Princeton Univ. F'24 COS 597b: Recent Advances in Graph Algorithms

Lecture 9: Deterministic Decremental SSSP and Approximate Min-Cost Flow in Almost-Linear Time [1]

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Summary: This lecture focuses on maintaining approximate shortest paths (SPs) in a decremental undirected graph G with efficient update times. The approach utilizes a robust core, EStree structures, and expander-based techniques to address the challenges of handling SPs under edge deletions. By increasing the "weight" or "multiplicity" of certain edges, we transform the graph into a low-diameter expander, enabling efficient maintenance of SPs. Key lemmas provide bounds on the total weight adjustments and demonstrate methods for preserving expander properties under deletions. Finally, the lecture outlines the algorithm for maintaining the robust core and EStree structures to ensure reliable and efficient updates throughout the decremental process.

Setting Decremental undirected graph G with the goal to maintain a $(1+\epsilon)$-approximate Shortest Paths (SP) from a source s in total update time $m \cdot n^{O(1)}$.

EStree:

- Maintains exact distances from a set S up to distance d in time $O(md)$.
- Maintains $(1+\epsilon)$-approximate SP using U edges in time $\tilde{O}(m \cdot U)$.

Robust Core:

- Given a decremental graph, find an initial set C^{init} of vertices such that $\text{diam}(C^{\text{init}}) \leq d$.
- Maintain a decremental set C, initialized to C^{init}, until $C = \emptyset$, such that:
  1. $\text{diam}(C) \leq d \cdot n^{O(1)}$.
  2. To leave C, (i.e. for each $v \in C^{\text{init}} \setminus C$), we have $|\text{Ball}(v, 2d) \cap C^{\text{init}}| \leq (1-\delta)|C^{\text{init}}|$.
- When removing v, ensure that its neighborhood doesn't contain the majority of vertices.

Goal: Deterministically maintain the robust core with a total update time of $|G| \cdot n^{O(1)}$.

- For simplicity, assume $C^{\text{init}} = V_G$.
- Natural attempt: pick u, maintain EStree $T(u)$ up to distance $O(d)$.
- If $|\text{Ball}(v, 2d)| > \frac{n}{2}$, remove v far; otherwise, rebuild EStree.

Prior Work:

- Pick a random vertex, maintain EStree $T(u)$ up to $O(d)$, and cut all v that are far from u.

- If the graph is an expander, build the EStree from the expander.

Main Idea: Increase the "weight" or "multiplicity" of vertices or edges to turn a low-diameter graph into an expander such that the total weight is minimized.

Definition: Let $k : E \to \mathbb{Z}^+$ be a weight function. A graph (G, k) is a ϕ-expander if, for all sets S with $|S| \leq \frac{n}{2}$,

$$\sum_{e \in E(S, \bar{S})} k(e) \geq \phi |S|.$$

If we weaken the condition by requiring the maximum degree to be at most 3, the expander condition shifts from ensuring a large weighted edge-boundary (counting all multi-edges incident on a set) to ensuring that enough vertices in the set each contribute a few edges to its boundary. This makes the requirement weaker and thus more efficient to achieve.

Goal: Find k such that (G, k) is a ϕ-expander, and the total weight $\sum_{e \in E} k(e) \leq mn^{O(1)}$.

Lemma 1 (Existence of Expander Weights): For any graph G such that $\mathrm{diam}(G) \leq d$, there exists a weight function k such that (G, k) is a ϕ-expander and

$$\sum_{e \in E} k(e) \leq \tilde{O}(d \cdot |G| \cdot \phi).$$

Proof:
Consider the following procedure:

- Set $k(e) = 1$ for all $e \in E$.
- While there exists S with $|S| \leq \frac{n}{2}$ and $\sum_{e \in E(S, \bar{S})} k(e) \leq \phi |S|$, for each $e \in E(S, \bar{S})$, set $k(e) \leftarrow 2 \cdot k(e)$.

Bounding the Total Weight: Charge the increase to set S such that each vertex $v \in S$ is charged $O(\phi)$ each time.

Bounding the Time that v is Charged: Fix shortest paths (SPs) from v to all other vertices.

- **Claim 1:** For every edge e, $k(e) \leq O(n)$, with $k(e)$ doubling at most $O(\log n)$ times.
- **Claim 2:** When v is charged, there exist at least $\frac{n}{2}$ vertices u such that at least one edge on the shortest path from v to u has had its weight doubled.

Thus, the total increase of k is $\tilde{O}(d \cdot n \cdot \phi)$, and the number of times v is charged is:

$$\leq \frac{O(n \cdot d \cdot \log n)}{n/2} \leq \tilde{O}(d),$$

resulting in a total increase of

$$k \leq \tilde{O}(n \cdot d \cdot \phi).$$

$\square$

Lemma 2: Let (G, k) be a $\frac{1}{n}$-expander. A decremental set $X \subseteq V$ can be maintained under edge deletions such that:

- $|X| \geq \frac{|G|}{2}$.
- $(G[X], k)$ remains a $\frac{1}{n^{O(1)}}$-expander as long as $\leq n^{-O(1)}$-fraction of edges, in terms of the total weight k, are deleted.

X can be maintained in total time $|G| \cdot n^{O(1)}$

Note: Under this definition, we can't guarantee $G[X]$ has small diameter.

Solution: 'Embed' an actual expander W with good edge expansion into G, and maintain the expansion of W.
'Embed' means that for each edge $(u, v) \in W$, there exists a short path in G from u to v, with paths disjoint in the multigraph G.

- Compute k such that (G, k) is an expander.
- Maintain X such that $(G[X], k)$ is an expander with low diameter and $|X| \geq \frac{|G|}{2}$.
- Maintain EStree (X) up to distance $O(d)$; remove any v such that $\text{dist}(x, v) > 2d$.
- If $|X| < \frac{|G|}{2}$, repeat the procedure on a subset of vertices with diameter $\tilde{O}(d)$ until $|G| < (1 - \delta) \cdot |\text{initial size}|$
- We can ensuring the repetition happens only $n^{O(1)}$ times.

Main Algorithm:

Maintain the robust core $\{C_i\}$ and EStree (C_i) up to distance $D_i \geq diam(C_i)$:

1. Every vertex u is close to some C_i.
2. Every vertex u lies within the $n^{O(1)}$ shell around some C_i.

When condition (1) fails, create a new core centered at u. The set $\{C_i\}$ defines a hop emulator.

References

[1] Aaron Bernstein, Maximilian Probst Gutenberg, and Thatchaphol Saranurak. Deterministic decremental sssp and approximate min-cost flow in almost-linear time. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1000–1008. IEEE, 2022.