COS 445 - PSet 2

Due online Tuesday, February 25th at 11:59 pm

Instructions:

- Some problems will be marked as *no collaboration* problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs to the correct locations![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- The cheatsheet gives problem solving tips, and tips for a "good proof" or "partial progress."
- Please reference the course collaboration policy here.

For convenience, we restate some definitions used in this problem set.

[1]We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Two Candidates, Two (Other) Rules (20 points, no collaboration)

For this problem, there are n voters and $m = 2$ candidates, and $n \geq 3$ is odd. Therefore, for a voting rule to have property X, it only needs to have property X when $m = 2$ and $n \geq 3$ is odd. For a voting rule to not have property X, there must exist a counterexample with $m = 2$ and $n \geq 3$ odd. Recall also the following definitions:

Definition 1 (Unanimous)**.** *F is* unanimous *if whenever everyone puts x as their first choice, F selects x.*

Definition 2 (Anonymous)**.** *Let σ be any permutation from $[n]$ to $[n]$. If F is anonymous, then $F(\succ_1, \dots, \succ_n) = F(\succ_{\sigma(1)}, \dots, \succ_{\sigma(n)})$.*

Definition 3 (Neutral)**.** *Let σ be any permutation from $[m]$ to $[m]$. If F is neutral, then $F(\sigma(\succ_1), \dots, \sigma(\succ_n)) = \sigma(F(\succ_1, \dots, \succ_n))$. Here, we have abused notation and let $\sigma(\succ_i)$ denote the preference $\succ$ where $\sigma(a) \succ \sigma(b)$ if and only if $a \succ_i b$.*

Part a (10 points)

Design a voting rule which **is** unanimous, **is not** anonymous, and **is** neutral (and briefly prove that it **is** unanimous, **is not** anonymous, and **is** neutral.).

Part b (10 points)

Design a voting rule which **is not** unanimous, **is** anonymous, and **is** neutral (and briefly prove that it **is not** unanimous, **is** anonymous, and **is** neutral).

Problem 2: Range Voting (40 points)

For this problem, like in lecture, there are $n \geq 2$ voters and $m \geq 2$ candidates.[2] Each voter i has a strict preference ordering $\succ_i$ over candidates.

Definition 4 (Range Voting)**.** *Ask each voter i to give a score $s_i(j) \in [0,1]$ to each candidate j.*[3] *The total score of candidate j is $\sum_{i=1}^n s_i(j)$. The candidate with the highest score wins (tie-breaking lexicographically).*

Below, we will use the notation $\vec{s}(j)$ to denote the list of all voters' scores for candidate j, $\vec{s}_{-i}(j)$ to denote the list of the scores for candidate j among all voters except *i and $\vec{s}_{-i}$ to denote the list of the scores for all candidates among all voters* except *i.*

Finally, Parts c and d will discuss 'best response' and 'dominated strategies': terms that typically are associated with payoff functions. For this problem, you do not need to know the precise payoff that Voter i gets when candidate a is selected, aside from the fact that Voter i has strictly higher payoff for candidates it prefers (i.e. if $a \succ_i b$, Voter i has strictly higher payoff if a is selected than b).

Hint: Any time you see a problem with new definitions, it's a good idea to write out some examples to confirm you understand them!

Part a: Unanimity (5 points)

Prove that Range Voting is Unanimous in the following sense: if every voter gives a a strictly higher score than all other candidates, then candidate a will be selected.

Part b: Dictatorship (5 points)

Prove that Range Voting is not a Dictatorship in the following sense: there does not exist a Voter i such that whenever Voter i gives a candidate a a strictly higher score than all other candidates, candidate a will be selected.

Part c: Very Weakly Strategyproof? (10 points)

Prove or Disprove the following claim:

Range Voting is Very Weakly Strategyproof in the following sense: for all $\vec{s}_{-i}$, if Player i knows *exactly* how all voters $\neq i$ are voting (that is, Voter i knows $\vec{s}_{-i}$), then Voter i has a best response $(s_i(1), \ldots, s_i(m))$ to $\vec{s}_{-i}$ such that for all candidates a, b: $a \succ_i b \Rightarrow s_i(a) \geq s_i(b)$ (that is, for all $\vec{s}_{-i}$, Voter i has a best response to $\vec{s}_{-i}$ such that Voter i gives its more preferred candidates at least as high a score as its less preferred candidates).

[2] That is, any claim you prove must be correct for all $n \geq 2, m \geq 2$. Any claim you disprove must be false for at least one $n \geq 2, m \geq 2$.

[3] That is, the syntax for Range Voting differes from a voting rule in that it asks for input of a different form – it doesn't ask for preferences, it asks for a list of scores.

Part d: Weaky Strategyproof? (20 points)

Prove or Disprove the following claim:

Range Voting is Weakly Strategyproof in the following sense: For any scores $(s_i(1), \ldots, s_i(m))$ such that there exists an a, b such that $s_i(a) > s_i(b)$ but $b \succ_i a$, there exists another list of scores $(s'_i(1), \ldots, s'_i(m))$ such that $(s'_i(1), \ldots, s'_i(m))$ weakly dominates $(s_i(1), \ldots, s_i(m))$ as a strategy in Range Voting. That is, scoring candidates out of order from your preferences is a weakly dominated strategy.

Problem 3: Representative Preferences (60 points)

This problem will involve the following four definitions (the first was studied in lecture, the second three are new to this problem). Throughout the problem, there are m candidates and n voters. A 'voter preference' is a strict ordering over the m candidates (from favorite to least favorite).

Definition 5 (Single-Peaked). *A set $V \neq \emptyset$ of voter preferences is* single-peaked *if there exists an ordering of the candidates $c_1 \ldots, c_m$ such that for all $\succ \in V$, the following holds: Let c_i denote the favorite candidate of $\succ$. Then for all $j < k \leq i$, $c_k \succ c_j$. Also, for all $j > k \geq i$, $c_k \succ c_j$.*

Definition 6 (Pairwise-Issue-Aligned). *A set $V \neq \emptyset$ of n voter preferences is said to be* pairwise-issue-aligned *if there exists an ordering of the n* preferences in V *$\succ_1, \ldots, \succ_n$ such that for all candidates a, b, there exists a threshold index $i_{a,b}$ such that either:*

- *For all $i \leq i_{a,b}$, $a \succ_i b$ and for all $i > i_{a,b}$, $b \succ_i a$, or*
- *For all $i \leq i_{a,b}$, $b \succ_i a$ and for all $i > i_{a,b}$, $a \succ_i b$.*

That is, V is pairwise-issue-aligned if there exists a way to order the preferences in V such that for all a, b, there exists a threshold index $i_{a,b}$ such that all preferences to the left of $i_{a,b}$ agree on a vs. b, and all preferences to the right of $i_{a,b}$ agree on a vs. b.[4]

Definition 7 (Representative). *Let W be a (multi)-set*[5] *of n voter preferences. We say that a preference ordering $\succ$ is* representative *of W if for all pairs of candidates a, b, $a \succ b$ if and only if a strict majority ($> n/2$) of voters in W prefer a to b. Note that a representative ordering does not necessarily exist.*

We say that $\succ$ is a strong representative *of W if $\succ$ is representative of W, and also $\succ \in W$.*

Definition 8 (Representable). *Let V be a set. We say that V is* (strongly) representable *if it holds that for all (multi-)sets W such that (i) all voters in W are also in V and (ii) $|W|$ is odd, W has a (strong) representative.*

Hint: Any time you see a new definition on a PSet, it is a good idea to work through examples to confirm you understand what the definitions mean!

Part a (10 points)

Let V be pairwise-issue-aligned. Prove that V is strongly representable.

Part b (25 points)

Provide an example of a set V that is strongly representable, but not pairwise-issue-aligned. Prove your example is correct.

Hint: An example exists with four candidates and $|V| = 4$. But you are allowed to use different parameters.

[4] Note the order of quantifiers: there must exist a *single* ordering of preferences in V that does not change for each a, b. But the index $i_{a,b}$ depends on a, b.

[5] A multi-set is just a set that is allowed to contain repetitions. This just means that two voters in W might have the same preference.

Part c (25 points)

Below is an incorrect proof that whenever V is single-peaked, V is strongly representable. Find the specific logical claim in the proof below that is incorrect. Quote (or paraphrase) the exact line below and prove that it's false (by counterexample). Your solution should actually identify a flaw in the logic below, and not simply declare that the first line is false.

> This solution will rely on the following lemma.
>
> **Lemma 9.** *Let V be single-peaked for ordering $c_1, \ldots, c_m$. For any i, consider deleting candidate c_i from all preference in V, to get set V' of n preferences over $m-1$ candidates. Then V' is still single-peaked for ordering $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m$.*
>
> *Proof.* Consider any $\succ \in V$, and let $\succ' \in V'$ denote the preference over $m-1$ candidates obtained by deleting c_i.
>
> Consider first the case that the favorite candidate of $\succ$ is c_j for $j \neq i$. Then c_j is still the favorite candidate of $\succ'$, because it is not deleted. Furthermore, observe that for all ℓ, k we have $c_j \succ' c_k \Leftrightarrow c_j \succ c_k$. In particular, this means that for all $k < \ell \leq j$, $c_\ell \succ' c_k$ (because $c_\ell \succ c_k$), and also that for all $k > \ell \geq j$, $c_\ell \succ' c_k$ (again because $c_\ell \succ c_k$). This means that $\succ'$ is single-peaked for the desired ordering.
>
> Consider now the case that the favorite candidate of $\succ$ is c_i. Observe that the favorite candidate $c_{i'}$ of $\succ'$ must be c_{i-1} or c_{i+1} (because c_{i-1} is preferred to all candidates to the left, and c_{i+1} is preferred to all candidates to the right). But now observe again that any two candidates to the left of the new peak *are also to the left of the old peak*, and any two candidates to the right of the new peak *are also to the right of the new peak*. Therefore, it again holds that for all $j < k \leq i'$, $c_k \succ' c_j$ (because $c_k \succ c_j$), and also that for all $j > k \geq i'$, $c_k \succ' c_j$ (because $c_k \succ c_j$). Therefore, $\succ'$ is again single-peaked for the desired ordering. $\square$
>
> Now, we get into the solution. We will build the strong representative $\succ$ recursively as follows. At all points, the candidates are sorted so that $c_1, \ldots, c_m$ is the ordering satisfying the single-peaked property. Initialize the remaining candidates to be the set of all candidates, and initialize $\succ$ to be empty. Sort the voters in W according to their favorite candidate, and refer to the Median voter as the voter who is $((n+1)/2)^{th}$ in the ordering.
>
> 1. Let a denote the Median voter's favorite remaining candidate.
> 2. Set the next-favorite candidate of $\succ$ to be a.
> 3. Remove a from the remaining candidates and go back to step 1 (unless there are no further remaining candidates, in which case terminate).
>
> This process is clearly well-defined, and produces a full ordering over the n candidates. We now wish to show that the resulting $\succ$ is strong representative of W. To see this, we prove the following lemma:
>
> **Lemma 10.** *Let candidate a be added as the next-favorite candidate of $\succ$ when set R of candidates remain. Then for all $b \in R$, a strict majority of voters in W prefer a to b.*

Proof. Consider any $b \in R$ which lies before a in the single-peaked ordering. Then the Median voter's favorite candidate in R is a, and all voters to the right of the Median must have a favorite candidate further right of a. By the single-peaked property, this means that all such voters prefer a to b (because a lies between b and their peak). Therefore, a strict majority of voters in W prefer a to b. Similarly, if b lies after a in the single-peaked ordering, all voters to the left of the Median have a favorite candidate further left of a. Again by the single-peaked property, this means that all such voters prefer a to b and again a strict majority of voters in V prefer a to b. $\square$

Corollary 11. $\succ$ *is strong representative of* W.

Proof. Consider any pair a, b of candidates such that $a \succ b$. This means that a was set as the next favorite candidate of $\succ$ while b was still a remaining candidate. By Lemma 10, this means that a strict majority of candidates prefer a to b (analogous reasoning holds if $b \succ a$). For any pair of candidates, either $a \succ b$ or $b \succ a$. In either case, a strict majority of candidates agree with $\succ$, and therefore $\succ$ is representative of W.

To see that $\succ$ is a strong representative of W, simply observe that $\succ$ is built by iteratively taking the next-favorite candidate of the Median voter, and therefore $\succ$ is exactly the Median voter of W (which is in W). $\square$