Princeton Univ. F'24 COS 597b: Recent Advances in Graph Algorithms

Lecture 5: Maximum Flow by Augmenting Paths in $n^{2+o(1)}$ Time

Lecturer: *Huacheng Yu* Scribe: *Hee Yun Suh & Haichen Dong*

Definition 1 (Maximum Flow Problem). *Given a directed graph $G = (V, E)$ with integral capacities $c : E \to \{1, 2, \cdots, U\}$, source $s \in V$ and sink $t \neq s \in V$, one is asked to output a maximum flow from s to t.*

Theorem 1. *There exists an algorithm that solves the maximum flow problem within $n^{2+o(1)} \log U$ time.*

In this note, we focus on the case where $U = 1$, i.e., the input graph is unit-capacitated. Also, it suffices to find an $1/n^{o(1)}$-approximate flow algorithm for directed graphs, as the exact algorithm then follows by repeating the approximate algorithm $n^{o(1)}$ times on the residual graph.

1 Preliminary

1.1 General Flow Instance

A general flow instance with a capacitated graph $G = (V, E, c)$ consists of a source vector $\Delta : V \to \mathbb{R}_{\geq 0}$ and a sink vector $\nabla : V \to \mathbb{R}_{\geq 0}$, where $\|\Delta\|_1 = \|\nabla\|_1$ and $\text{supp}(\Delta) \cap \text{supp}(\nabla) = \varnothing$.

A flow $f : E \to \mathbb{R}_{\geq 0}$ routes (Δ, ∇) if f sends Δ amount of flow to ∇. We may allow $f(e) > c(e)$ for some $e \in E$, and the congestion of f is defined as $\text{cong}(f) = \max_{e \in E} \left| \frac{f(e)}{c(e)} \right|$. Let $G_f = (V, E_f)$ be the residual graph with respect to flow f.

1.2 Push-Relabel Algorithm

In the push-relabel algorithm, we maintain a preflow $f : E \to \mathbb{R}_{\geq 0}$, labels $l : V \to \mathbb{Z}_{\geq 0}$ such that $l(u) \leq l(v) + 1$ for all $(u, v) \in E_f$. While there exists some vertex v with strictly positive excess and attempt either to:

- push(u): if there exists v such that $(u, v) \in E_f$ and $l(u) = l(v) + 1$, push 1 unit of flow through (u, v).
- relabel(v): otherwise set $l(u) \leftarrow \min_{(u,v) \in E_f} l(v) + 1$.

1.3 Expander Graphs

Definition 2. *Let $G = (V, E)$ be a directed unweighted graph, and define the volume of $S \subseteq V$ as $\text{vol}(S) = \sum_{u \in S} \deg(u)$. A cut $(S, V \setminus S)$ is ϕ-sparse if*

$$\min\{|E(S, V \setminus S)|, |E(V \setminus S, S)|\} \leq \phi \cdot \min\{\text{vol}(S), \text{vol}(V \setminus S)\}$$

G is a ϕ-expander if there does not exist a ϕ-sparse cut.

2 Main Results

2.1 Weighted Push-Relabel Algorithm

Intuitively, if an edge is *infrequent* in the maximum flow, it can be less admissible to allow more efficiency. In this paper, every edge is weighted by $w : E \to \mathbb{N}_{\geq 1}$. We maintain lables $l(u) \leq l(v) + w(u, v)$ for any $(u, v) \in E$. An edge $(u, v) \in E$ is considered admissible only if $l(u) = l(v) + w(u, v)$, and when relabeling, the label of $u \in V$ is set as $l(u) \leftarrow \min_{(u,v)\in E_f}(l(v) + w(u, v))$.

Theorem 2. *Suppose there exists a flow f such that every flow path P in f has $w(P) = \sum_{e\in P} w(e) \leq h$ for some h, then weighted push-relabel algorithm finds an $\Omega(1)$-approximate flow f' such that $|f'| \geq \Omega(|f|)$ in time $\tilde{O}(m + h \cdot \sum_{e\in E} \frac{1}{w(e)})$.*

The proof of Theorem will be deferred to the next lecture. Now we focus on the criteria of good weight functions and how to find them.

Good Weight Our goal becomes to find a good weight function $w : E \to \mathbb{R}_{\geq 0}$ such that:

- $\sum_{e\in E} \frac{1}{w(e)} \leq n^{1+o(1)}$, and
- $h \leq n^{1+o(1)}$, i.e., there exists an approximate max flow f such that $w(P) \leq n^{1+o(1)}$ for every flow path $P \in f$.

As a result, the time complexity in Theorem 2.1 becomes $\tilde{O}(m + h \cdot \sum_{e\in E} \frac{1}{w(e)}) = n^{2+o(1)}$.

Directed Acyclic Graph. Here we present an examples of good weight function in directed acyclic graphs. For a DAG $G = (V, E)$, we first compute the topological order of vertices $\tau : V \to [|V|]$. Then we set the weights as $w(u, v) = \tau(v) - \tau(u)$. For any flow path $P \in f$ where f is an s-t-flow, we have $w(P) = \tau(t) - \tau(s) \leq n$. Furthermore, it holds that

$$\sum_{e\in E} \frac{1}{w(e)} = \sum_{u\in V} \sum_{v\in V:(u,v)\in E} \frac{1}{w(u, v)} \leq \sum_{u\in V} \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{u\in V} O(\log n) \leq n^{1+o(1)}.$$

Therefore, such weight w is good and we can compute an $\Omega(1)$-approximate max flow in $n^{2+o(1)}$ time by Theorem 2.1.

Existence of Good Weight. The example above also leads to the existence of good weight functions for any general graph G. Let f be a maximum flow of G. We keep removing cycles in f until f defines a DAG. Then we assign weights as above for edges $e \in f$, and large weights like $w(e) = n^2$ for all other edges $e \notin f$.

2.2 Good Weight Function via Directed Expander Decomposition

Lemma 3. *Let $G = (V, E)$ be directed and $\phi = 1/n^{o(1)}$, then V can be decomposed into $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ such that*

- $\sum_{i<j} |E(U_i, U_j)| \leq \tilde{O}(\phi \cdot m)$, *and*

- *each $G[U_i]$ is ϕ-expander.*

Proof Sketch. Starting with G, we keep removing the sparse direction of a sparse cut, and recurse on both sides. □

Simple Expanders To utilize the above lemma, note that every partition is ϕ-expander, we first focus on constructing good weights when the whole graph is ϕ-expander.

Lemma 4. *For a ϕ-expander graph $G = (V, E)$, there exists an $\Omega(1)$-approximate maximum flow such that every flow path uses at most $\tilde{O}(1/\phi)$ edges.*

Proof. Let $l = 10 \log n/\phi$. We keep finding (s, t) argument paths of length $\leq l$ until no such path exists. Let f be the resulting flow, and f^* be the maximum flow. Suppose $|f^*| \geq 2|f|$ for contradiction.

Consider any (s, t) cut $(S, \bar{S})$, there is $\left|E_f(S, \bar{S})\right| = \left|E(S, \bar{S})\right| - |f|$. And by the max-flow min-cut theorem, we have $\left|E(S, \bar{S})\right| \geq |f^*| \geq 2|f|$. Therefore,

$$\left|E_f(S, \bar{S})\right| = \left|E(S, \bar{S})\right| - |f| \geq \frac{1}{2}\left|E(S, \bar{S})\right|.$$

Define balls in the residual graph as $B_i = \{x \in V : \mathrm{dist}_{G_f}(s, x) \leq i\}$ and $B'_i = \{x \in V : \mathrm{dist}_{G_f}(x, t) \leq i\}$ for $i = 0, 1, 2, \cdots$. By definition, $\mathrm{vol}(B_0) \geq 1$. Furthermore, since G is ϕ-expander, as long as $\mathrm{vol}(B_{i-1}) \leq \frac{\mathrm{vol}(V)}{2}$, we have

$$\mathrm{vol}(B_i) \geq (1 + \frac{\phi}{2}) \,\mathrm{vol}(B_{i-1}).$$

Therefore, it holds that $\mathrm{vol}(B_{l/2}) \geq (1 + \phi/2)^{l/2} \geq \mathrm{vol}(V)/2$. Same argument implies $\mathrm{vol}(B'_{l/2}) \geq \mathrm{vol}(V)/2$ as well.

Since $\mathrm{vol}(B_{l/2}) + \mathrm{vol}(B'_{l/2}) \geq \mathrm{vol}(V)$, there exists an (s, t) path in G_f with length at most l, which raises contradiction. Therefore, f is a $\frac{1}{2}$-approximate maximum flow. □

DAG of Expanders Then we consider a more generalized case. For graph $G = (V, E)$ with decomposition $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ where every SCC $U_i \in \mathcal{U}$ is a ϕ-expander, but there can be DAG edges $D \subseteq E$ between the SCCs.

Suppose $\mathcal{U}$ is topologically ordered. First we compute a topological order $\tau : V \to [n]$ for G by assigning $\{1, 2, \cdots, |U_1|\}$ to vertices $u \in U_1$, then $\{|U_1| + 1, \cdots, |U_1| + |U_2|\}$ to vertices $u \in U_2$, and so on. It can be verified that $\tau(u) < \tau(v)$ for all $(u, v) \in D$, and the topological order is contiguous within every SCC.

For every $(u, v) \in E$, we set $w(u, v) = |\tau(u) - \tau(v)|$. Now we verify that such w is a good weight function.

- Since w is defined with some topological function, by the same analysis for DAGs, we have $\sum_{e \in E} \frac{1}{w(e)} \leq n^{1+o(1)}$.
- Suppose f is an approximate maximum flow, and P is a flow path in f.

– Edges $P \cap D$, i.e., the DAG edges. Note that $\tau(v) - \tau(u) > 0$ for every $(u, v) \in D$. We have $\sum_{e \in P \cap D} w(e) \leq 2n = O(n)$ since the sum of decrease in τ within SCCs can be at most n.
– Edges $P \cap (E \setminus D)$, i.e., the expander edges. For every $U_i \in \mathcal{U}$, note that $w(u, v) \leq |U_i|$ for all $u, v \in U_i$. By Lemma 4, an $\Omega(1)$-approx max flow uses at most $\tilde{O}(1/\phi)$ edges. As a result, the sum of weights in such flow path is no more than $\tilde{O}(|U_i|/\phi)$, and the sum of weights within all expanders is $\tilde{O}(n/\phi)$.

Therefore, we constructed a good weight function for graphs that are DAG of expanders.