of either $U(-\gamma, \theta/n)$ or $V(-\gamma, \theta/n)$, where γ and ρ are now in general complex. For $T(-\gamma, -\rho, \theta/n)$, ρ is some point on $L_2(-\gamma, 2\theta/n)$ that also lies on the boundary of either $U(-\gamma, \theta/n)$ or $V(-\gamma, \theta/n.)$ The subregion $S(-1, \pi/4)$ of $V(-1, \pi/4)$ is shown in Fig. 11 and the subregion $T(-1 + j, -4.7 + j\,3.3, \pi/4)$ of $V(-1 + j, \pi/4)$ is shown in Fig. 12.

The important thing about the regions $U(-\gamma, \theta/n)$ and $V(-\gamma, \theta/n)$ is that all the properties that were developed for $S(-\gamma, \theta/n)$ also hold for these regions, as we shall now show. Corresponding to Theorem 5 we have that if $W(s)$ is rational with n poles and n zeros and all the poles and zeros lie in $V(-\gamma, \theta/n)$, then $|\arg W(j\omega)| \leq \theta$ for all ω. A similar comment holds for $U(-\gamma, \theta/n)$. (A special case of this is the property for $U(-\gamma, \pi/2n)$ and $V(-\gamma, \pi/2n)$ corresponding to Theorem 2.)

To prove this fact we shall show that $|\arg [W(j\omega)]^2|$ is bounded by 2θ for all ω. The demonstration will be for $V(-\gamma, \theta/n)$, the one for $U(-\gamma, \theta/n)$ being essentially the same. Let $s = -\nu$ be any zero of $W(s)$ and let $s = -\xi$ be any pole of $W(s)$. Then $[W(j\omega)]^2$ is a product of terms each of which has the form

$$\frac{(j\omega + \nu)(j\omega + \bar{\nu})}{(j\omega + \xi)(j\omega + \bar{\xi})} = \left[\frac{(j\omega + \nu)}{(j\omega + \gamma)} \frac{(j\omega + \bar{\nu})}{(j\omega + \bar{\gamma})}\right]\left[\frac{(j\omega + \gamma)}{(j\omega + \xi)} \frac{(j\omega + \bar{\gamma})}{(j\omega + \bar{\xi})}\right], \tag{29}$$

where ν and ξ may be either real or complex. The phase function $\Phi_1(\omega)$ of the first bracketed factor on the right-hand side of (29) satisfies

$$-2\theta/n \leq \Phi_1(\omega) \leq 0 \tag{30}$$

as we shall now show. By definition of $V(-\gamma, \theta/n)$, a P path may be drawn from $s = -\gamma$ through $s = -\nu$ and finally through some point $s = -\zeta$ on $L_2(-\gamma, 2\theta/n)$. (Here we assume without loss of generality that $(Im\,\gamma)(Im\,\nu) \geq 0$.) By the property of P paths and by the definition of $L_2(-\gamma, 2\theta/n)$

$$-2\theta/n \leq \arg\left[\frac{(j\omega + \zeta)\,(j\omega + \bar{\zeta})}{(j\omega + \gamma)(j\omega + \bar{\gamma})}\right] \leq 0$$

for $\omega > 0$. We can generate the function in the first bracket in (29) by moving the zeros from $-\zeta$ and $-\bar{\zeta}$ to $-\nu$ and $-\bar{\nu}$. But this is a translation in the positive direction on a P path and therefore the phase function increases for each positive ω. This establishes (30).

Similarly the phase function $\Phi_2(\omega)$ of the second bracketed factor on the right-hand side of (29) satisfies $0 \leq \Phi_2(\omega) \leq 2\theta/n$ for all ω. Hence the phase function $\Phi_1(\omega) + \Phi_2(\omega)$ of (29) is bounded above and below by $\pm 2\theta/n$ and, since there are n factors such as (29) in $[W(j\omega)]^2$, $|\arg [W(j\omega)]^2| \leq 2\theta$ for all ω, which is what we wished to prove.

Having this result we may extend Theorems 3, 4, 6 and 7, and Corollary 7 (a) to the larger regions U and V by repeating their proofs after replacing $S(-\gamma, \theta/n)$ by either $U(-\gamma, \theta/n)$ or $V(-\gamma, \theta/n)$. The same may be said for $T(-\gamma, -\rho, \theta/n)$. This is the principal conclusion of this paper and it is subsumed by the following theorem:

Theorem 8: Theorems 2 through 7 and Corollary 7 (a) still hold when $S(-\gamma, \theta/n)$ is replaced by either $U(-\gamma, \theta/n)$, $V(-\gamma, \theta/n)$, or $T(-\gamma, -\rho, \theta/n)$.

V. Largest Regions

In the previous sections we have defined regions which have the following property: If n poles and n zeros are placed within one of these regions, then the phase function of the resulting rational function is bounded in magnitude on the $j\omega$ axis by some angle, say θ. The question arises as to whether we can define regions, say $R_n(\theta)$, which are largest in the sense that they are not proper subsets of other regions which also have the desired property. Although the possible variety of such regions would seem to be limitless, the authors have been able to find but one example of them, and that only for $n = 2$ and $\theta = \pi/2$. For further details about this result see Steiglitz and Zemanian.[5]

VI. C Paths

Up to now we have placed poles and zeros in n-fold alternation only on P paths. This suggests the question as to whether any other paths can be defined with similar properties. In this section we shall develop such paths; they are related to the P paths but are not exactly the same.

Consider a circle whose center is on the real axis and which lies partly in the left-half s plane and partly in the right-half s plane. We shall define a C path as that part of the circle which lies in the closed left-half s plane.

[5] K. Steiglitz. and A. H. Zemanian, "Sufficient conditions on pole and zero locations for rational positive-real functions," Dept. of Elect. Engrg., New York University, N. Y., Tech. Rept. 400-34; August, 1961.

The negative direction on a C path always points toward the left. Fig. 13 illustrates a C path.

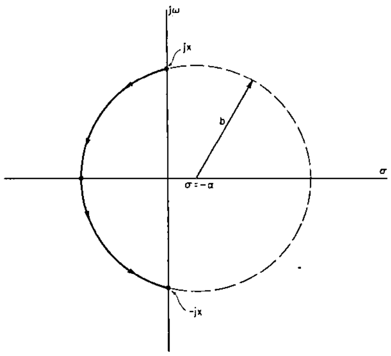

Fig. 13—A C path. The arrowheads point in the negative direction. The center of the circle is at $\sigma = -a$ and its radius equals b.

Corresponding to Lemma 2, we have

Lemma 5: Let $Q(s) = (s + \alpha)^2 + \beta^2$ *be a real quadratic having zeros in the left-half s plane* ($\alpha, \beta \geq 0$). *Also, let these zeros move in the negative direction on a C path that intersects the* $s = j\omega$ *axis at* $s = \pm jx$ ($x > 0$). *Then the phase function* $\Phi_Q(\omega)$ *of* $Q(j\omega)$ *decreases monotonically for every fixed* ω *in the interval* $x < \omega < \infty$ *and increases monotonically for every fixed* ω *in the interval* $0 < \omega < x$.

Proof: Letting $d\Phi_Q$ be the increment of Φ_Q at a fixed ω as the zeros of $Q(s)$ move on the C path, we have as in the proof of Lemma 2 that for $\omega > 0$ the sign of $d\Phi_Q$ is the same as the sign of

$$(\beta^2 - \alpha^2 - \omega^2)\, d\alpha - 2\alpha\beta\, d\beta. \tag{31}$$

Assuming that the circle which defines the C path has a center at $\sigma = -a$ and a radius equal to b and noting that $\beta \neq 0$ we may convert (31) into

$$(b^2 - a^2 - \omega^2)\, d\alpha = (x^2 - \omega^2)\, d\alpha. \tag{32}$$

Hence for $\omega = x$, $d\Phi_Q = 0$; for $0 < \omega < x$, $d\Phi_Q > 0$; for $\omega > x$, $d\Phi_Q < 0$. Q.E.D.

Let us define n-fold alternation of poles and zeros on a C path in precisely the same way as we did for P paths. We may now extend Lemma 3 to C paths as follows.

Lemma 6: Let $W(s)$ *have kn poles and kn zeros which possess n-fold alternation on a C path that intersects the* $s = j\omega$ *axis at* $\pm jx$. *Let* $\Phi_1(\omega), \Phi_2(\omega), \cdots, \Phi_{2k}(\omega)$ *denote the same phase contributions as they did in Lemma 3 and let* $\Phi_W(\omega) = \arg W(j\omega)$. *If zeros are the starting elements, then*

$$-n\pi/2 \leq -|\Phi_1(\omega)| \leq -|\Phi_1(\omega)| + |\Phi_{2k}(\omega)| \leq \Phi_W(\omega) \leq 0 \qquad (0 < \omega < x),$$

$$0 \leq \Phi_W(\omega) \leq |\Phi_1(\omega)| - |\Phi_{2k}(\omega)| \leq |\Phi_1(\omega)| \leq n\pi/2 \qquad (x < \omega).$$

If poles are the starting elements, then

$$0 \leq \Phi_W(\omega) \leq |\Phi_1(\omega)| - |\Phi_{2k}(\omega)| \leq |\Phi_1(\omega)| \leq n\pi/2 \qquad (0 < \omega < x),$$

$$-n\pi/2 \leq -|\Phi_1(\omega)| \leq -|\Phi_1(\omega)| + |\Phi_{2k}(\omega)| \leq \Phi_W(\omega) \leq 0 \qquad (x < \omega).$$

This lemma is proven in precisely the same way as was Lemma 3 except that the two intervals $0 < \omega < x$ and $x < \omega$ are considered separately.

Armed with Lemmas 5 and 6 we can now extend Theorem 3 as follows:

Theorem 9: Let $W(s)$ *have m poles and m zeros all occurring in either* $S(-\gamma, \pi/2n)$, $T(-\gamma, -\rho, \pi/2n)$, $U(-\gamma, \pi/2n)$, *or* $V(-\gamma, \pi/2n)$. *Of these m poles and m zeros, let q poles and q zeros* ($0 \leq q \leq m$) *possess n-fold alternation on a C path that passes through* $s = \pm jx$ *and let this alternation start with poles. Let the rest of the poles and zeros possess n-fold alternation on the same or another C path that passes through* $s = \pm jx$ *where the alternation now starts with zeros. Then* $W(s)$ *is positive-real.*

The proof of this theorem is almost identical to that of Theorem 3 and will not be given.

Because the behavior of the phase function depends on the point $\omega = x$, Theorem 4 cannot be extended to C paths. We can however obtain extensions of Theorems 6 and 7 by replacing the regions stated in Theorem 9 by $S(-\gamma, \theta/n)$, $T(-\gamma, -\rho, \theta/n)$, $U(-\gamma, \theta/n)$, and $V(-\gamma, \theta/n)$.

Acknowledgment

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