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6.4 Maximum Flow

- *introduction*
- *Ford–Fulkerson algorithm*
- *maxflow–mincut theorem*
- *analysis of running time*
- *applications*

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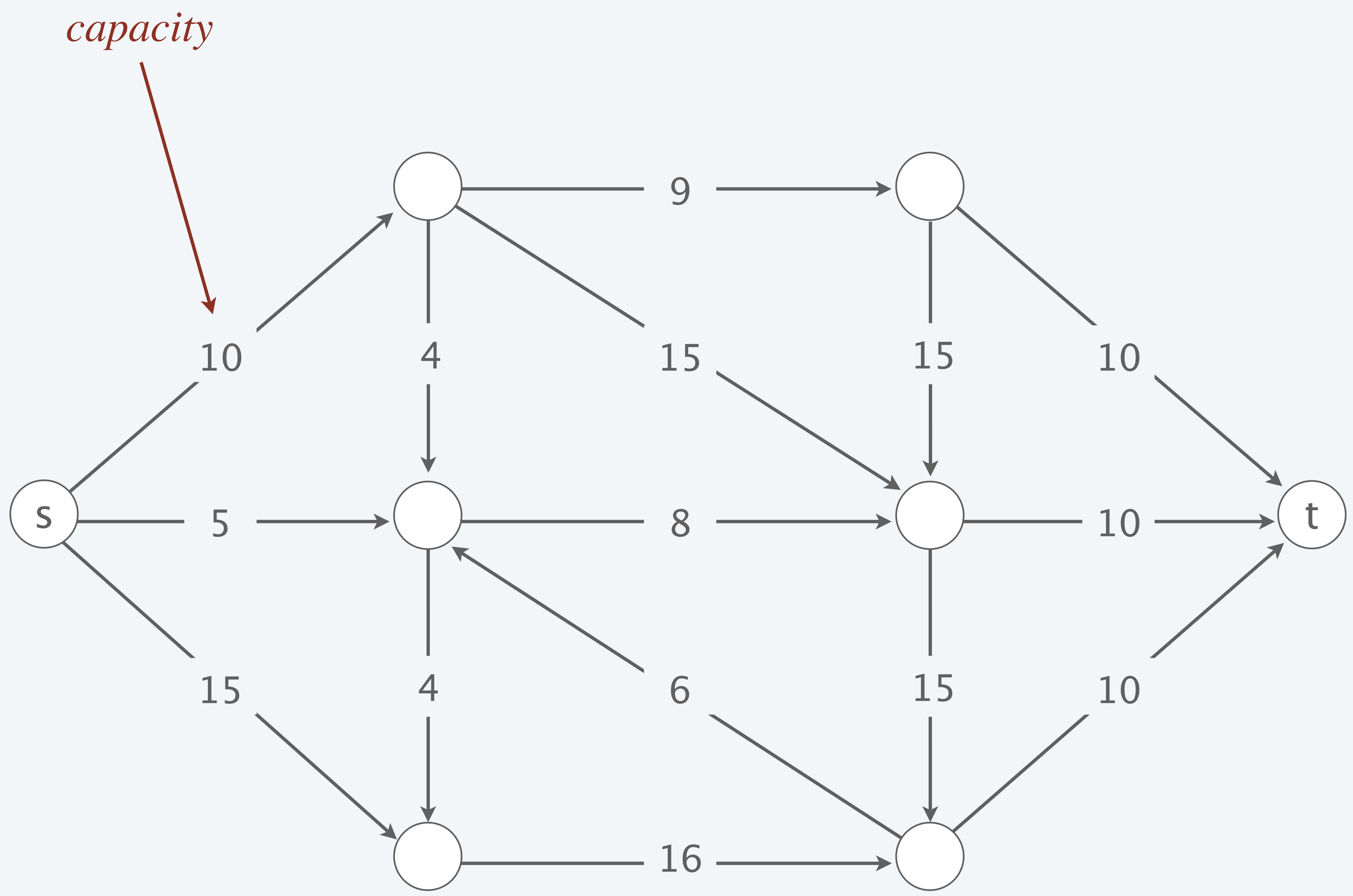
Mincut problem

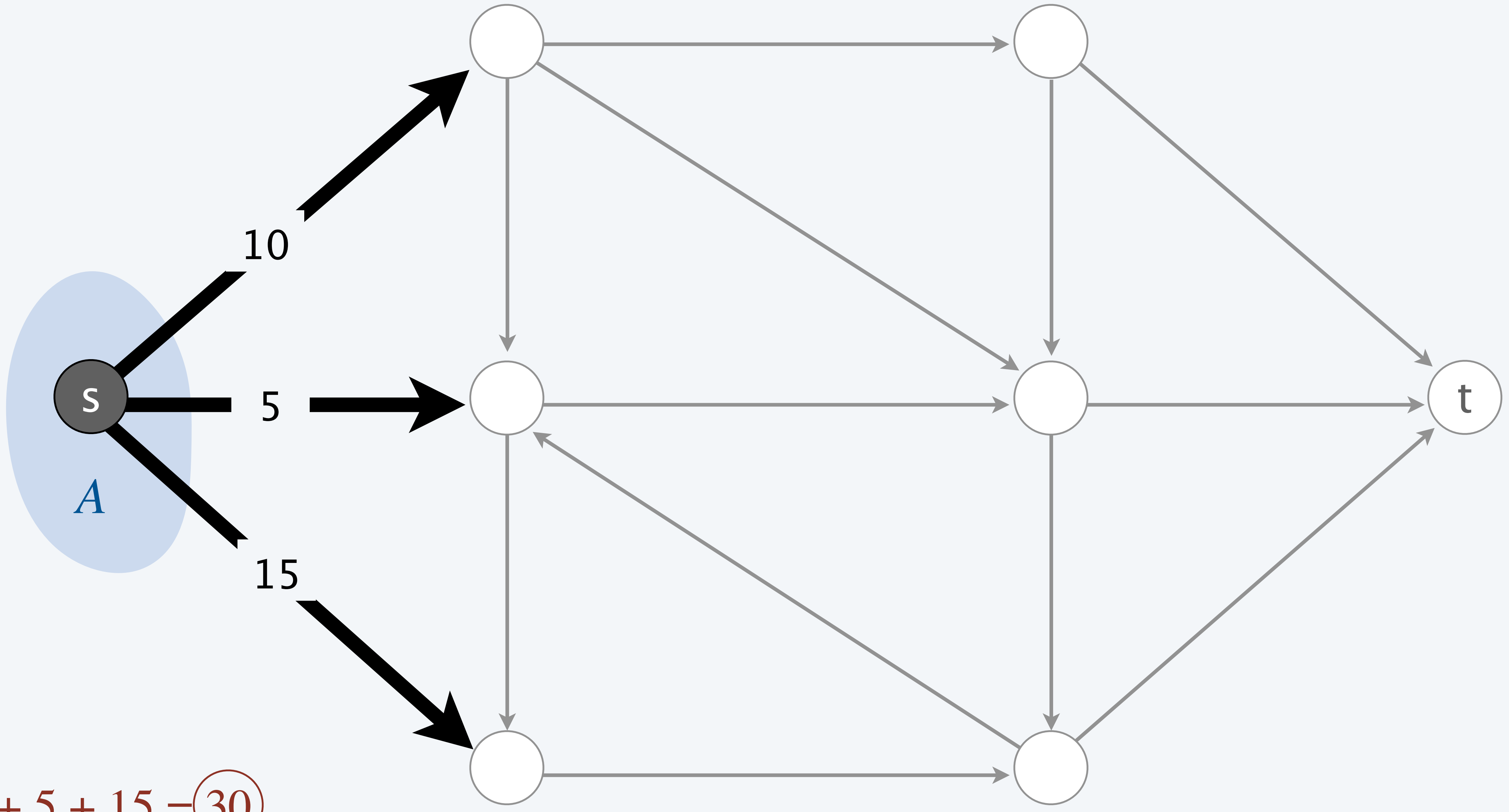
Input. A digraph with positive edge weights, source vertex s, and target vertex t.



Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

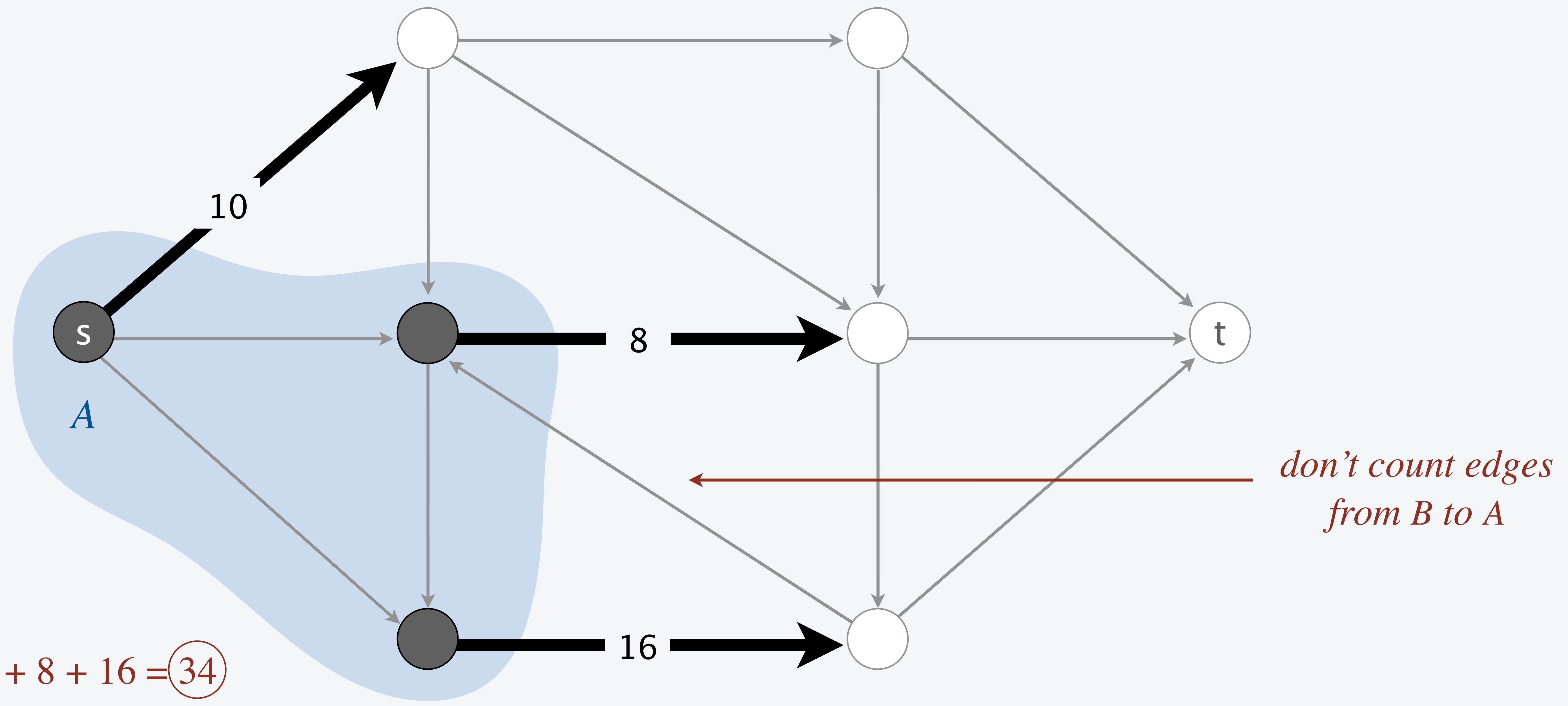


capacity = 10 + 5 + 15 = 30

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.



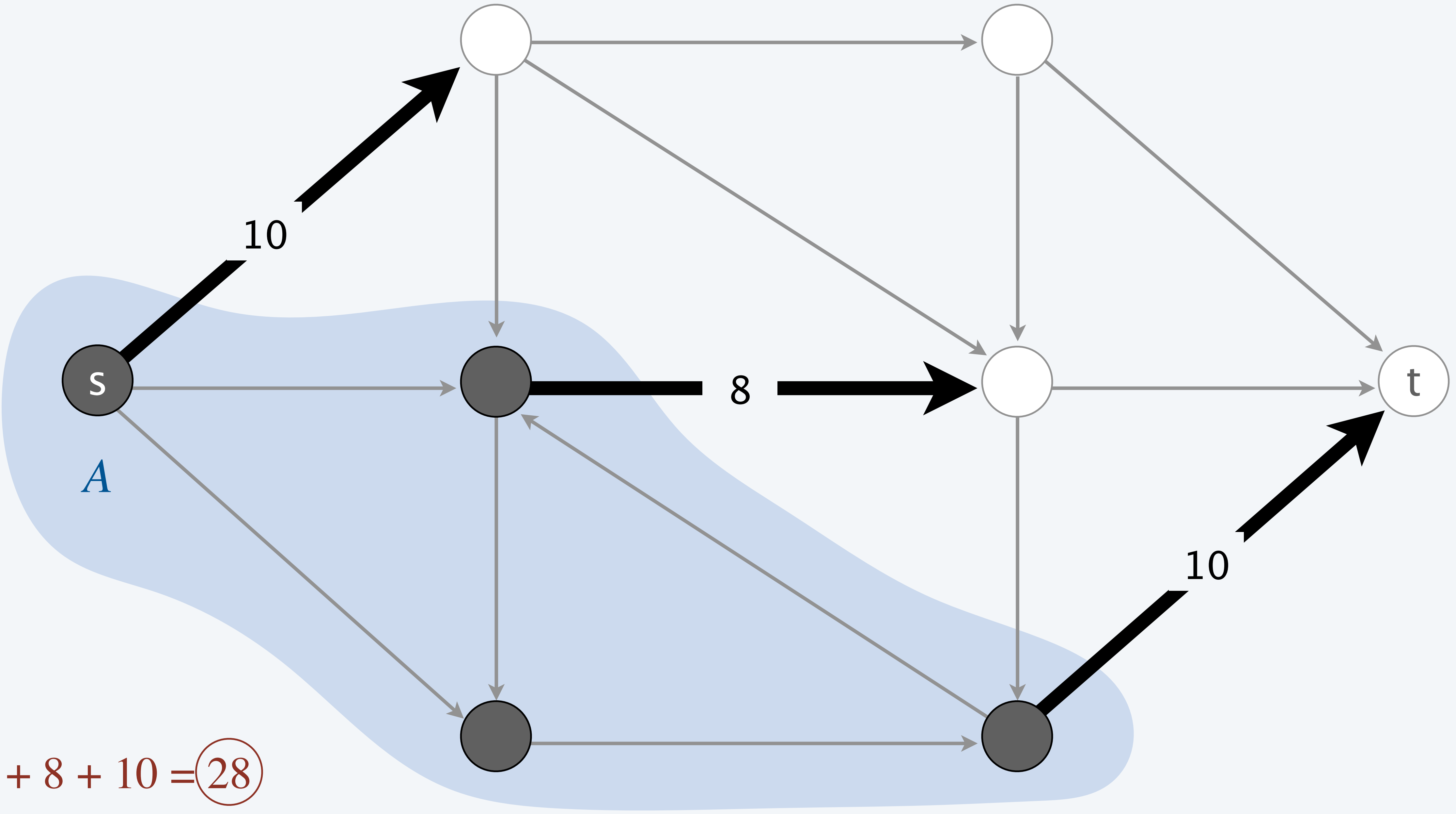
capacity = 10 + 8 + 16 = 34

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

Minimum st-cut (mincut) problem. Find an *st*-cut of minimum capacity.



capacity = 10 + 8 + 10 = 28

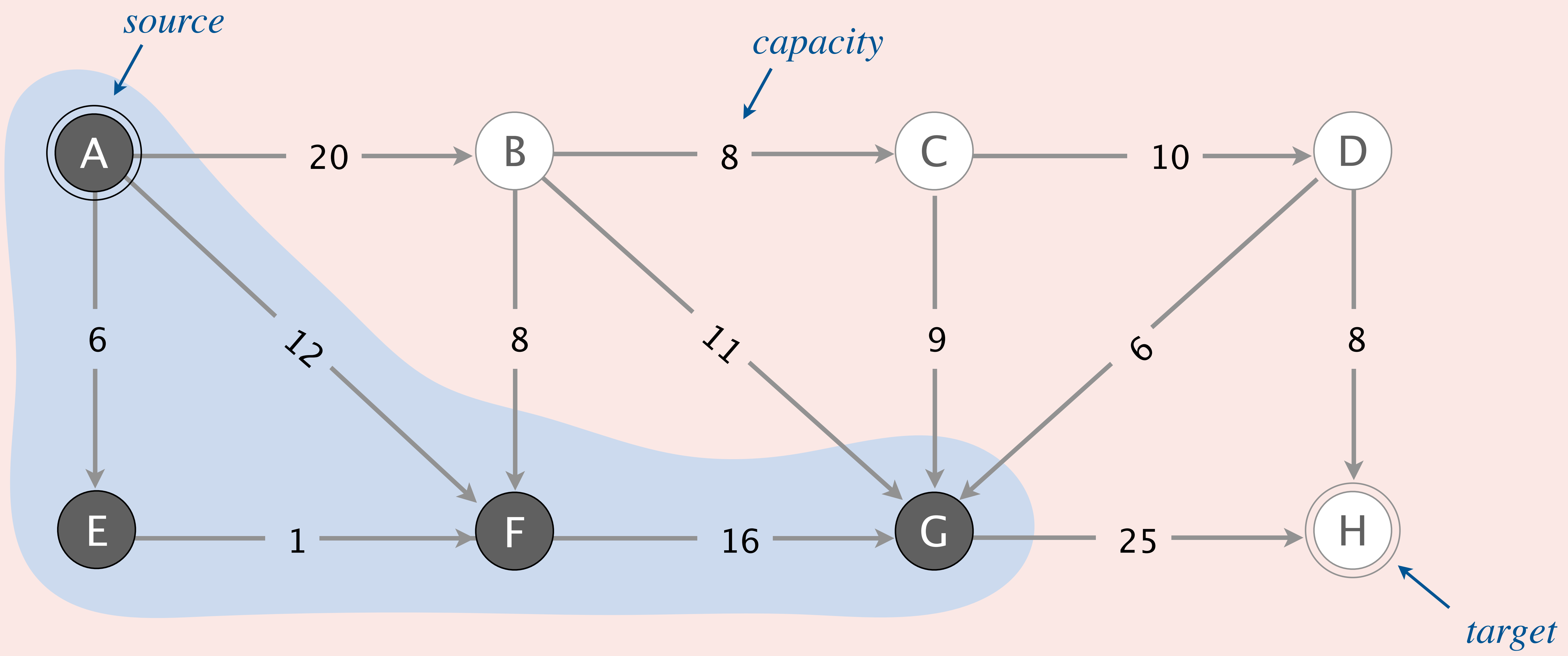
What is the capacity of the cut $\{A, E, F, G\}$?

A. 11 (20 + 25 − 8 − 11 − 9 − 6)

B. 34 (8 + 11 + 9 + 6)

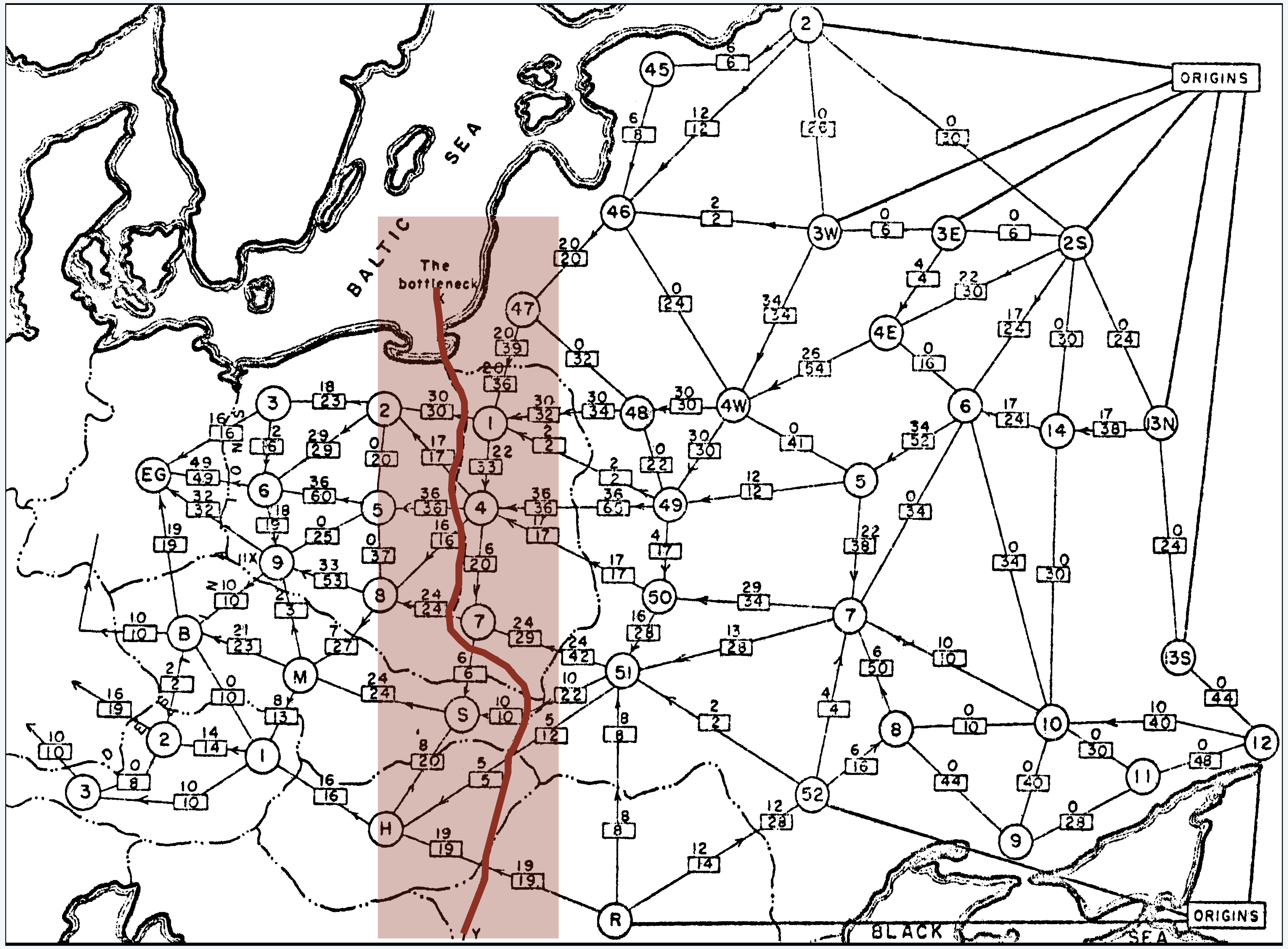
C. 45 (20 + 25)

D. 79 (20 + 25 + 8 + 11 + 9 + 6)



Mincut application (RAND 1950s)

“Free world” goal. Disrupt rail network (if Cold War turns into real war).



rail network connecting Soviet Union with Eastern European countries

(map declassified by Pentagon in 1999)

Maxflow problem

Though maximum flow algorithms have a long history, revolutionary progress is still being made.

BY ANDREW V. GOLDBERG AND ROBERT E. TARJAN

Efficient Maximum Flow Algorithms

gorithms in more detail. We restrict ourselves to basic maximum flow algorithms and do not cover interesting special cases (such as undirected graphs, planar graphs, and bipartite matchings) or generalizations (such as minimum-cost and multi-commodity flow problems).

Before formally defining the maximum flow and the minimum cut problems, we give a simple example of each problem: For the maximum flow example, suppose we have a graph that represents an oil pipeline network from an oil well to an oil depot. Each arc has a capacity, or maximum number of liters per second that can flow through the corresponding pipe. The goal is to find the maximum number of liters per second (maximum flow) that can be shipped from well to depot. For the minimum cut problem, we want to find the set of pipes of the smallest total capacity such that removing the pipes disconnects the oil well from the oil depot (minimum cut).

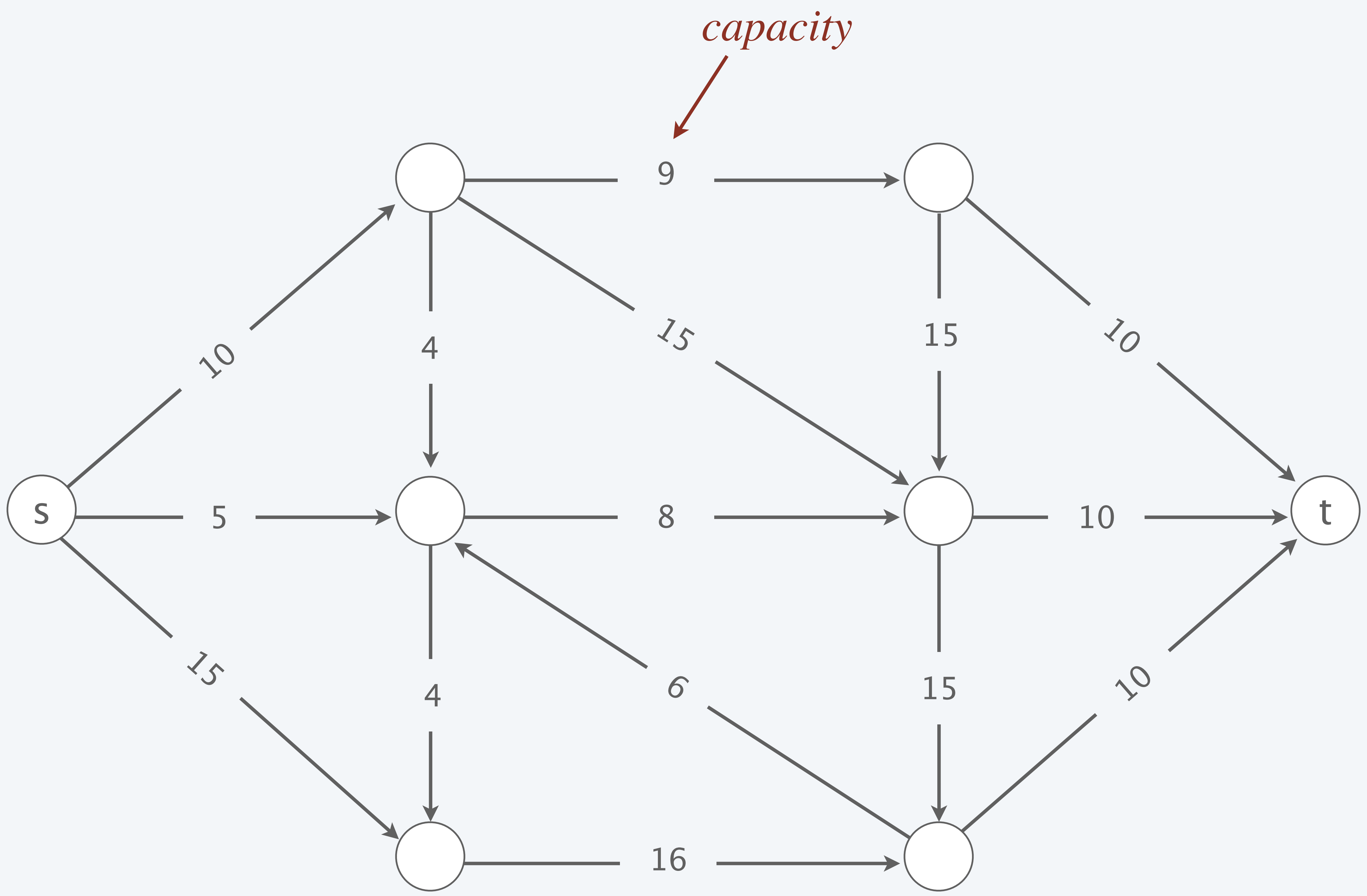
The maximum flow, minimum cut

Efficient Maximum Flow Algorithms by Andrew Goldberg and Bob Tarjan

https://vimeo.com/100774435

Maxflow problem

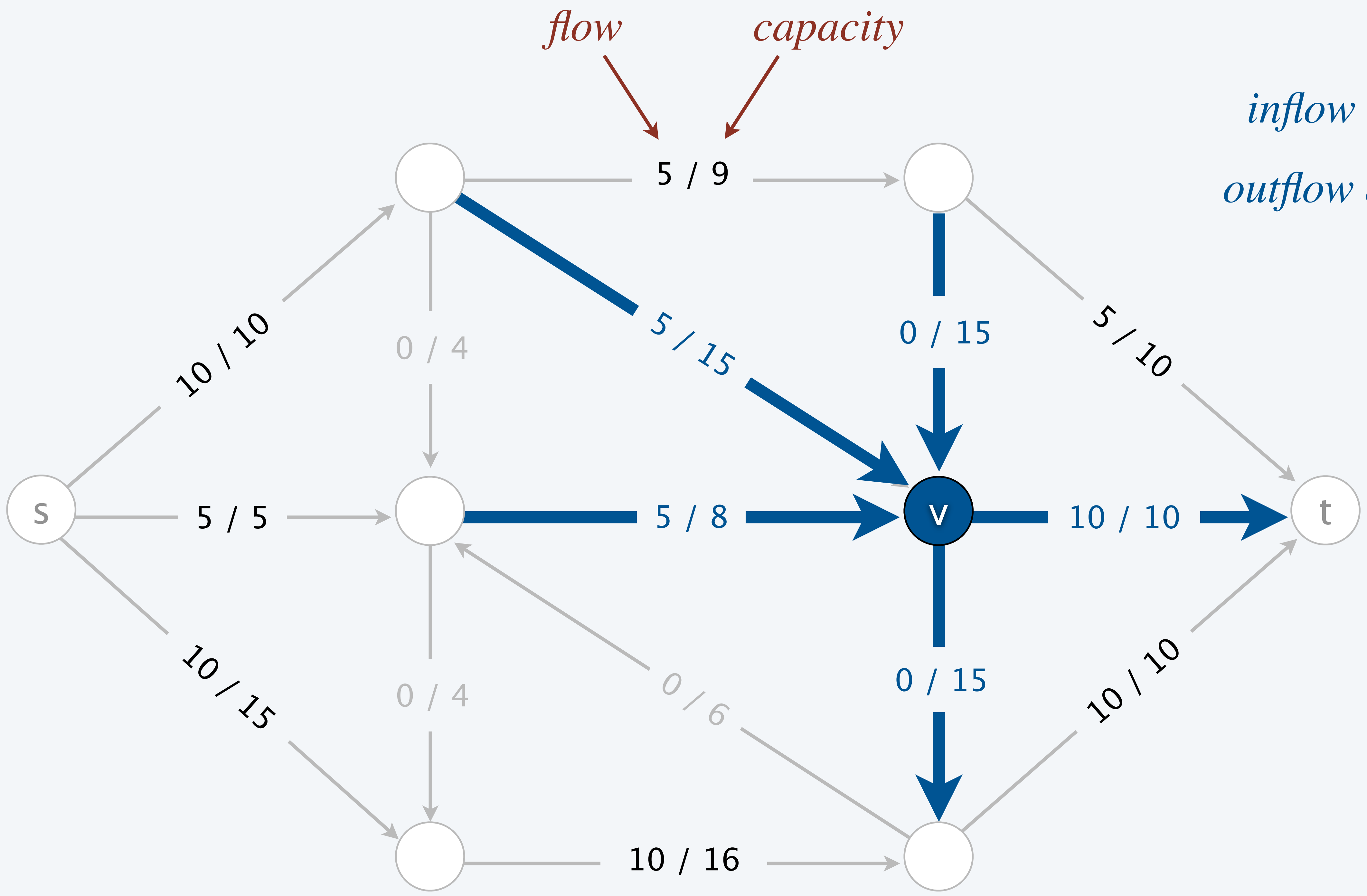
Input. A digraph with positive edge weights, source vertex s, and target vertex t.



Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:

- Capacity constraints: $0 \le$ edge's flow $\le$ edge's capacity.
- Flow conservation constraints: inflow = outflow at every vertex (except s and t).



inflow at v = 5 + 5 + 0 = 10

outflow at v = 10 + 0 = 10

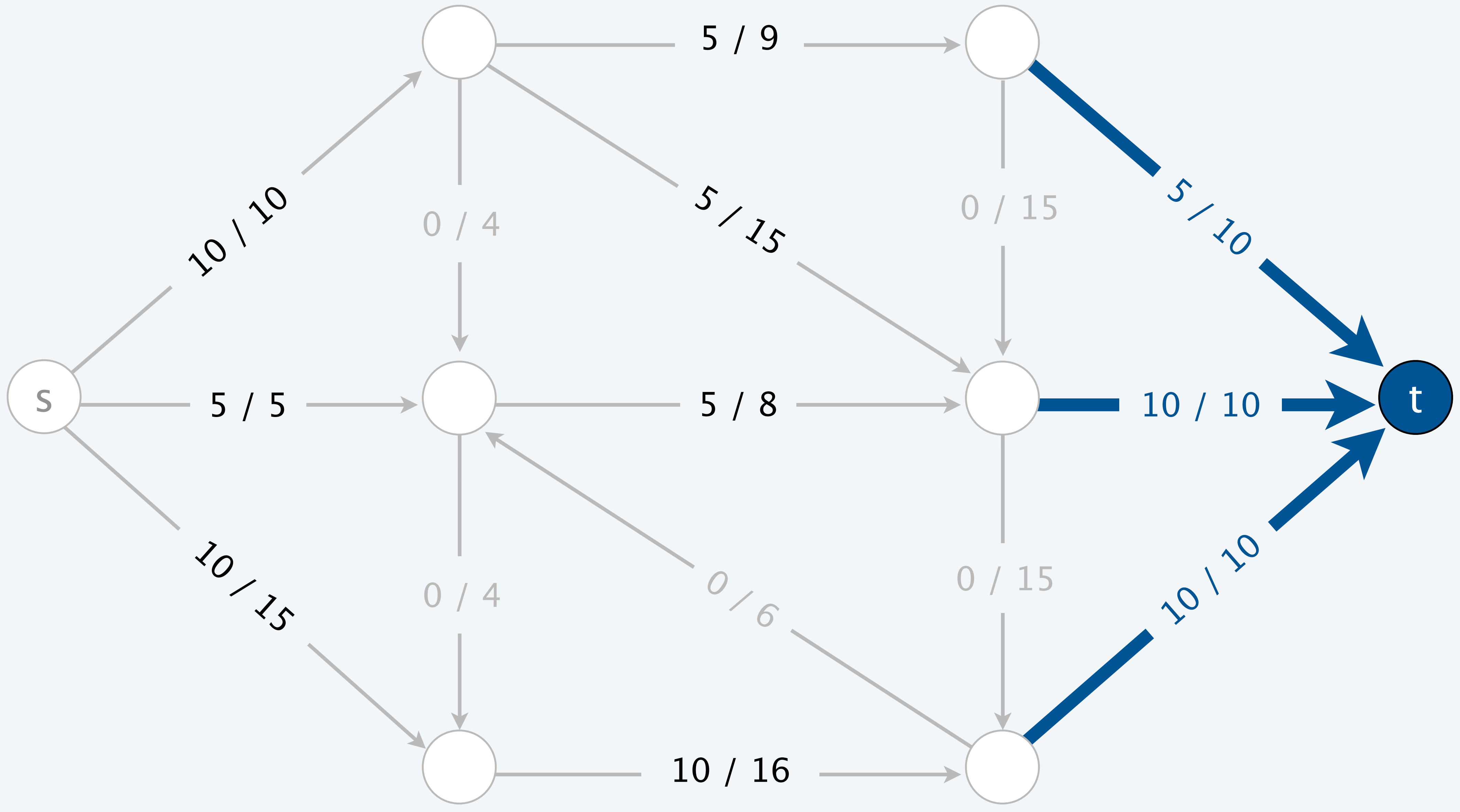
Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:

- Capacity constraints: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Flow conservation constraints: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

we assume no edges incident to s or from t



value $= 5 + 10 + 10 = 25$

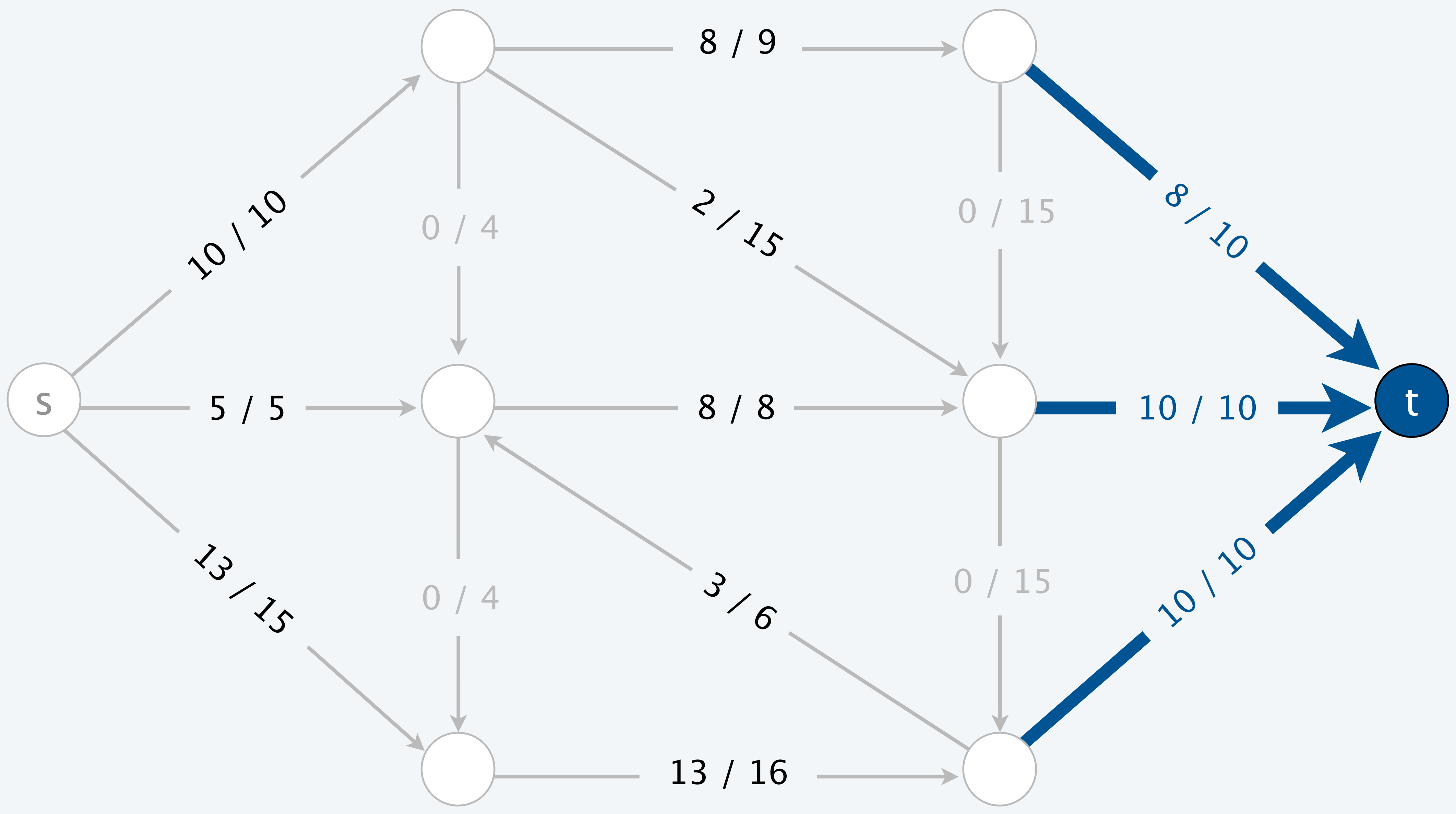
Maxflow problem

Def. An st-flow (flow) is an assignment of values to the edges such that:

- Capacity constraints: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Flow conservation constraints: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

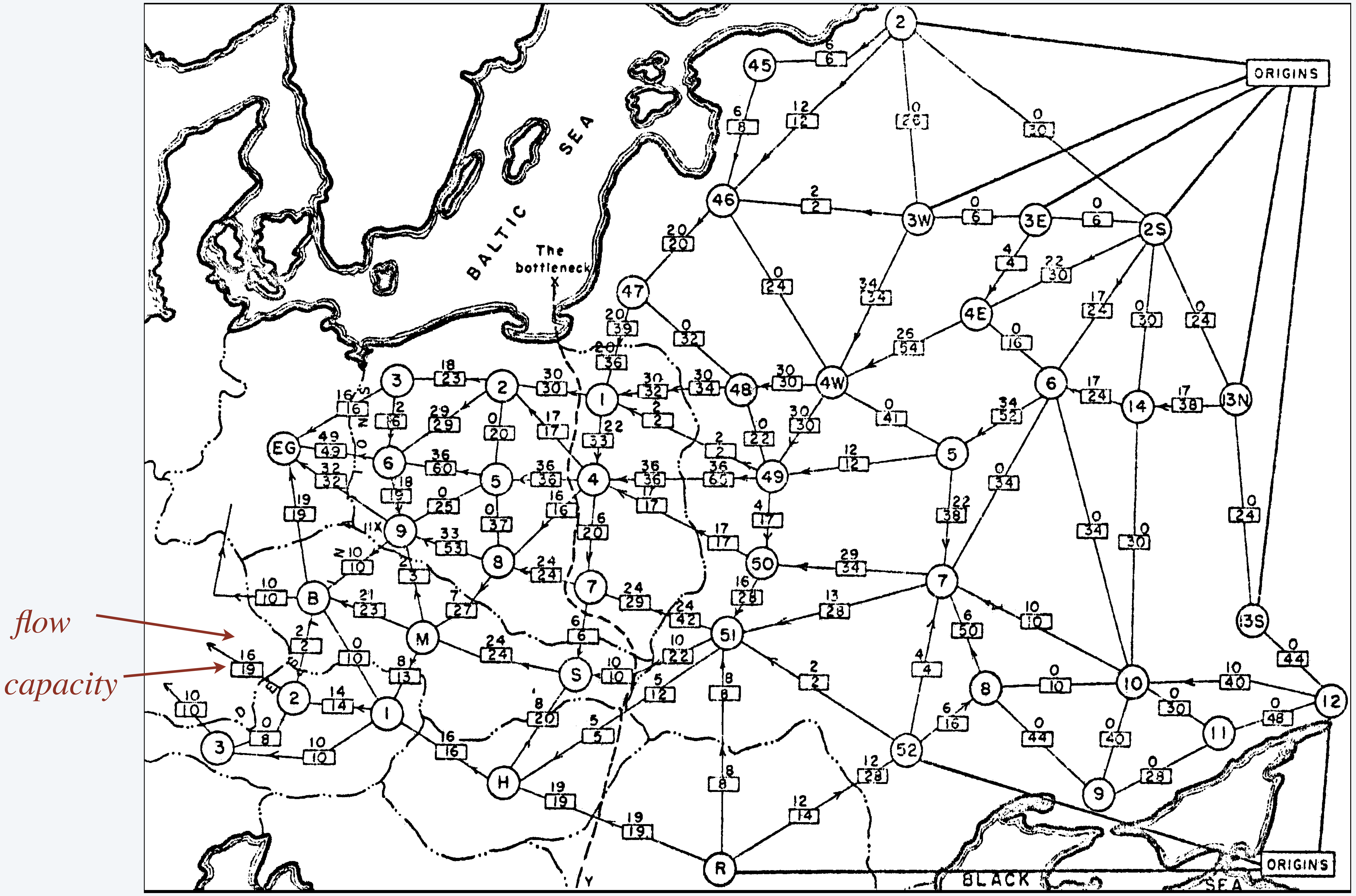
Maximum st-flow (maxflow) problem. Find a flow of maximum value.



$value = 8 + 10 + 10 = 28$

Maxflow application (Tolstoǐ 1930s)

Soviet Union goal. Maximize flow of supplies to Eastern Europe.



rail network connecting Soviet Union with Eastern European countries

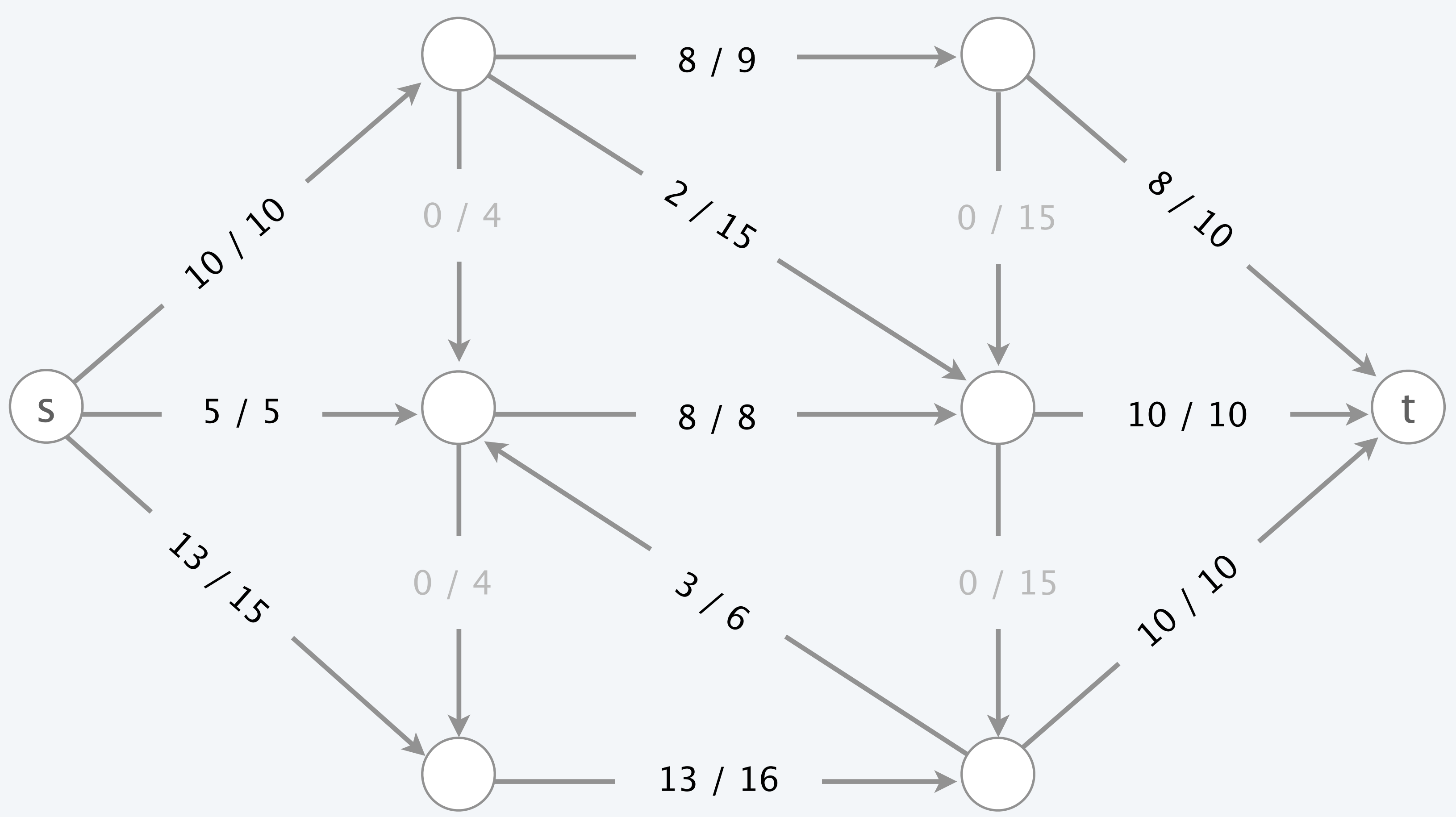
(map declassified by Pentagon in 1999)

Summary

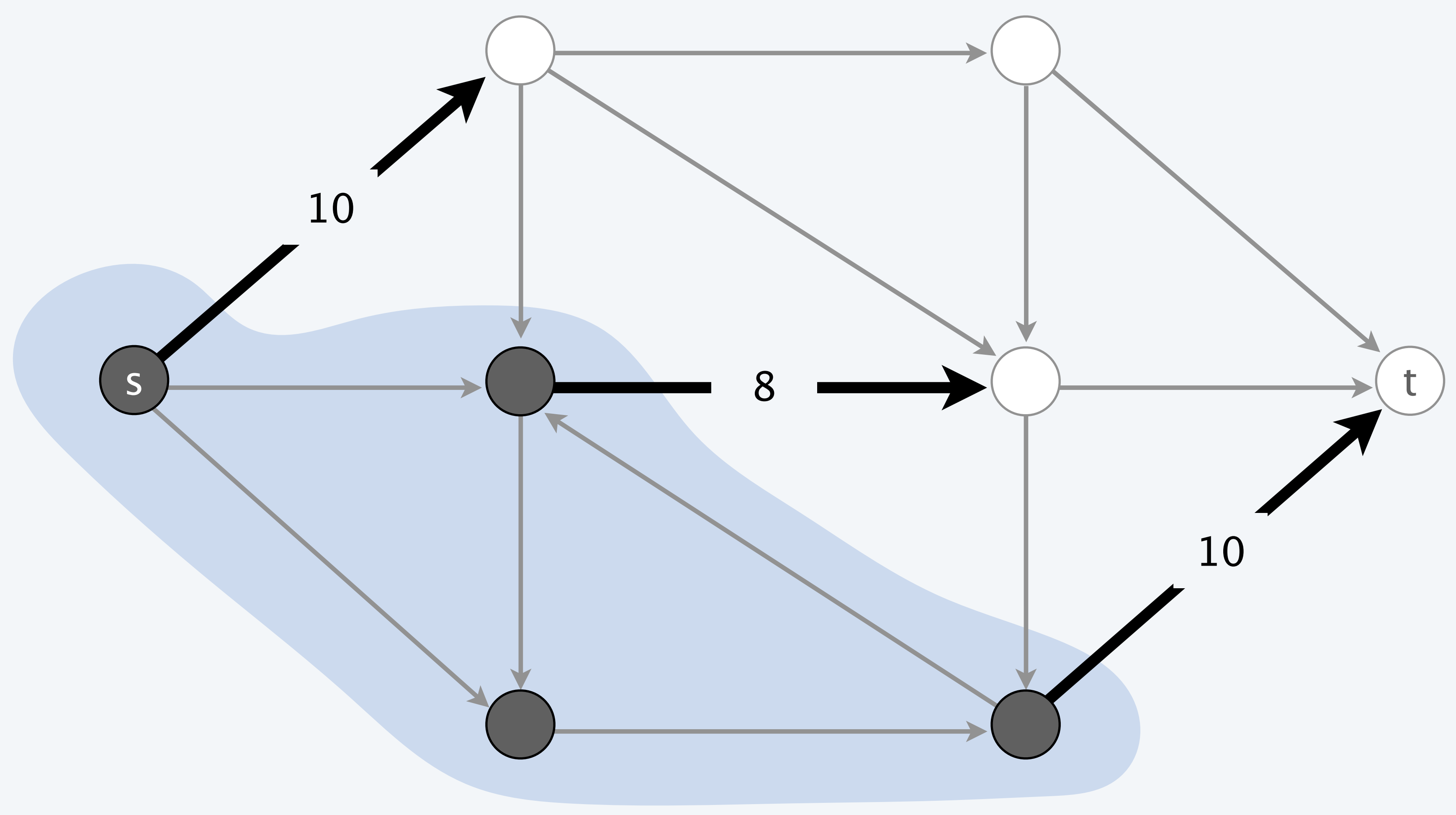
Input. A digraph with positive edge weights, source vertex s, and target vertex t.

Mincut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.



value of flow = 28



capacity of cut = 28

Remarkable fact. These two problems are dual! [stay tuned]

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6.4 Maximum Flow

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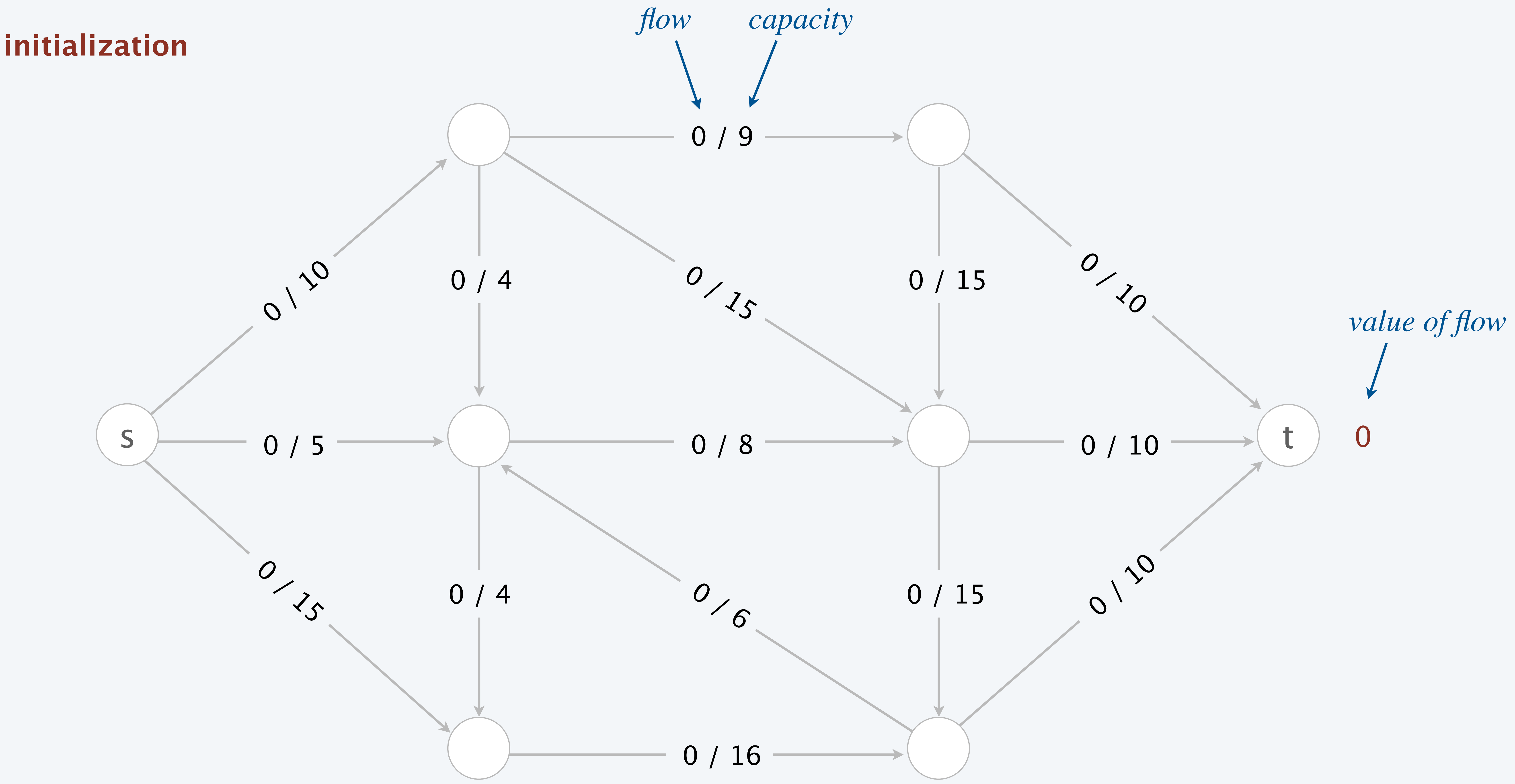
Ford–Fulkerson algorithm demo

Initialization. Start with 0 flow.



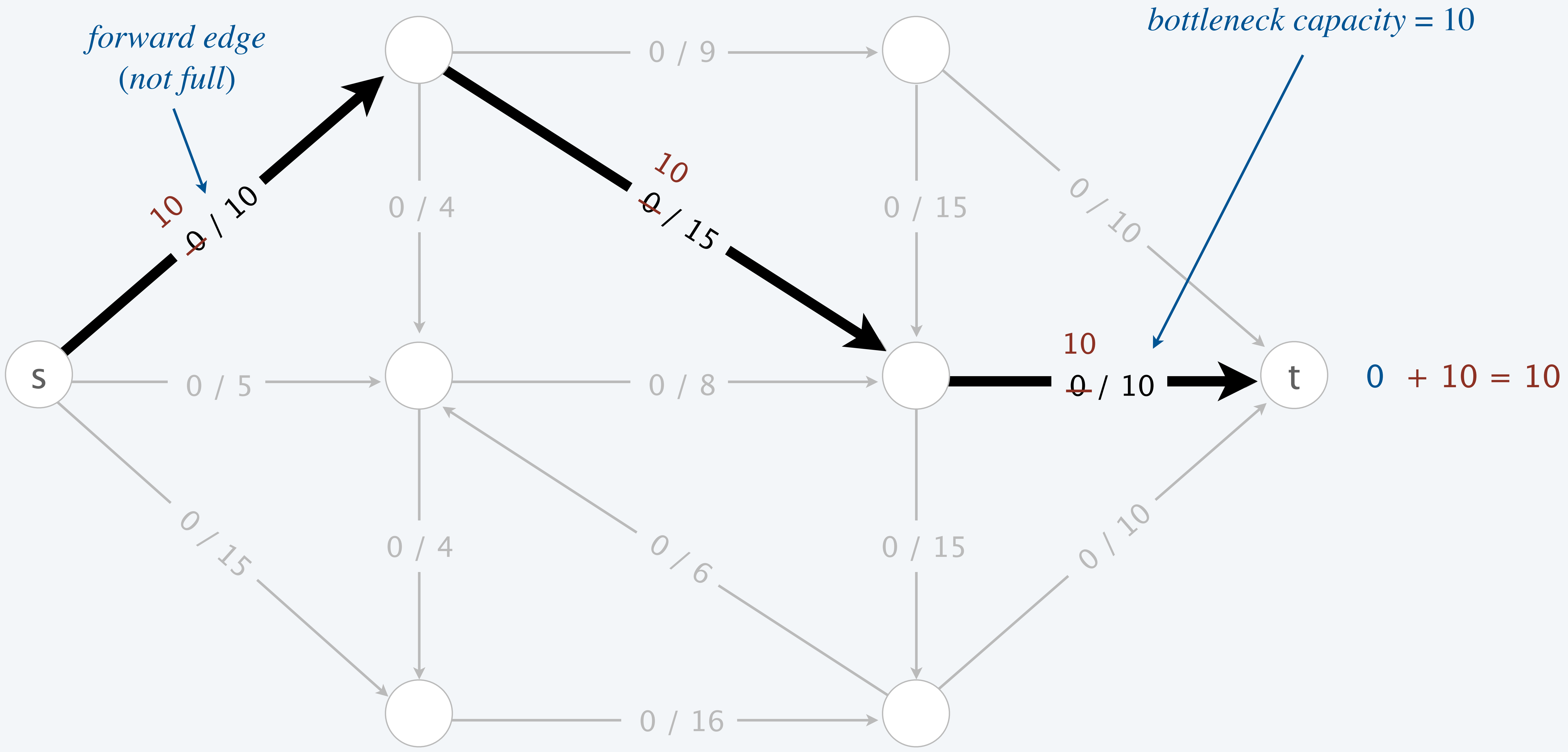
Ford–Fulkerson algorithm demo

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

impact: increases value of flow, while maintaining capacity and flow conservation constraints

1st augmenting path

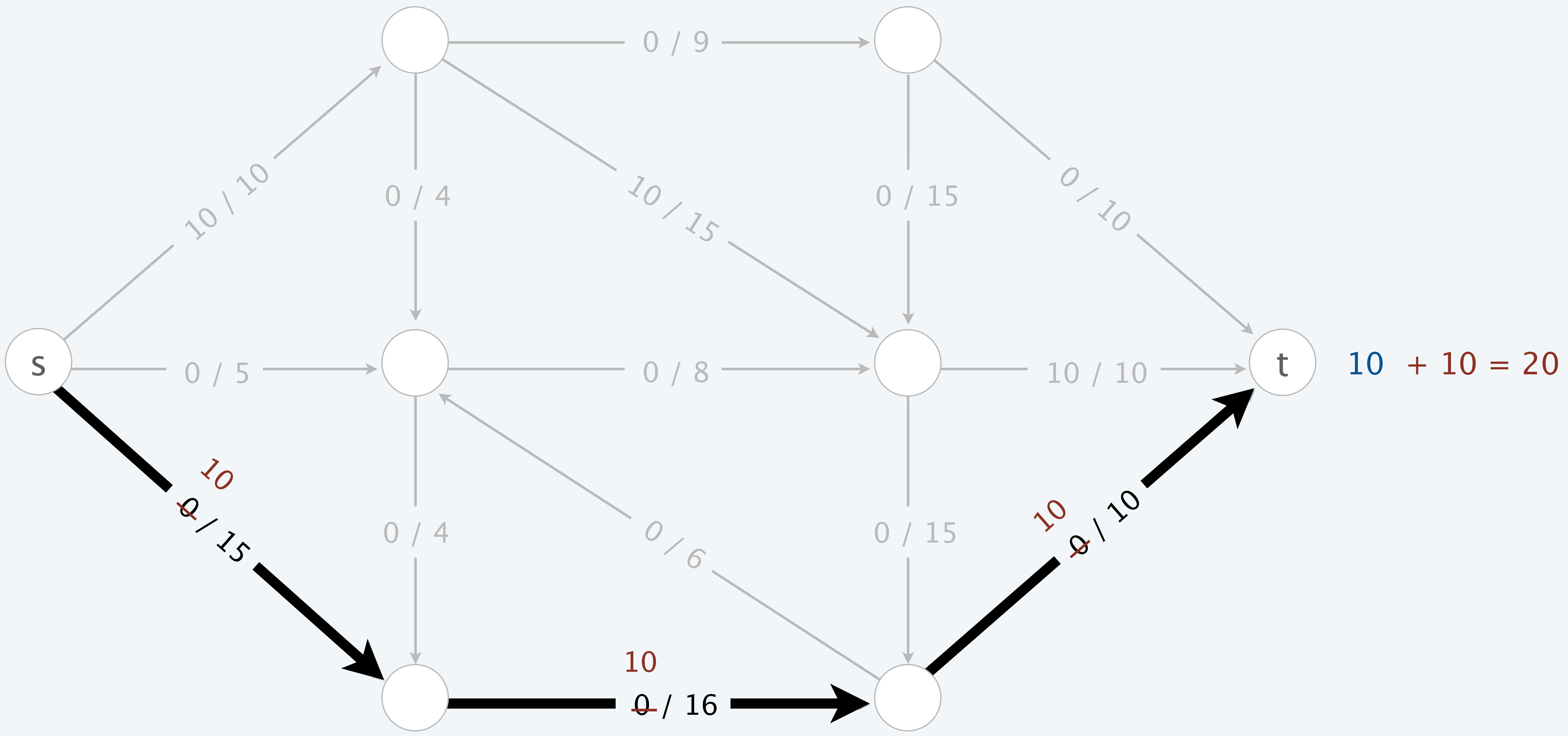


Ford–Fulkerson algorithm demo

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

2nd augmenting path



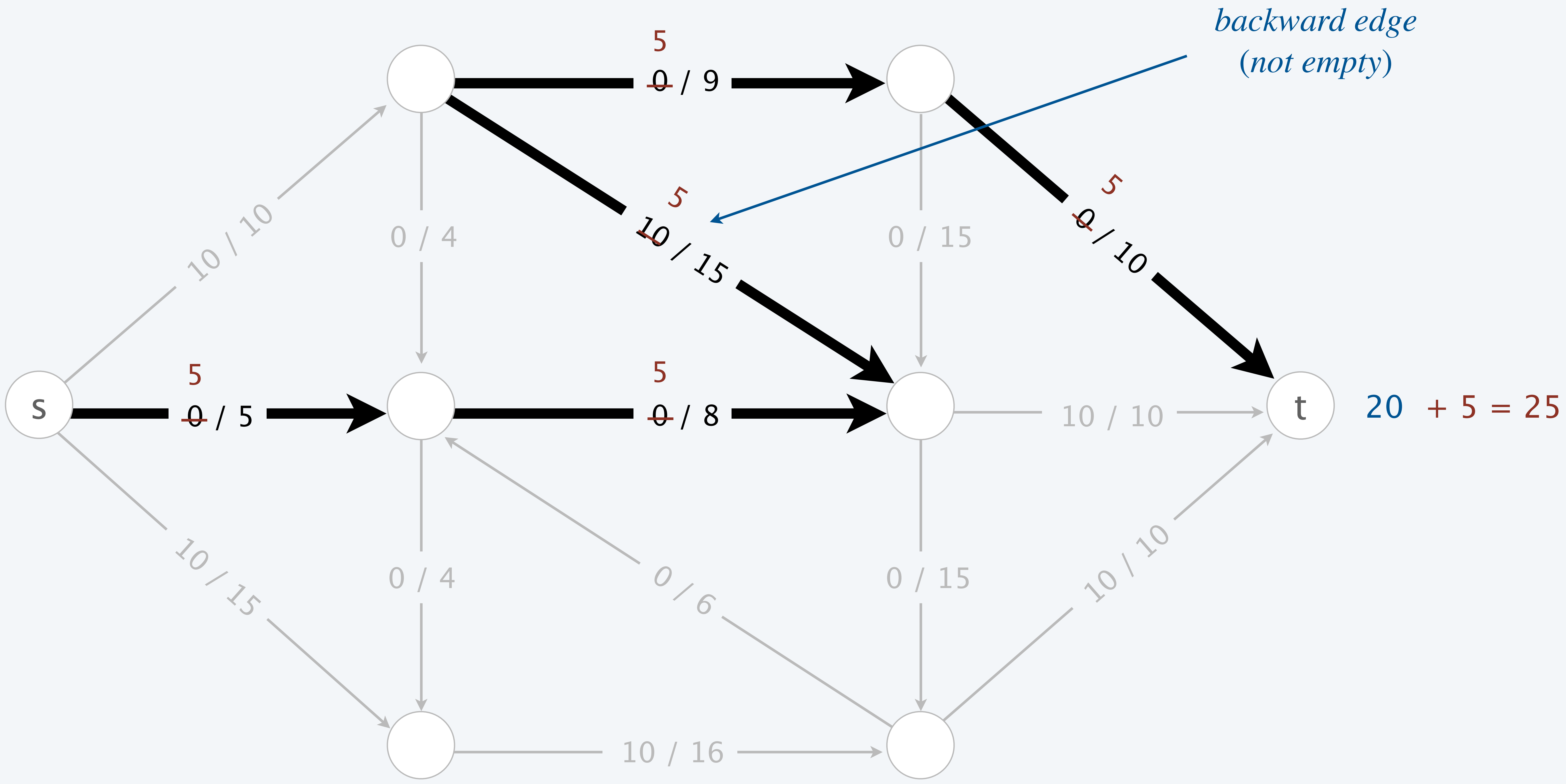
Ford–Fulkerson algorithm demo

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

impact: increases value of flow, while maintaining capacity and flow conservation constraints

3rd augmenting path

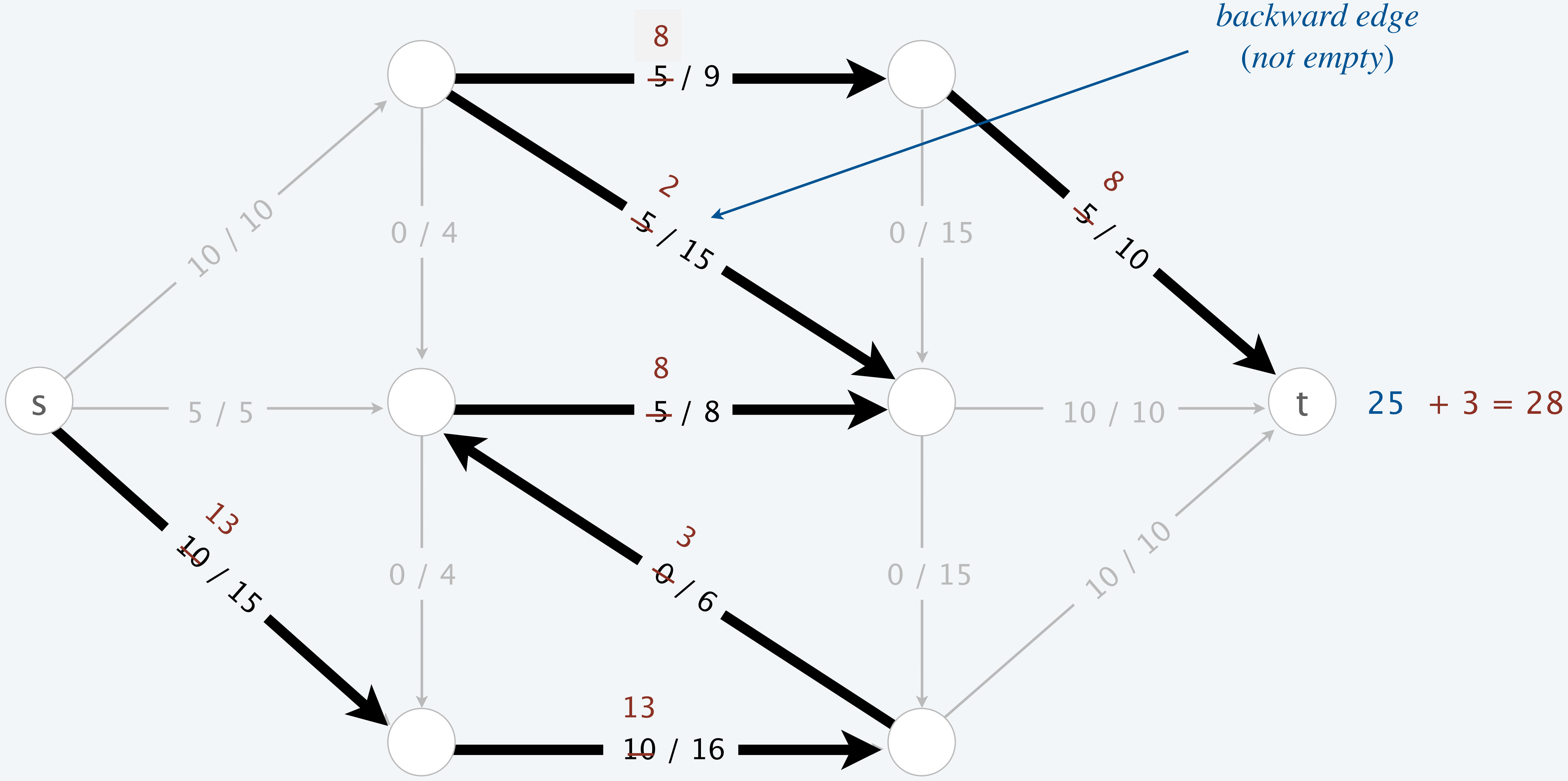


Ford–Fulkerson algorithm demo

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

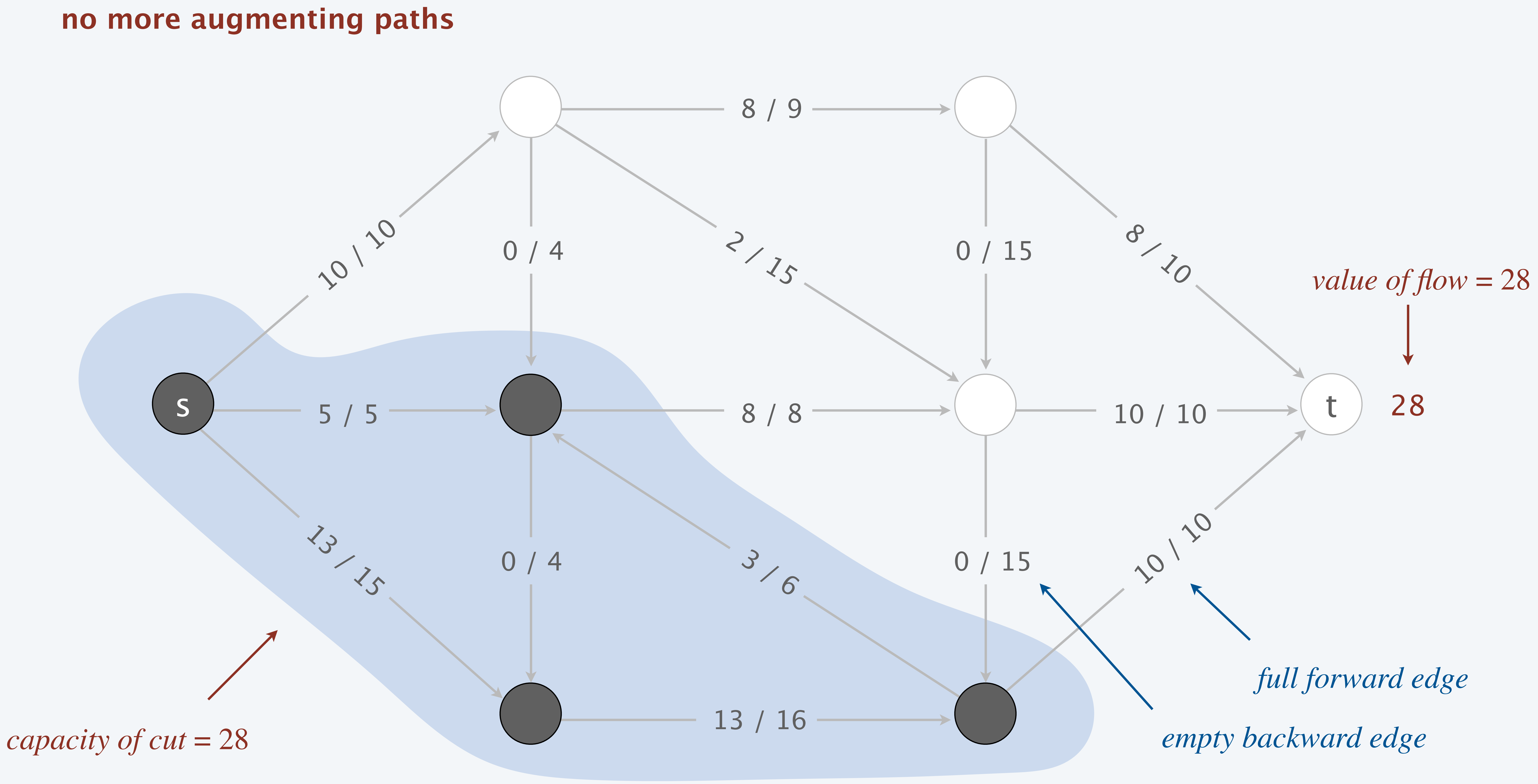
4th augmenting path



Ford–Fulkerson algorithm demo

Termination. All paths from s to t are blocked by either a

- Full forward edge.
- Empty backward edge.



Maxflow: quiz 2

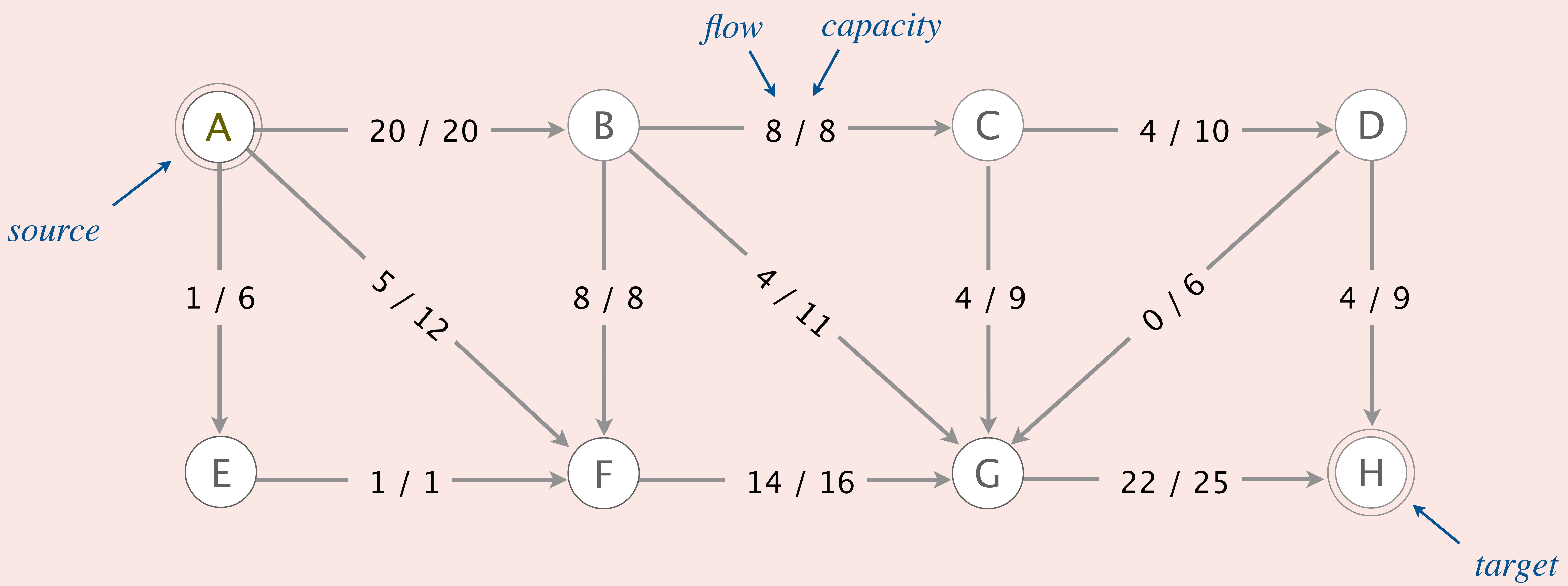
Which is an augmenting path with respect to the given flow?

A. $A \to F \to G \to D \to H$

B. $A \to F \to B \to G \to C \to D \to H$

C. Both A and B.

D. Neither A nor B.



Maxflow: quiz 3

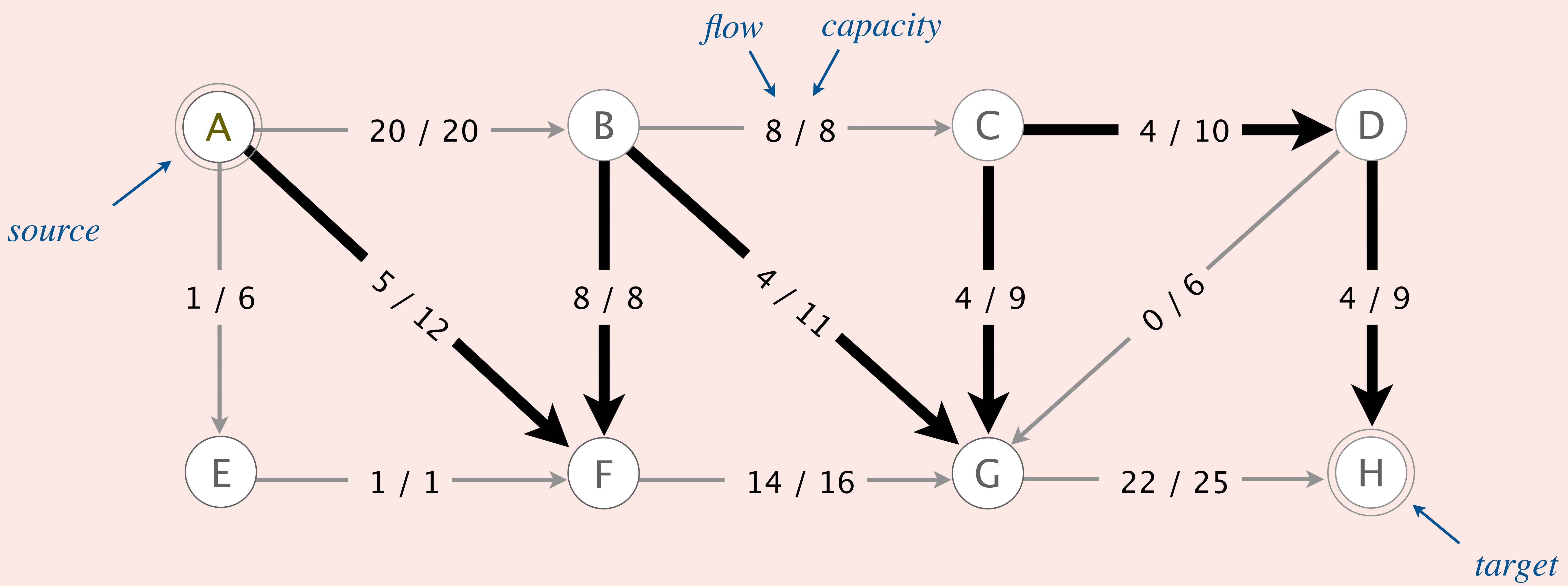
What is the bottleneck capacity of the augmenting path $A \to F \to B \to G \to C \to D \to H$ **?**

A. 4

B. 5

C. 6

D. 7



Ford–Fulkerson algorithm

Ford–Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path P**
- **compute bottleneck capacity of P**
- **update flow on P by bottleneck capacity**

Fundamental questions.

- How to find an augmenting path?
- How many augmenting paths?
- Guaranteed to compute a maxflow?
- Given a maxflow, how to compute a mincut?

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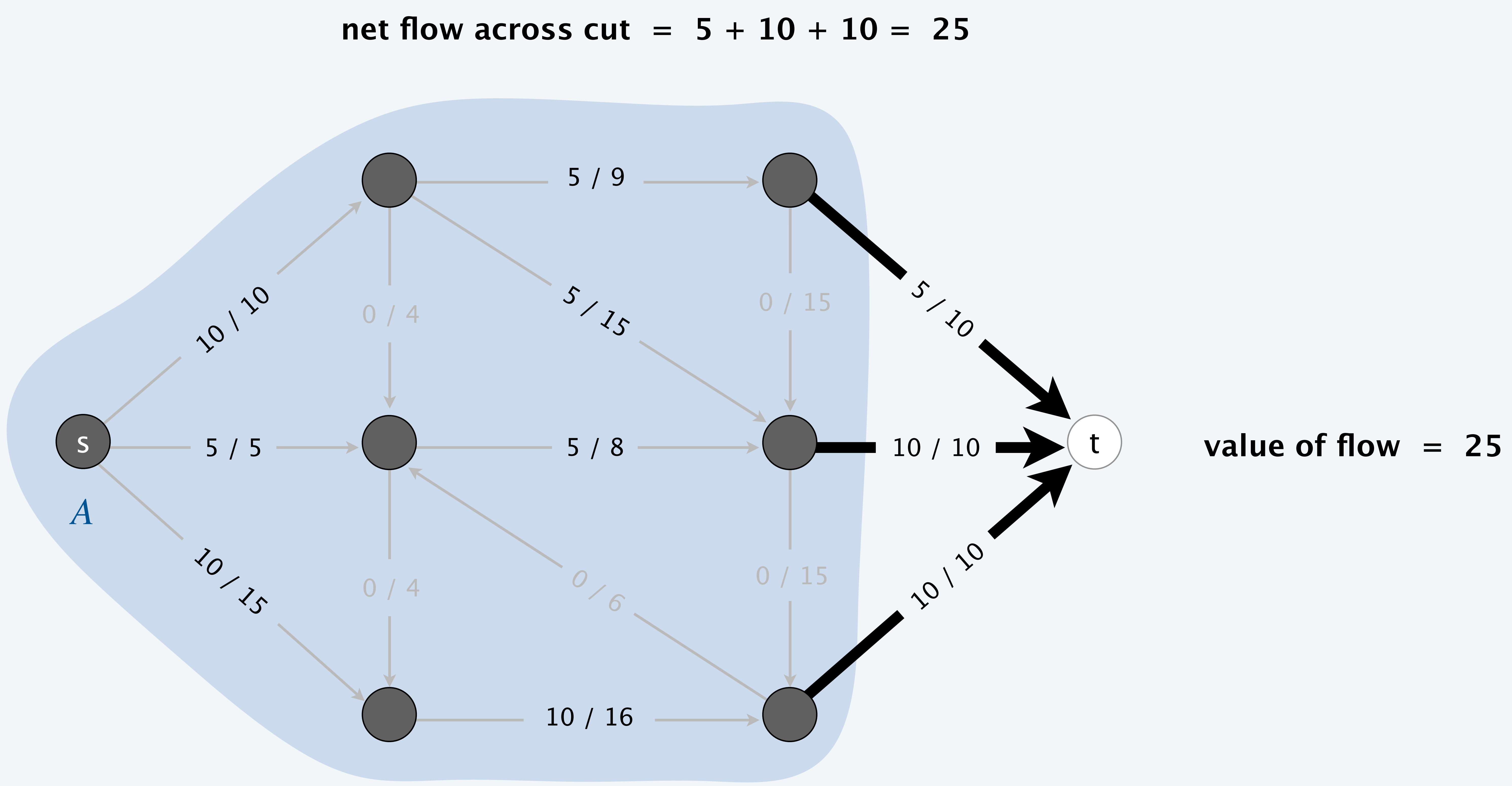
6.4 Maximum Flow

- introduction
- Ford–Fulkerson algorithm
- **maxflow–mincut theorem**
- analysis of running time
- applications

Relationship between flows and cuts

Def. Given a flow f, the net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.



Relationship between flows and cuts

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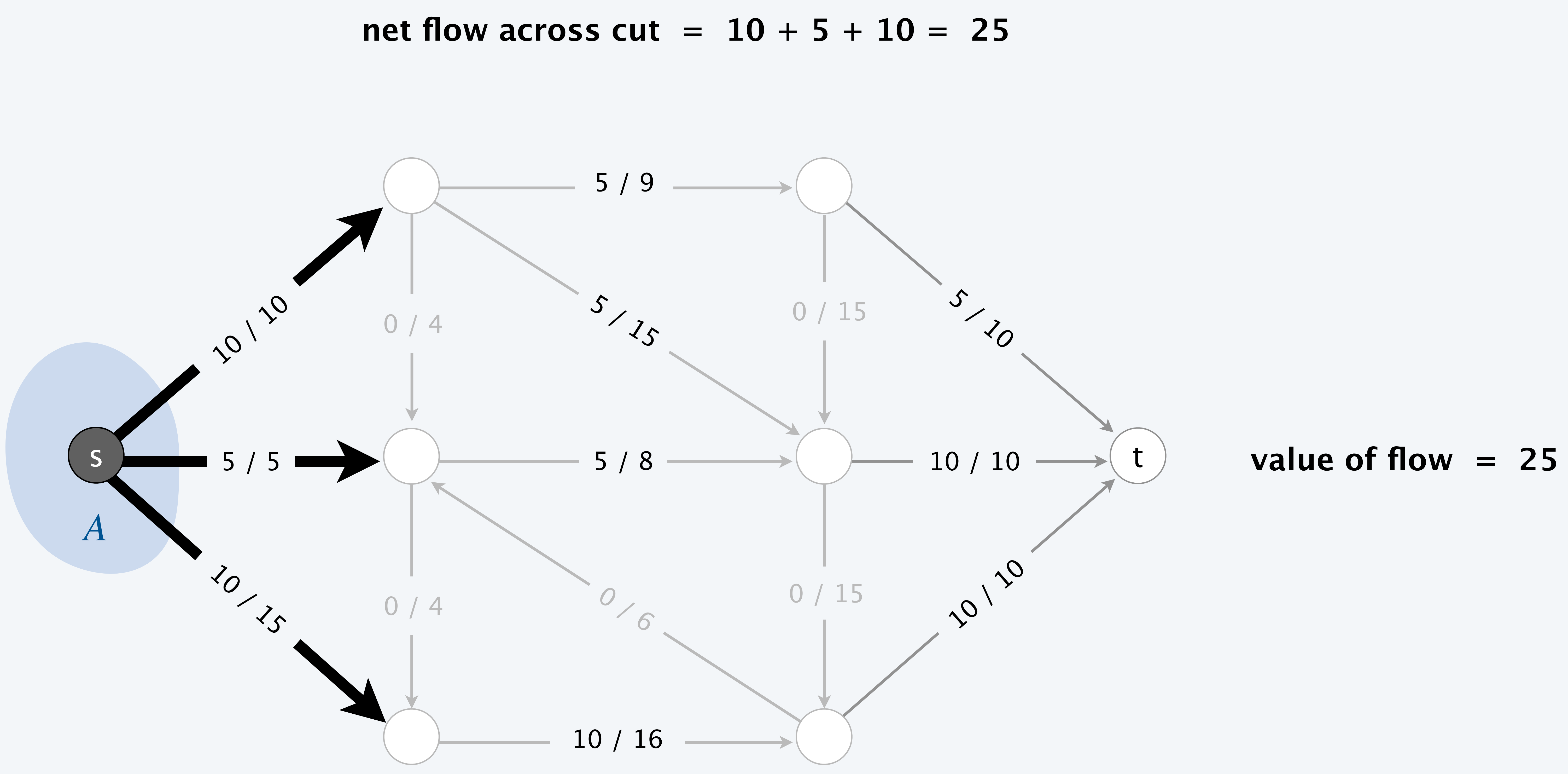
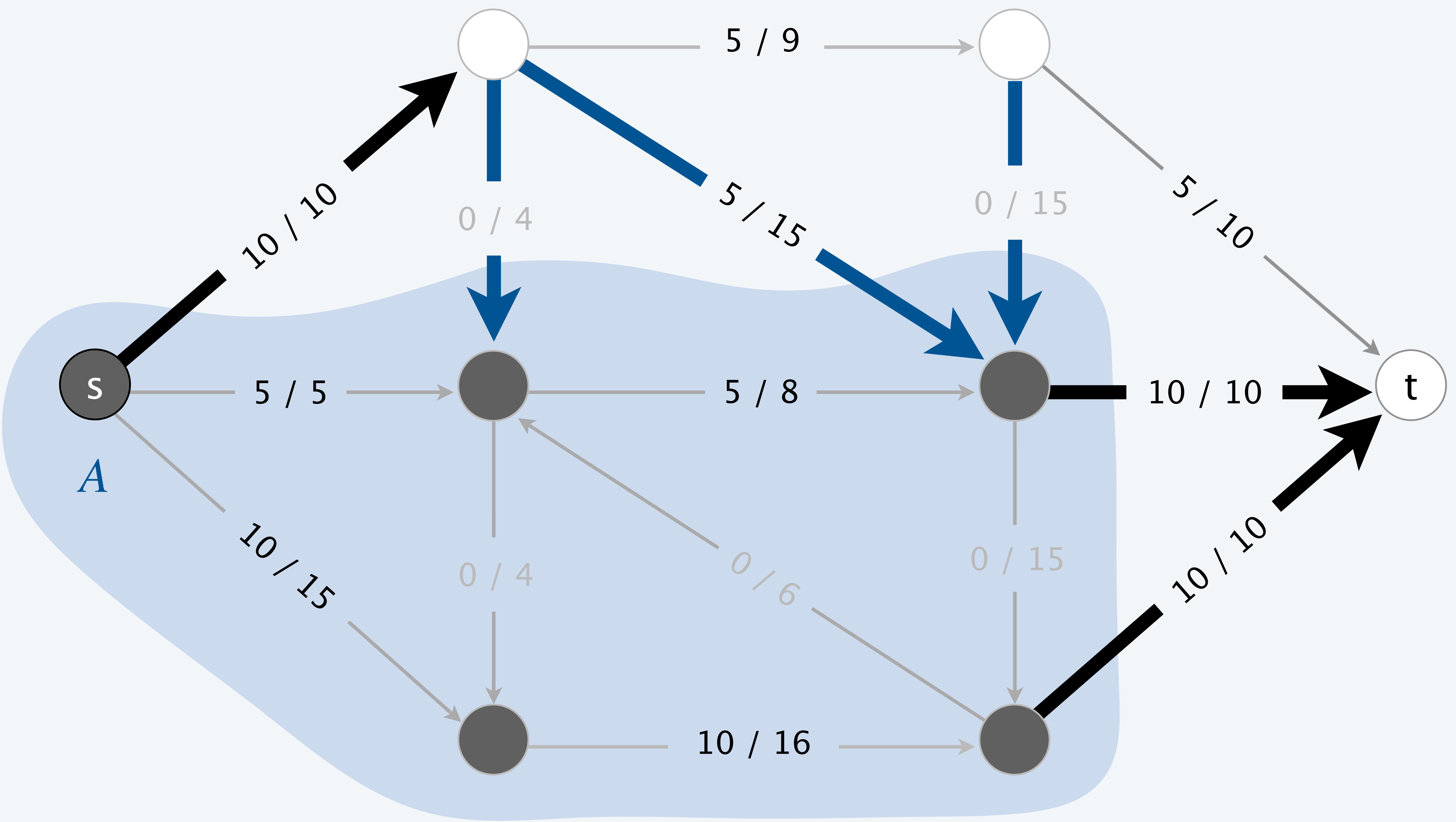


Relationship between flows and cuts

Def. Given a flow f, the net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

net flow across cut = (10 + 10 + 10) − (0 + 5 + 0) = 25



value of flow = 25

Maxflow: quiz 4

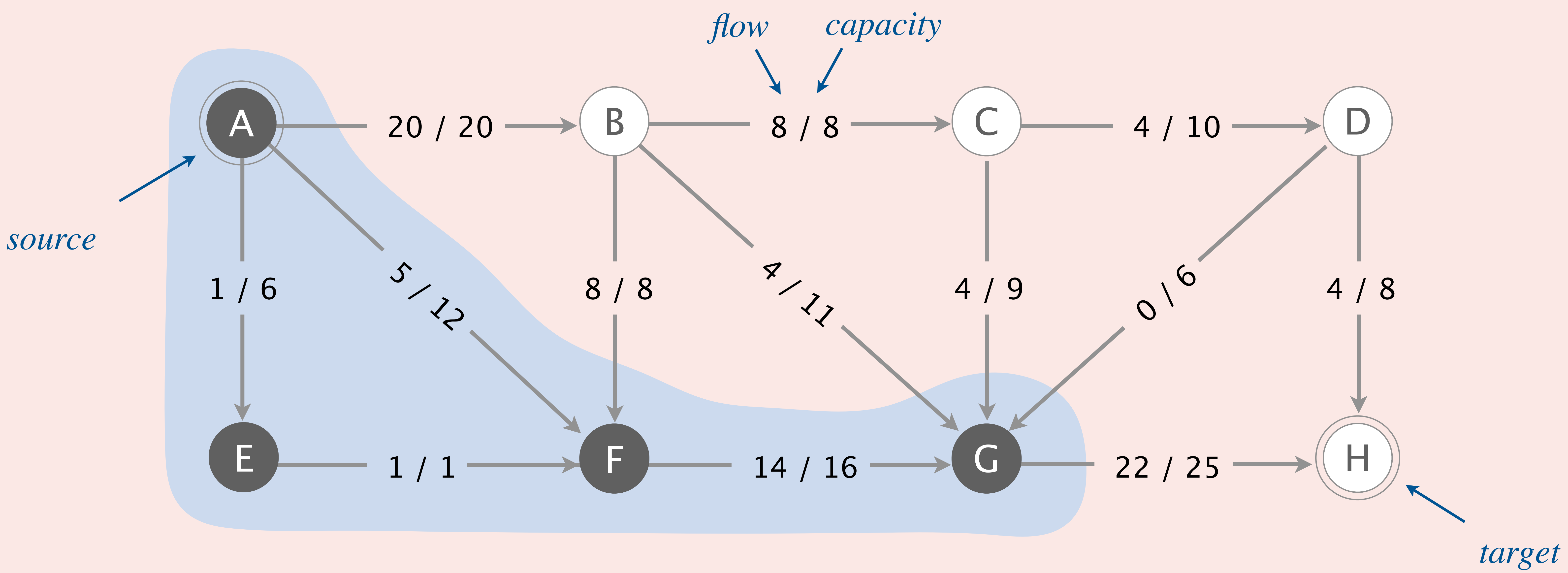
Given the flow below, what is the net flow across the cut $\{A, E, F, G\}$?

A. 11 (20 + 25 – 8 – 11 – 9 – 6)

B. 26 (20 + 22 – 8 – 4 – 4 – 0)

C. 42 (20 + 22)

D. 45 (20 + 25)



Relationship between flows and cuts

Flow-value lemma. Let f be any flow and let (A, B) be any cut.
Then, the net flow across the cut (A, B) equals the value of the flow f.

Intuition. Conservation of flow.

Pf. By induction on the number of vertices in B.

- Base case: $B = \{ t \}$.
- Induction step: remains true when moving any vertex v from A to B
  (because of flow conservation constraint for vertex v)

Corollary. Outflow from s = inflow to t = value of flow.

we assume no edges incident to s or from t

Relationship between flows and cuts

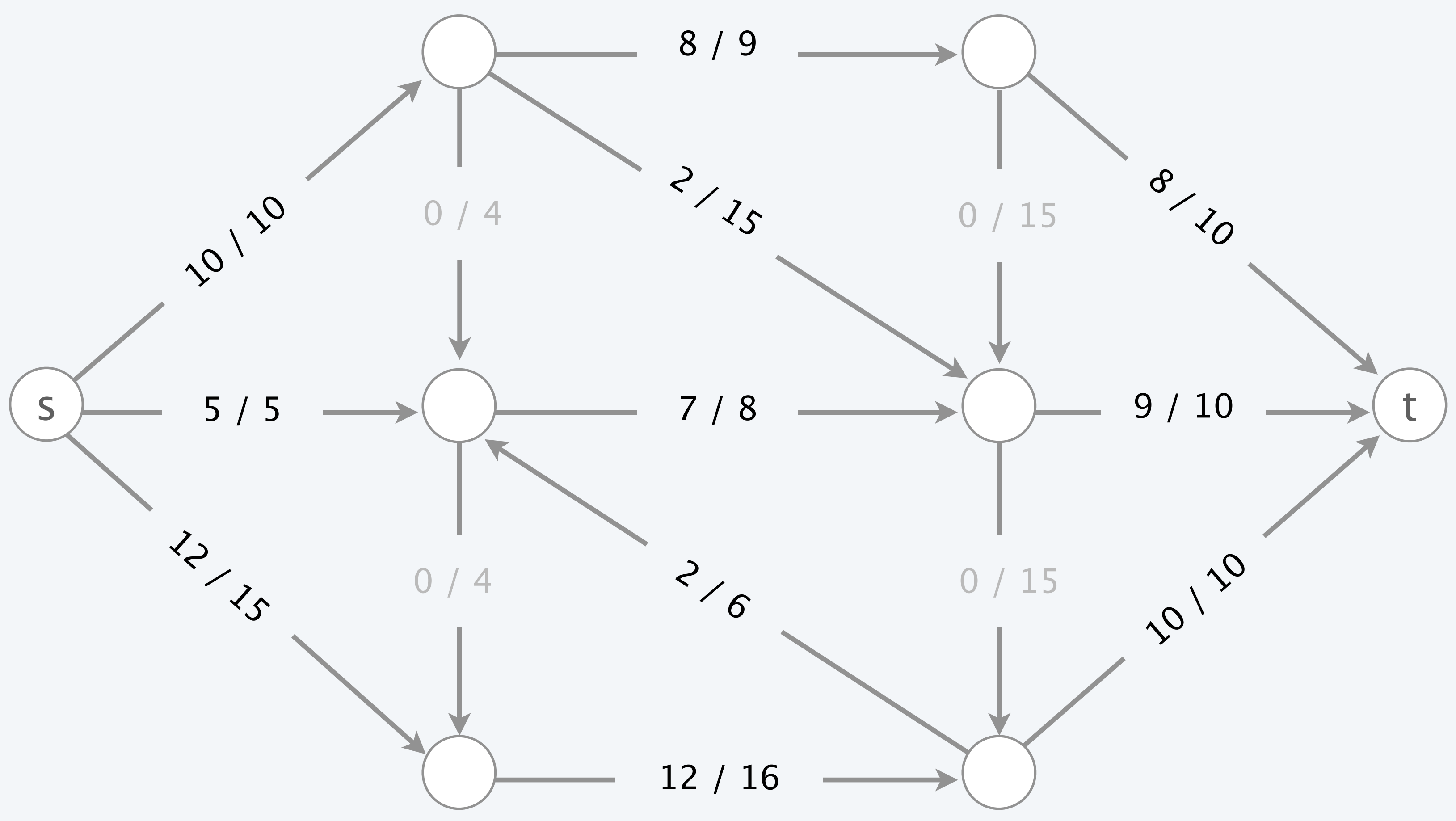
Weak duality. Let f be any flow and let (A, B) be any cut.

Then, the value of flow f ≤ the capacity of cut (A, B).

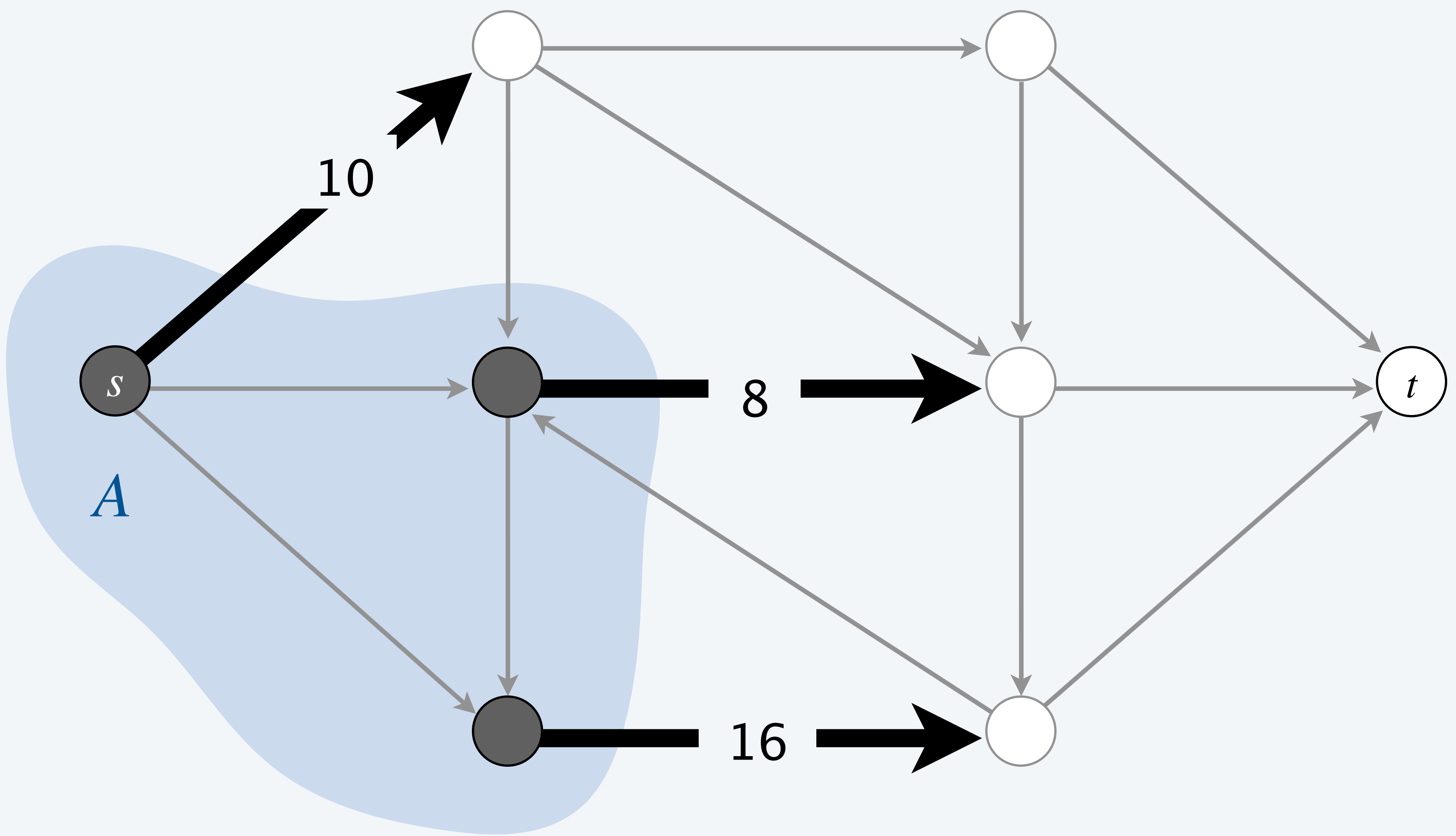
Pf. Value of flow f = net flow across cut (A, B) ≤ capacity of cut (A, B).

flow–value lemma *flow on each edge from A to B bounded by capacity*

Equivalent. Value of maxflow ≤ capacity of mincut.



value of flow f = 27



capacity of cut (A, B) = 34

Maxflow–mincut theorem

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut. ← *"strong duality"*

Augmenting path theorem. A flow f is a maxflow if and only if no augmenting paths.

Pf. For any flow f, the following three conditions are equivalent:

i. Flow f is a maxflow.

ii. There is no augmenting path with respect to flow f.

iii. There exists a cut whose capacity equals the value of flow f.

[i ⇒ ii] We prove contrapositive: ¬ii ⇒ ¬i.

- Suppose that there is an augmenting path with respect to flow f.
- Can improve f by sending flow along this path.
- Thus, f is not a maxflow. ▪

Maxflow-mincut theorem

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Augmenting path theorem. A flow f is a maxflow if and only if no augmenting paths.

Pf. For any flow f, the following three conditions are equivalent:

i. Flow f is a maxflow.

ii. There is no augmenting path with respect to flow f.

iii. There exists a cut whose capacity equals the value of flow f.

[iii ⇒ i]

- Let (A, B) be a cut whose capacity equals the value of flow f.
- Then, the value of any flow $f' \leq$ capacity of (A, B) $=$ value of f.
- Thus, f is a maxflow. ▪

weak duality *by assumption*

Maxflow–mincut theorem

[ii ⇒ iii]

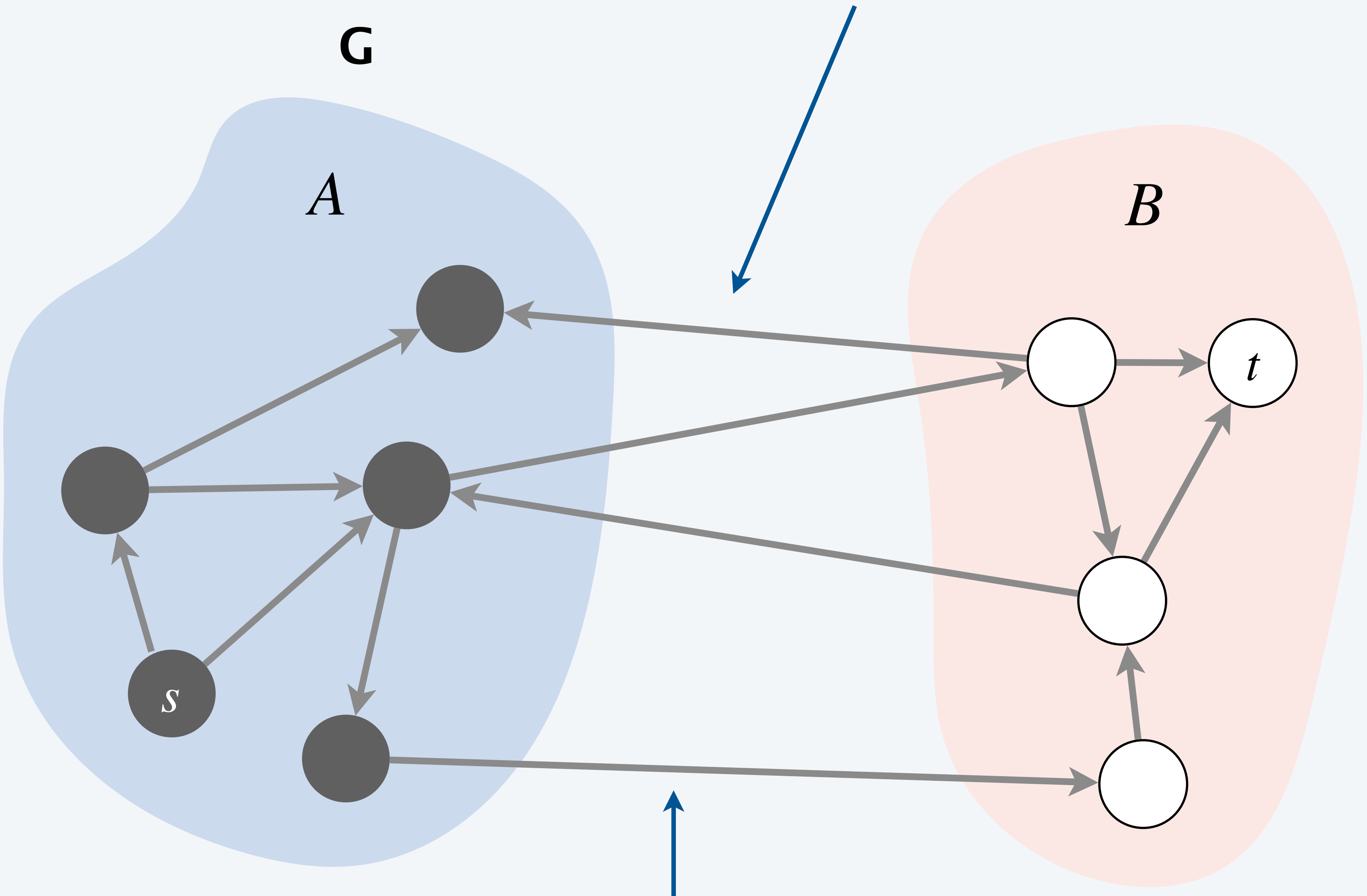
- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s via a path with no full forward or empty backward edges.
- By definition of cut (A, B), s is in A.
- By definition of cut (A, B) and flow f, t is in B.
- Capacity of cut (A, B) = net flow across cut

  = value of flow f. ▪

by construction of cut

flow–value lemma

backward edge from B to A (flow = 0)



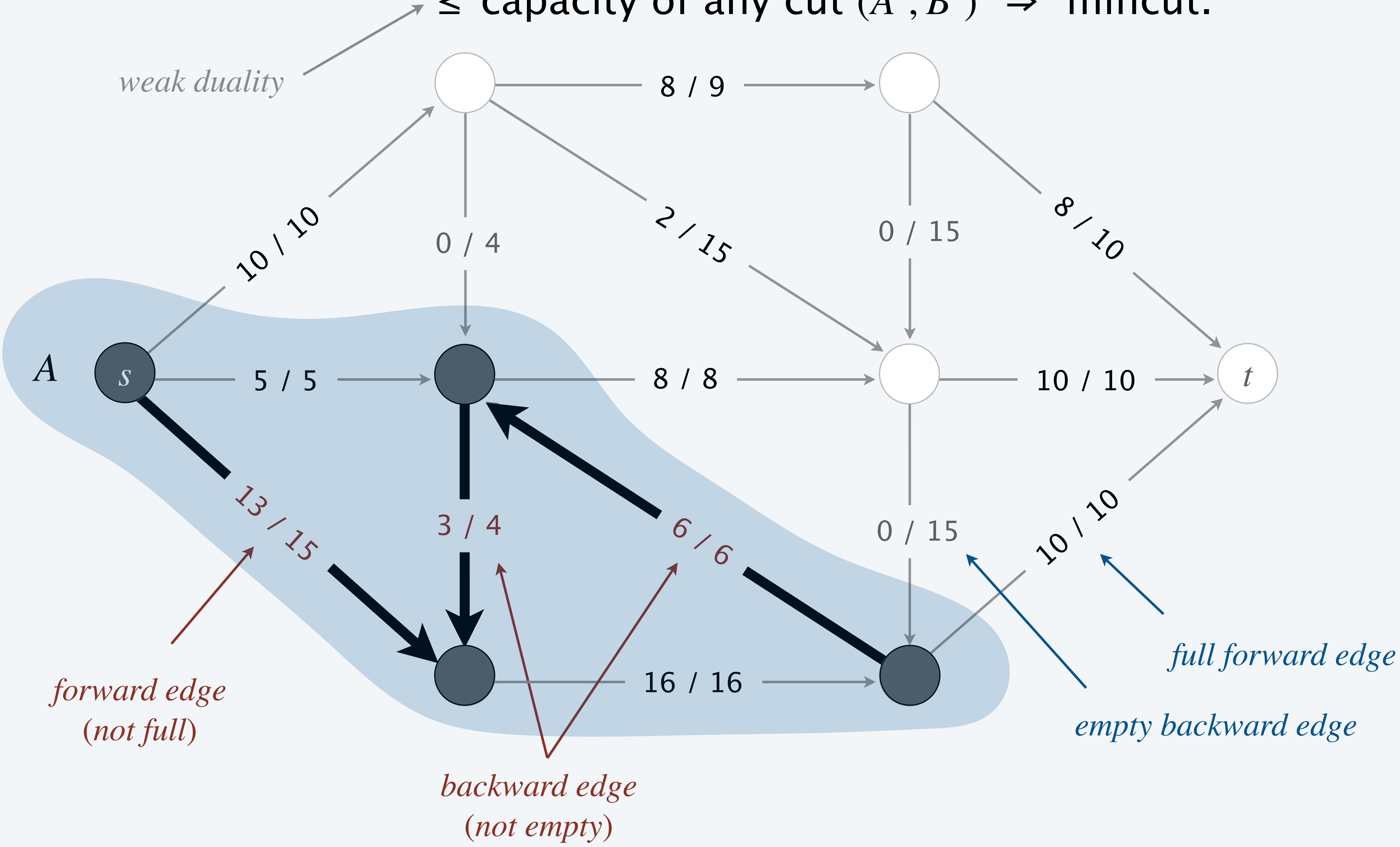
forward edge from A to B (flow = capacity)

Computing a mincut from a maxflow

To compute mincut (A, B) from maxflow f :

- By augmenting path theorem, no augmenting paths with respect to f.
- Compute A = { vertices connected to s by path with no full forward or empty backward edges }.
- Capacity of cut (A, B) = value of flow f

  ≤ capacity of any cut (A', B') ⇒ mincut.



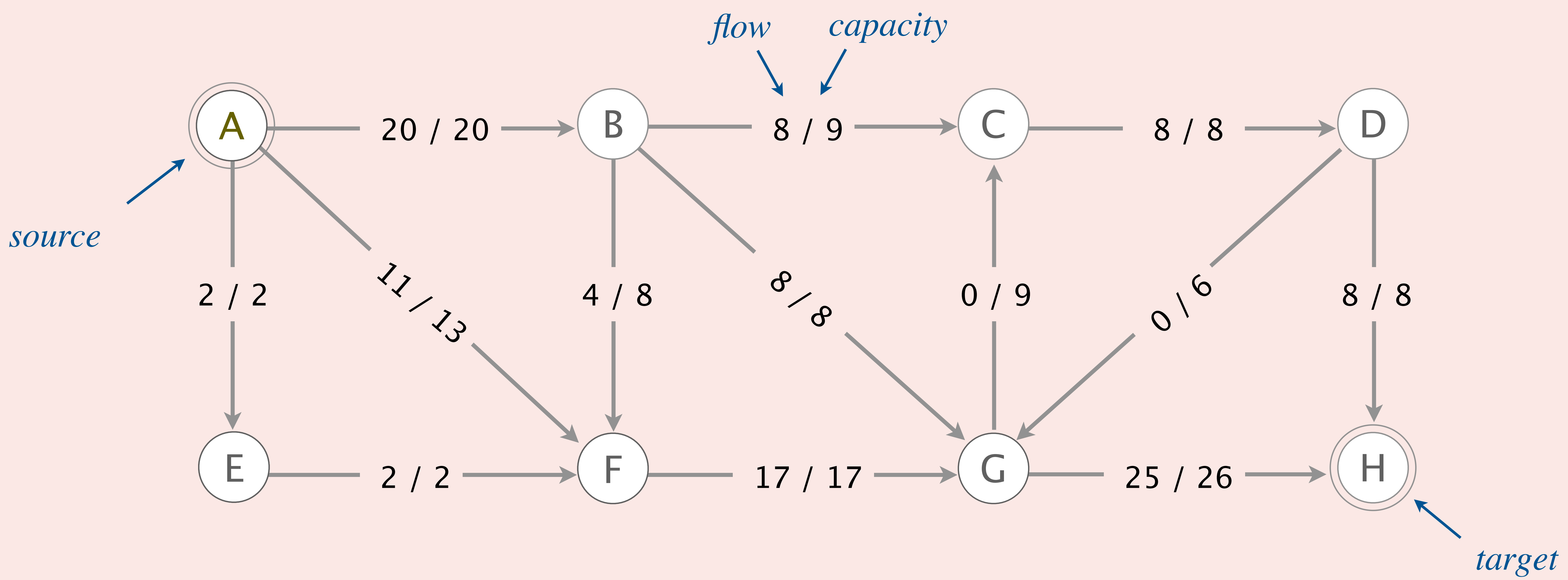
Given the following maxflow, which is a mincut?

A. $A = \{ A, F \}$.

B. $A = \{ A, B, C, F \}$.

C. $A = \{ A, B, C, E, F \}$.

D. None of the above.



Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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6.4 MAXIMUM FLOW

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- *maxflow–mincut theorem*
- *analysis of running time*
- *applications*

Ford–Fulkerson algorithm analysis (with integer capacities)

Important special case. Edge capacities are integers between 1 and U.

flow on each edge is an integer

Invariant. The flow is integral throughout Ford–Fulkerson.

Pf.

- Bottleneck capacity is an integer.
- Flow on an edge increases/decreases by bottleneck capacity. ▪

Proposition. Number of augmentations $\leq$ value of maxflow $\leq$ $E\,U$.

Pf. Each augmentation increases the value of the flow by at least one. ▪

critical for some applications (ahead)

Integrality theorem. There exists an integral maxflow.

Pf.

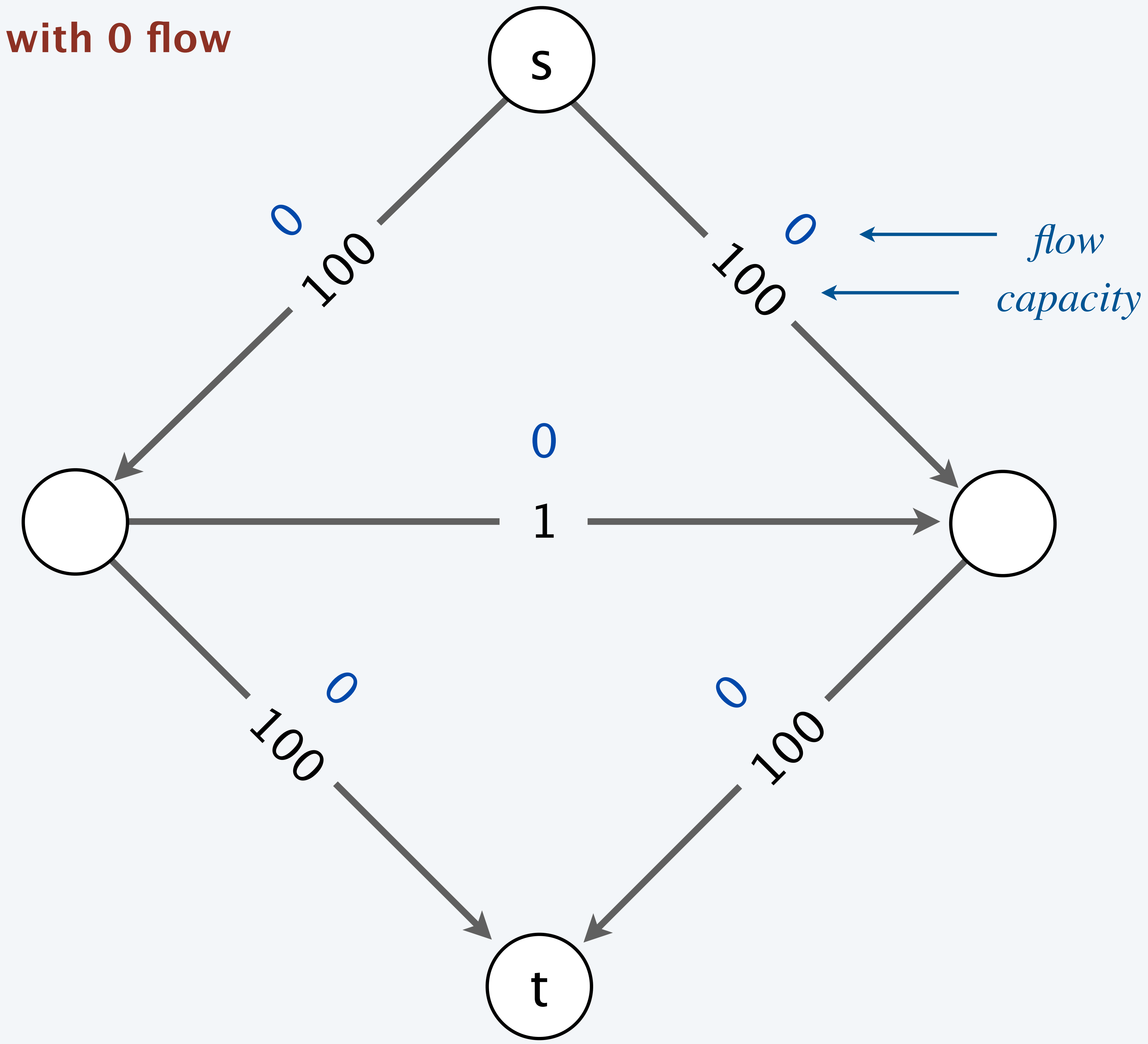
- Proposition + Augmenting path theorem $\Rightarrow$ Ford–Fulkerson terminates with a maxflow.
- Invariant $\Rightarrow$ That maxflow is integral. ▪

Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

even when capacities are integral

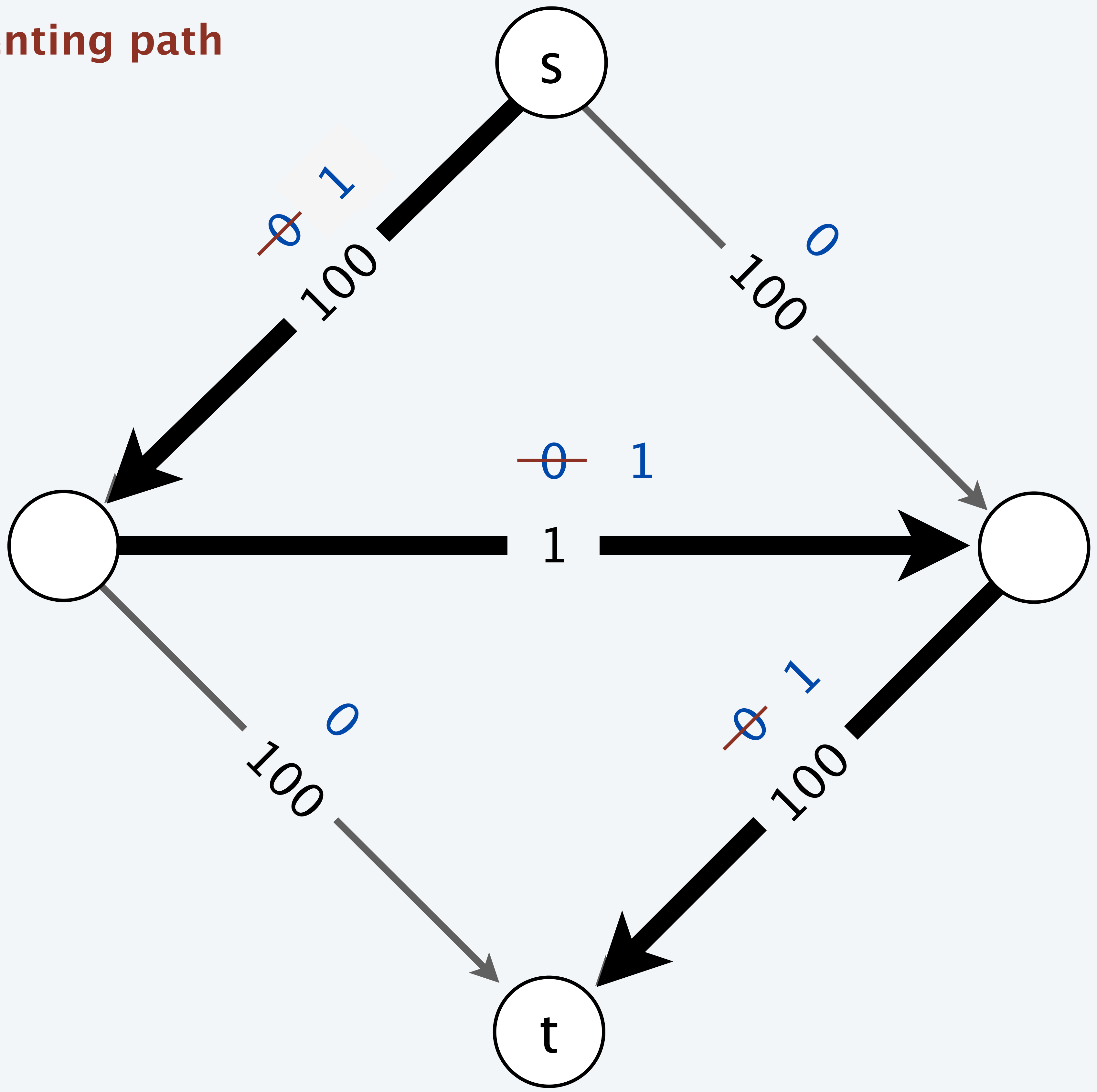
initialize with 0 flow



Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

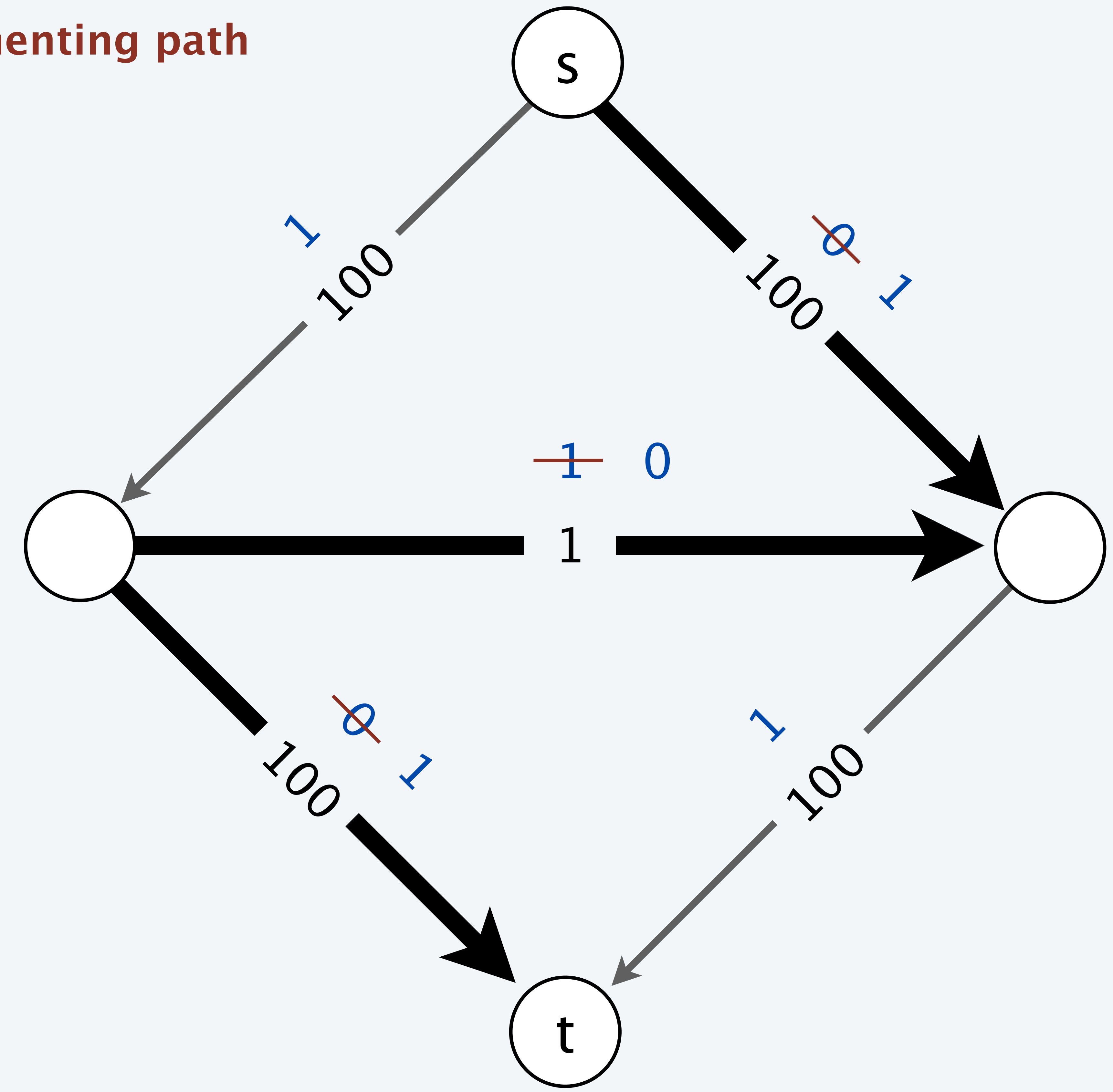
1st augmenting path



Bad case for Ford–Fulkerson

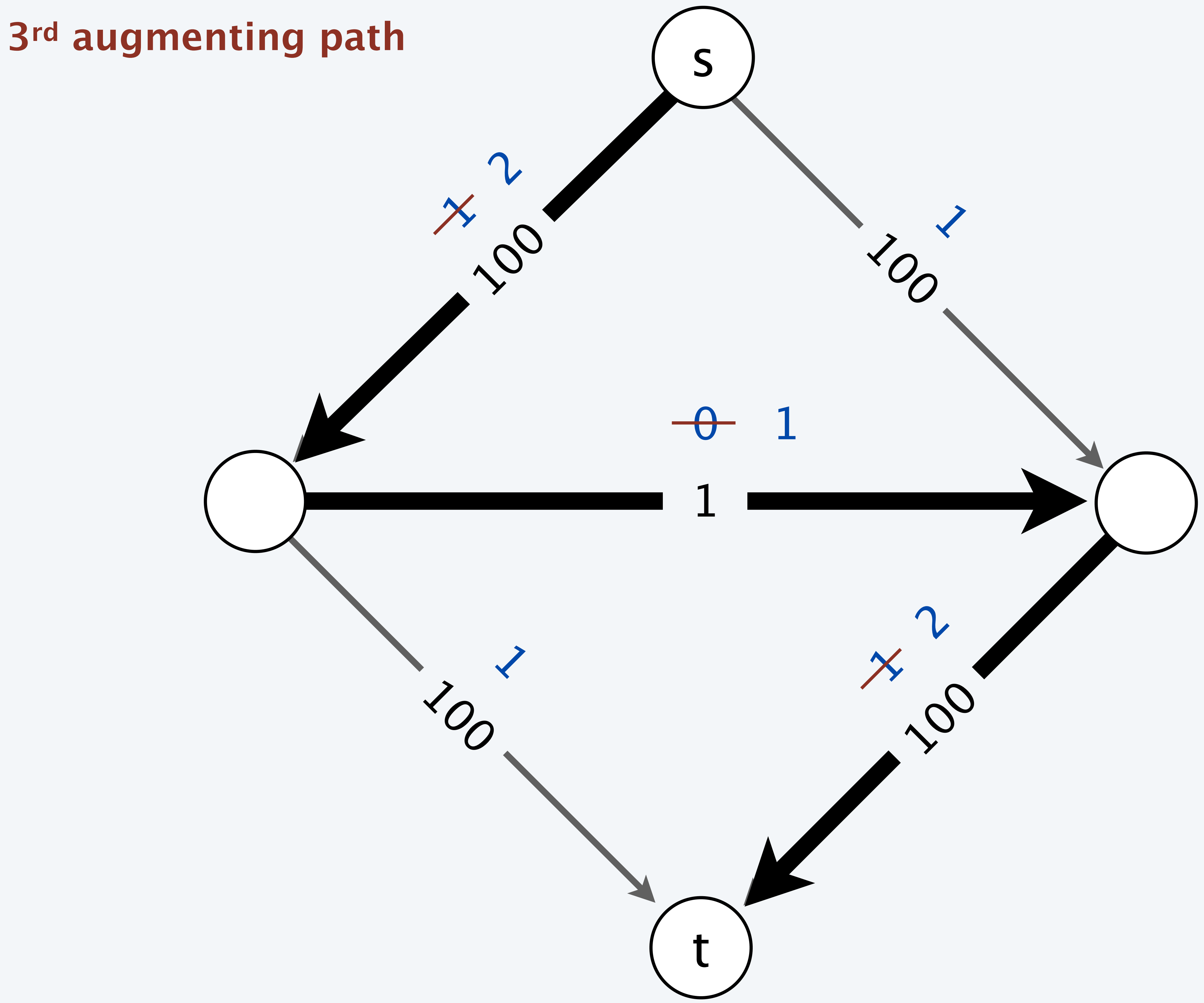
Bad news. Number of augmenting paths can be very large.

2nd augmenting path



Bad case for Ford–Fulkerson

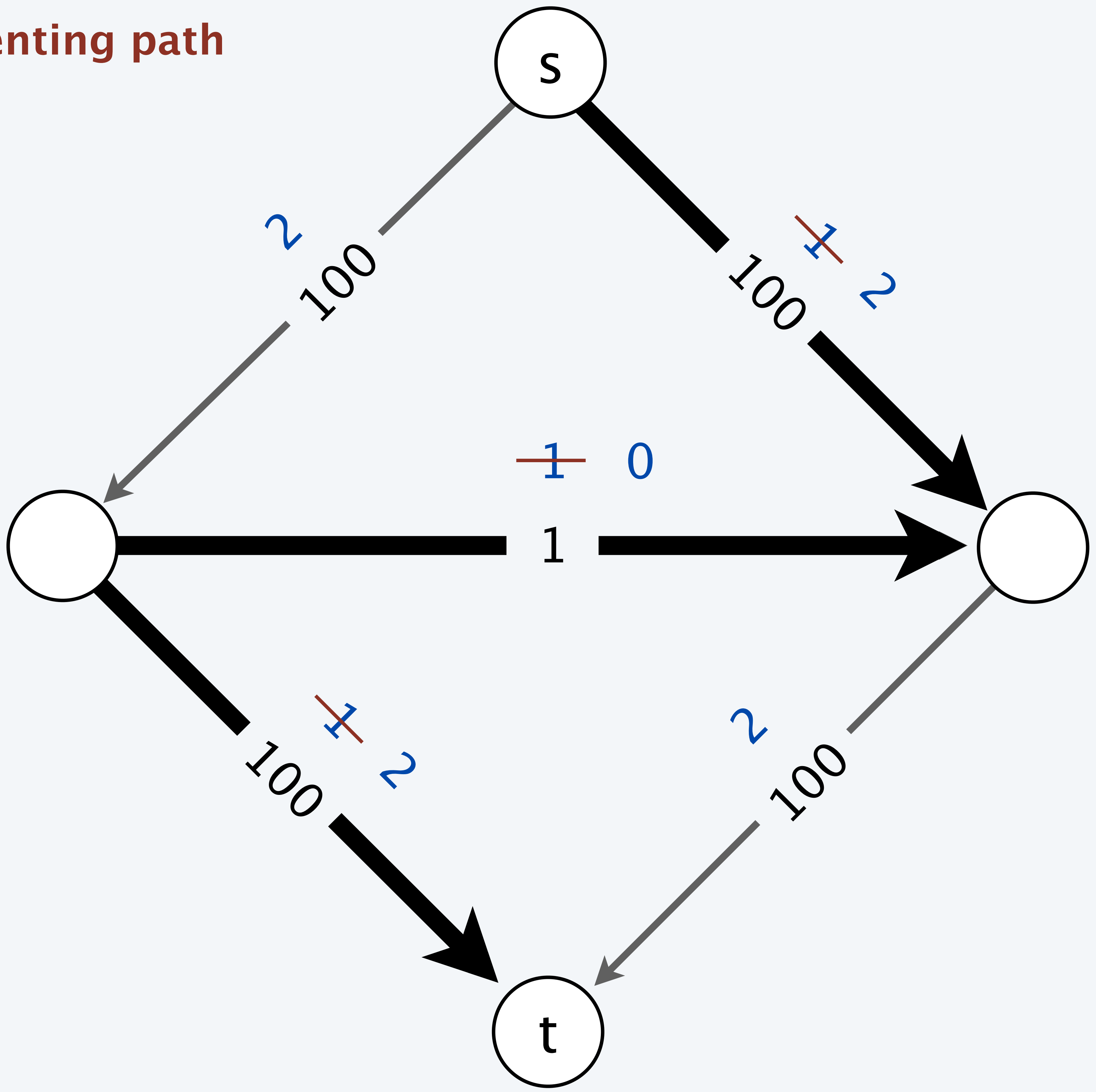
Bad news. Number of augmenting paths can be very large.



Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

4th augmenting path



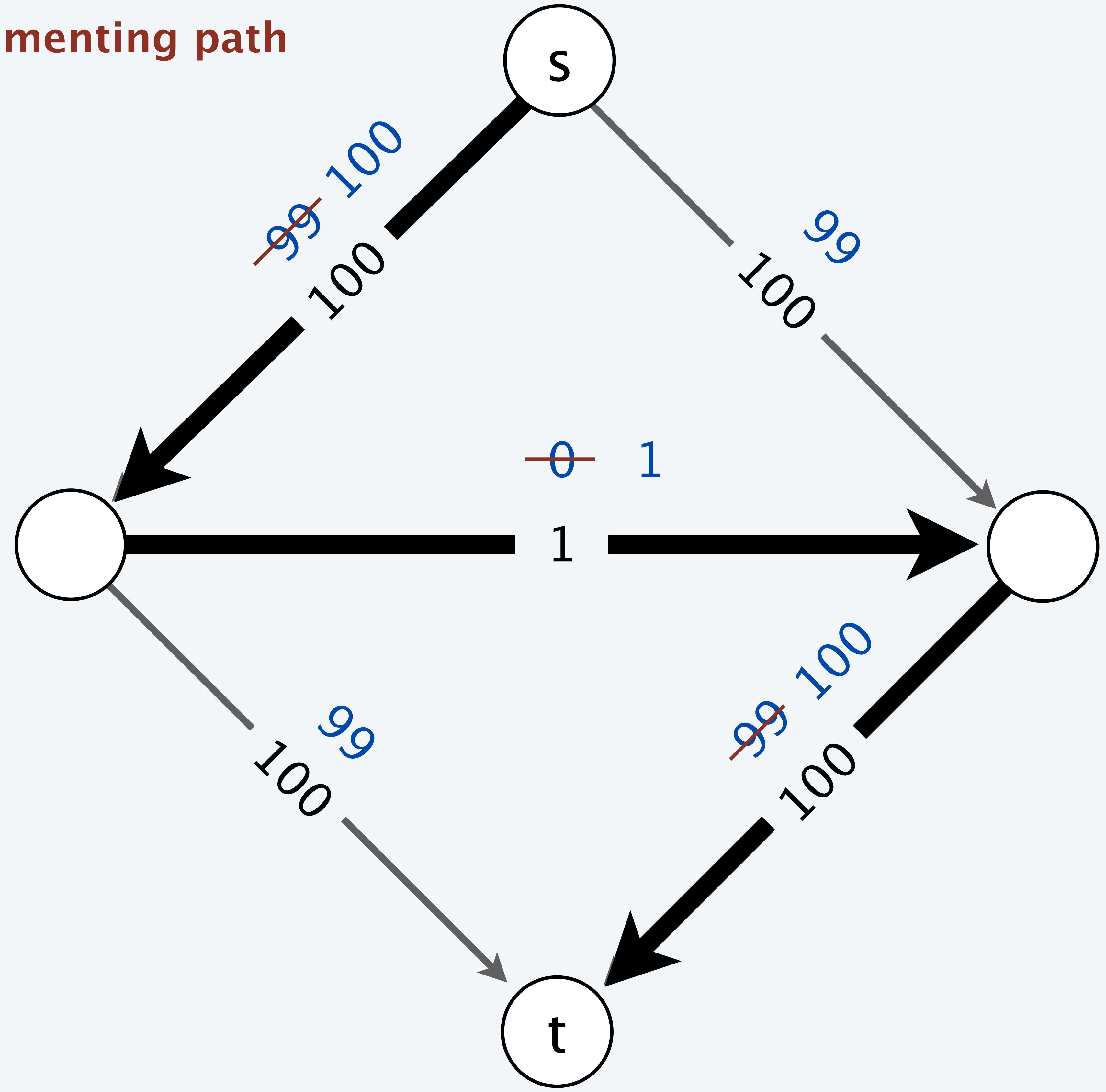
Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

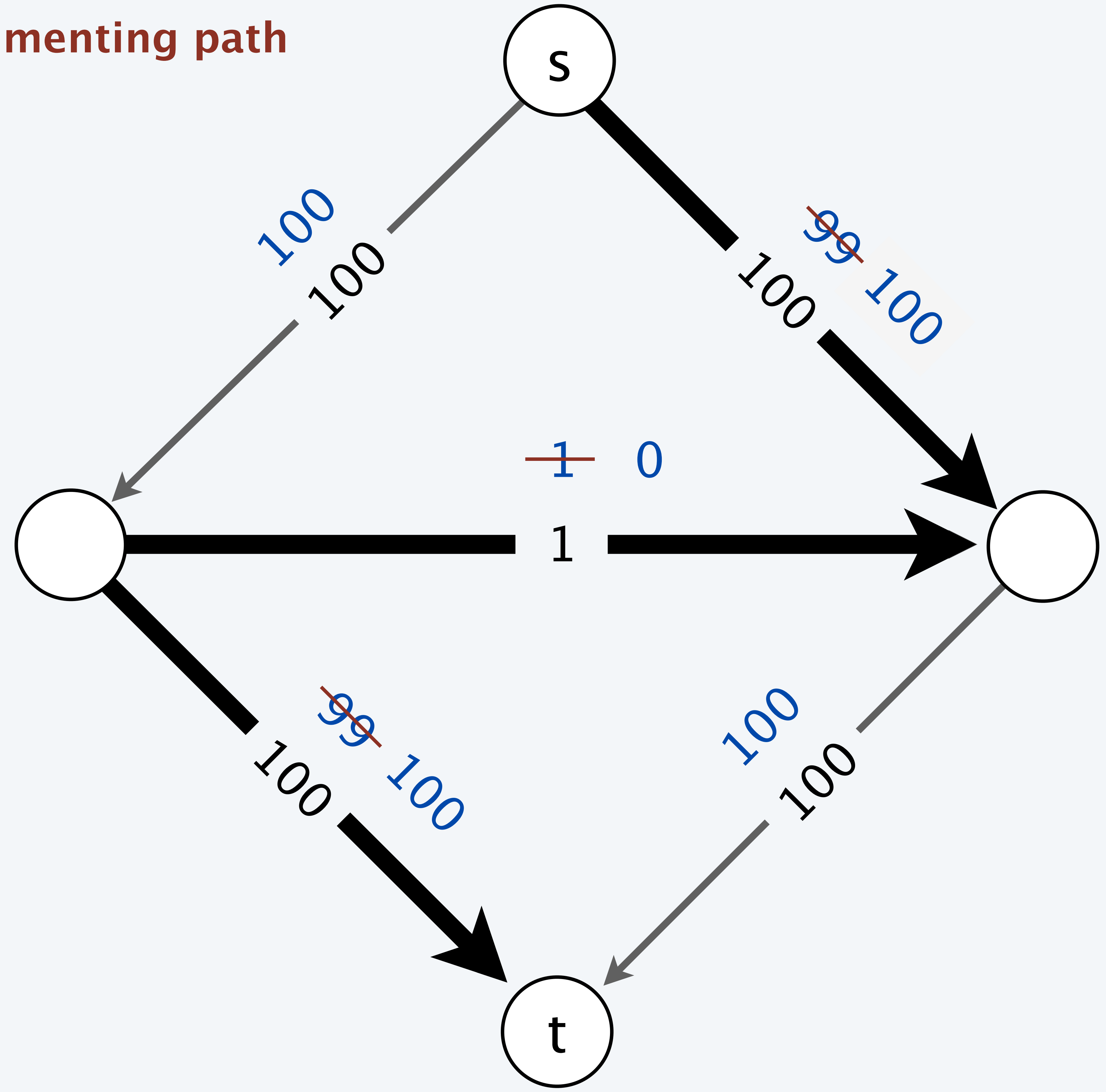
199th augmenting path



Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

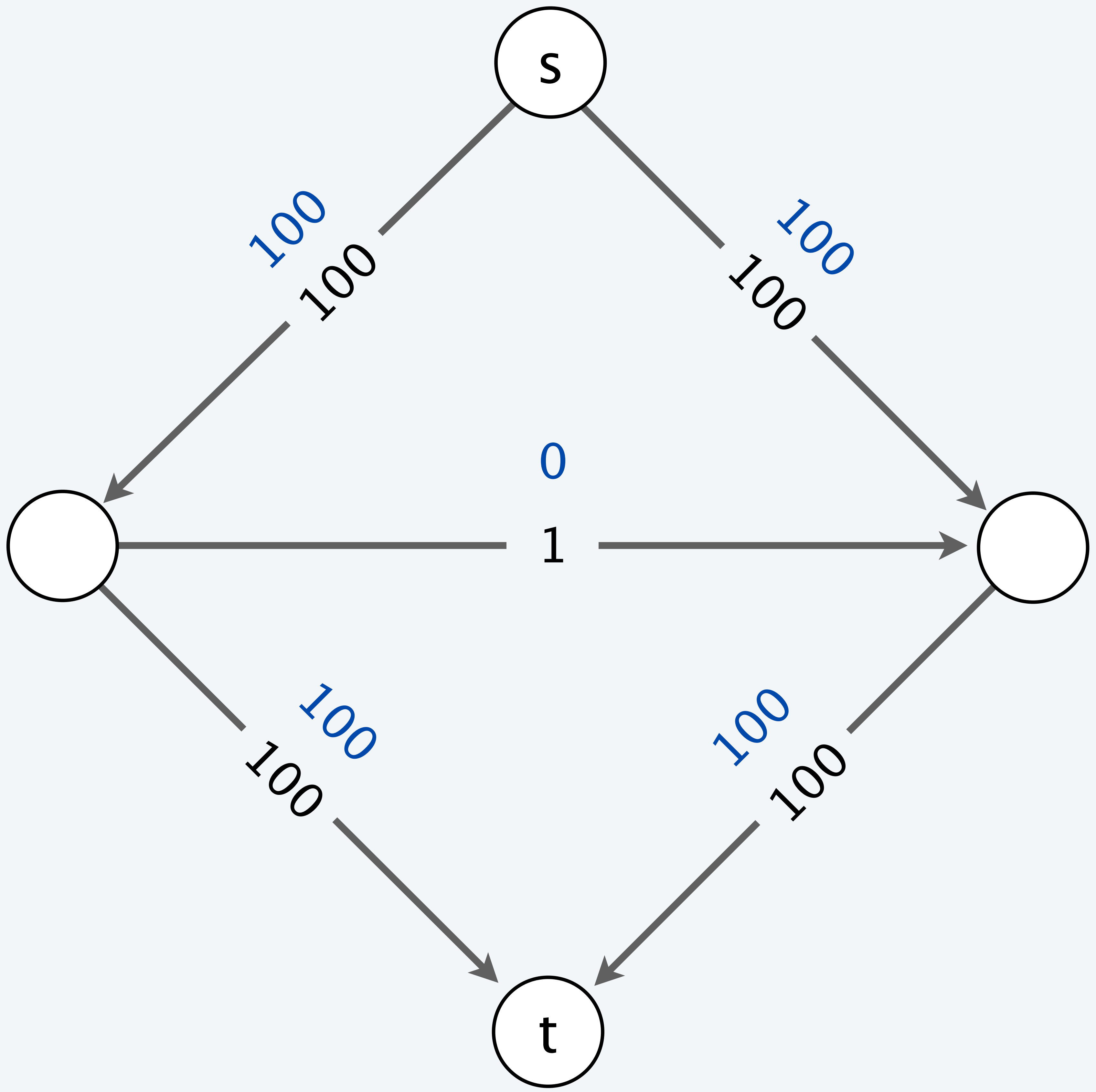
200th augmenting path



Bad case for Ford–Fulkerson

Bad news. Number of augmenting paths can be very large.

exponential in input size
$(V, E, \log U)$



How to choose augmenting paths?

Bad news. Some choices lead to exponential-time algorithms.

Good news. Clever choices lead to polynomial-time algorithms. ← *polynomial in input size* $(V, E, \log U)$

augmenting path	number of iterations	implementation
DFS path	$\le E\,U$	*stack*
random path	$\le E\,U$	*randomized queue*
shortest path (fewest edges)	$\le \frac{1}{2}\,E\,V$	*queue*
fattest path (max bottleneck capacity)	$\le E \ln(E\,U)$	*priority queue*

flow network with V vertices, E edges, and integer capacities between 1 and U

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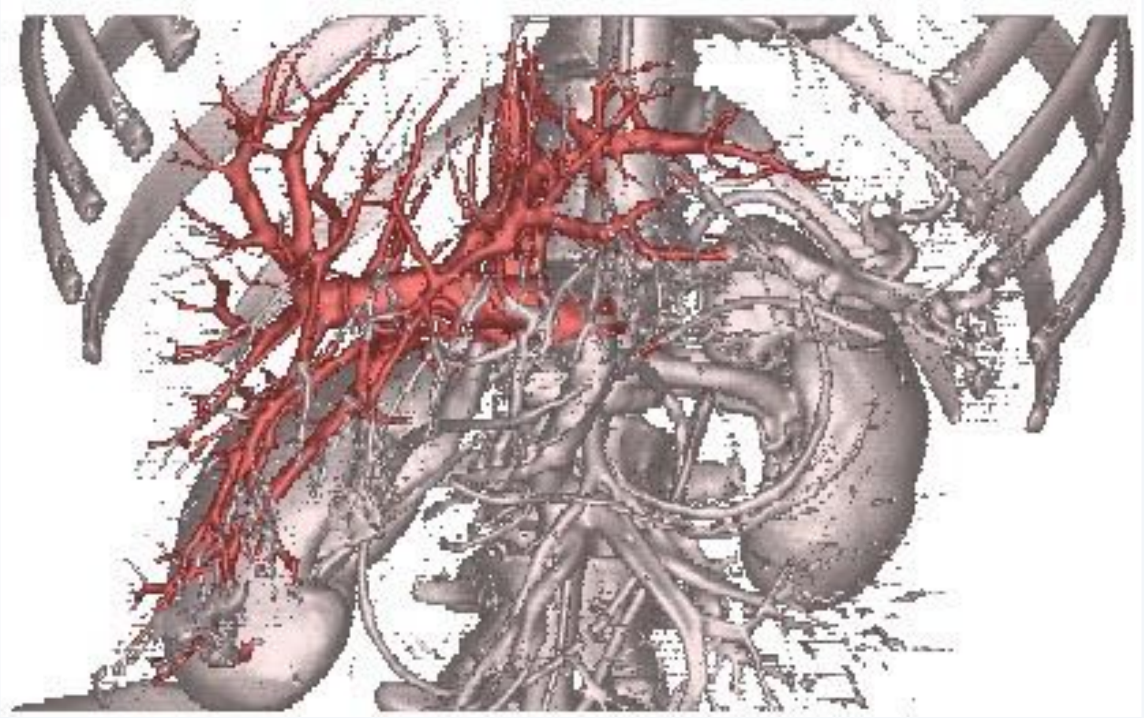
Maxflow and mincut applications

Maxflow/mincut is a widely applicable problem-solving model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.



liver and hepatic vascularization segmentation

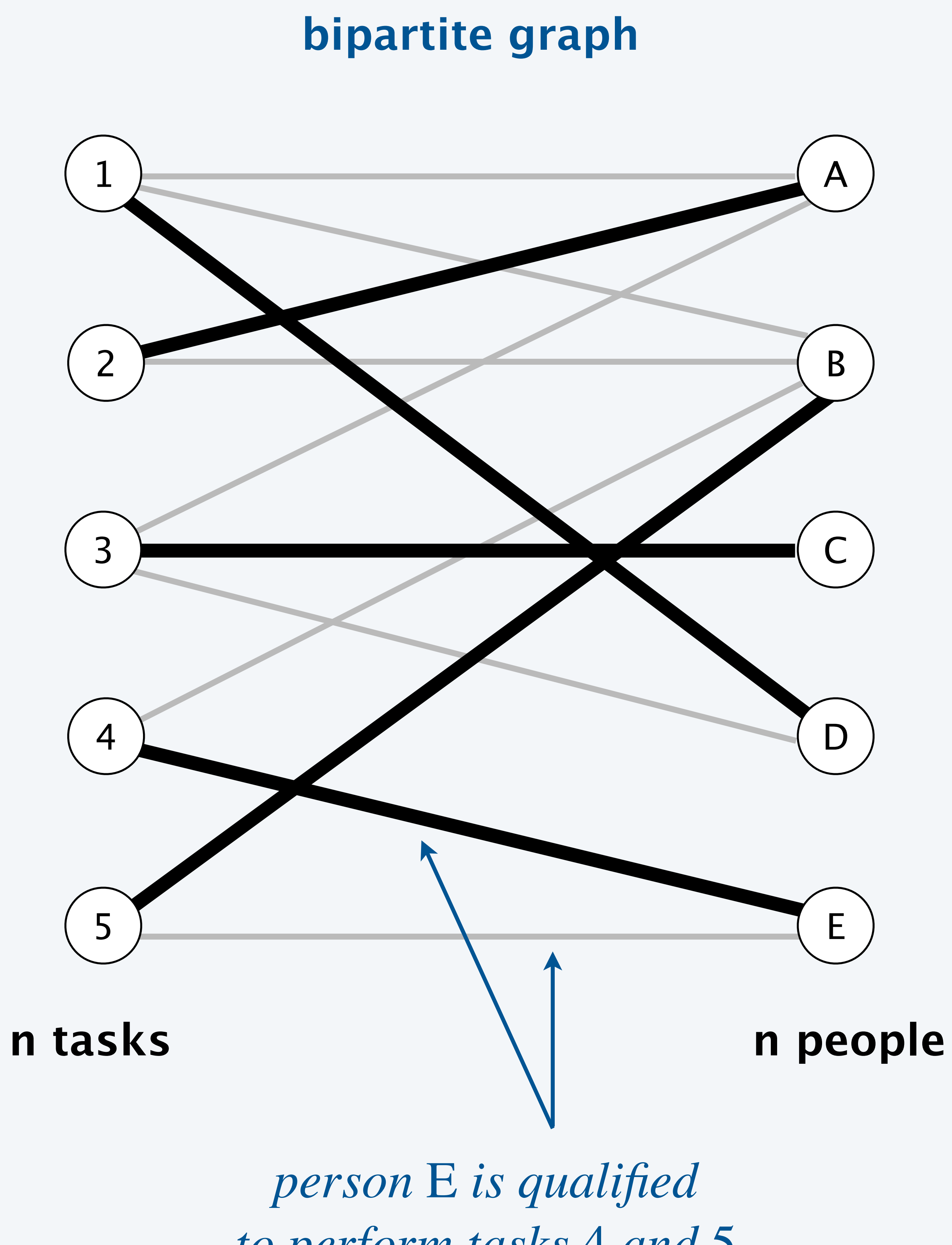
Bipartite matching problem

Problem. Given n people and n tasks, assign the tasks to people so that:

- Every task is assigned to a qualified person.
- Every person is assigned to exactly one task.

Bipartite matching problem

Problem. Given a bipartite graph, find a perfect matching (if one exists).



perfect matching

1–D

2–A

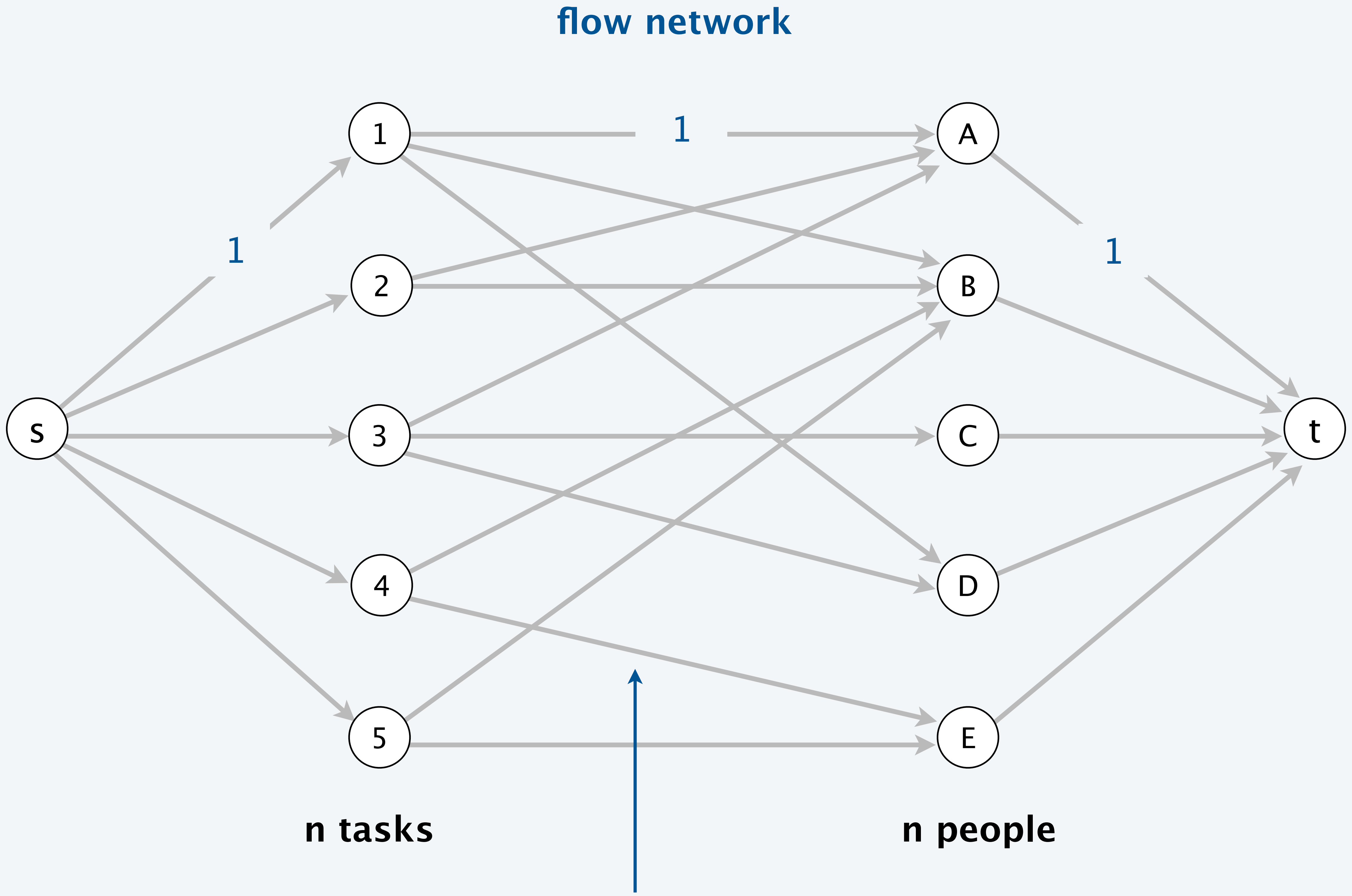
3–C

4–E

5–B

Maxflow formulation of bipartite matching

- Create source s, target t, one vertex i for each task, and one vertex p for each person.
- Add edge from s to each task i of capacity 1.
- Add edge from each person p to t of capacity 1.
- Add edge from task i to qualified person p of capacity 1.



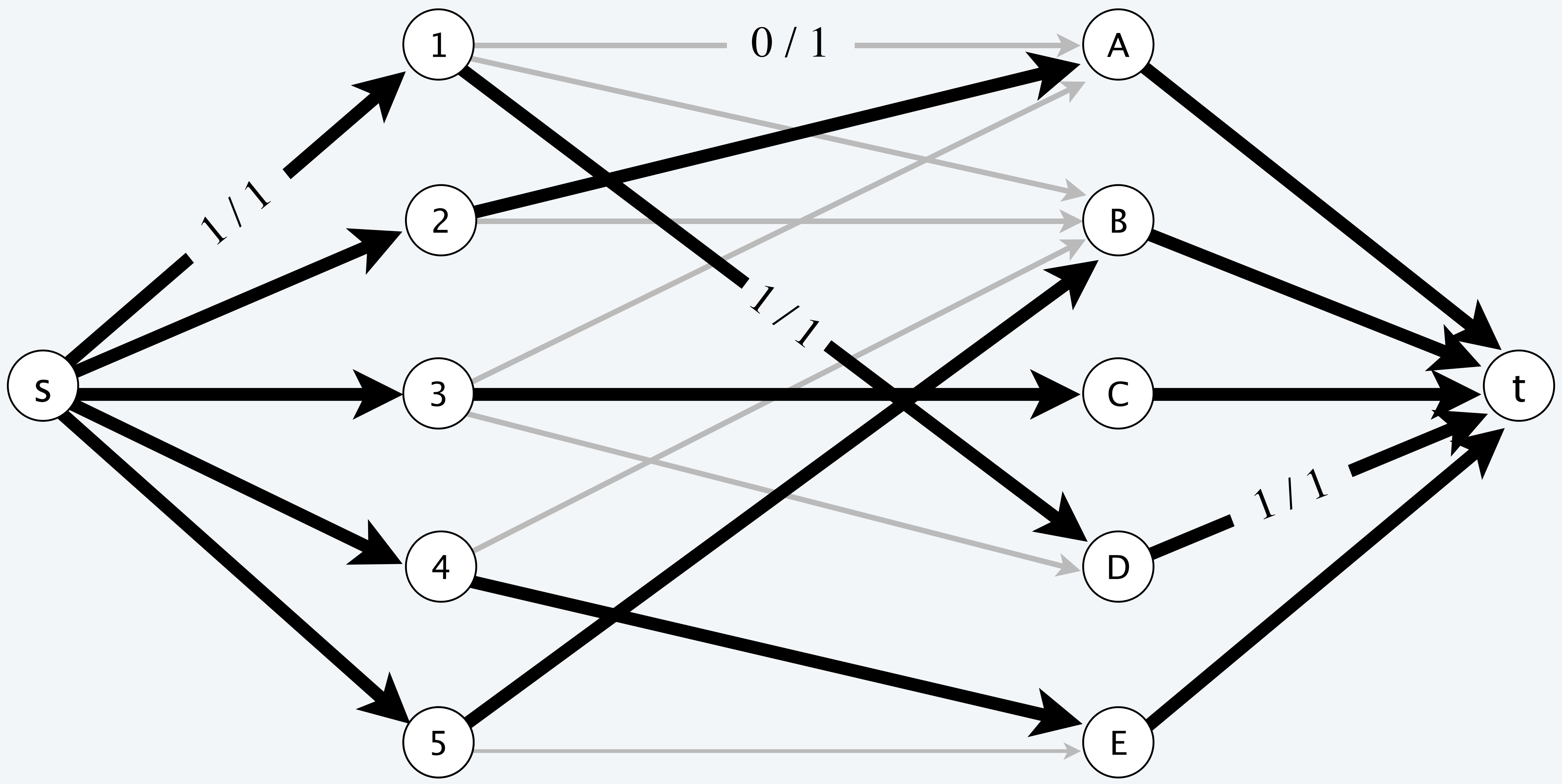
interpretation: flow on edge 4→E = 1 *means assign task* 4 *to person* E

Maxflow formulation of bipartite matching

1–1 correspondence between perfect matchings in bipartite graph
and integral flows of value n in flow network.

Integrality theorem + 1–1 correspondence ⇒ Maxflow formulation is correct.

flow network



n tasks

n people

In the worst case, how many augmenting paths does the Ford–Fulkerson algorithm consider in order to find a perfect matching in a bipartite graph with n vertices per side?

A. $\Theta(n)$

B. $\Theta(n^2)$

C. $\Theta(n^3)$

D. $\Theta(n^4)$

Maximum flow algorithms: theory highlights

year	method	worst case	discovered by
1955	*augmenting paths*	$O(E^2 U)$	Ford–Fulkerson
1970	*shortest augmenting paths*	$O(E^2 V),\ O(E V^2)$	Edmonds–Karp, Dinitz
1974	*blocking flows*	$O(V^3)$	Karzanov
1983	*dynamic trees*	$O(E V \log V)$	Sleator–Tarjan
1988	*push–relabel*	$O(E V \log (V^2 / E))$	Goldberg–Tarjan
1998	*binary blocking flows*	$O(E^{3/2} \log (V^2 / E) \log U)$	Goldberg–Rao
2013	*compact networks*	$O(E V)$	Orlin
2014	*interior-point methods*	$\tilde{O}(E V^{1/2} \log U)$	Lee–Sidford
2016	*electrical flows*	$\tilde{O}(E^{10/7} U^{1/7})$	Mądry
2022	*min ratio cycles*	$O(E^{1+\varepsilon} \log U)$	CKLPGS
20*xx*		????	

max-flow algorithms with E edges, V vertices, and integer capacities between 1 and U

Summary

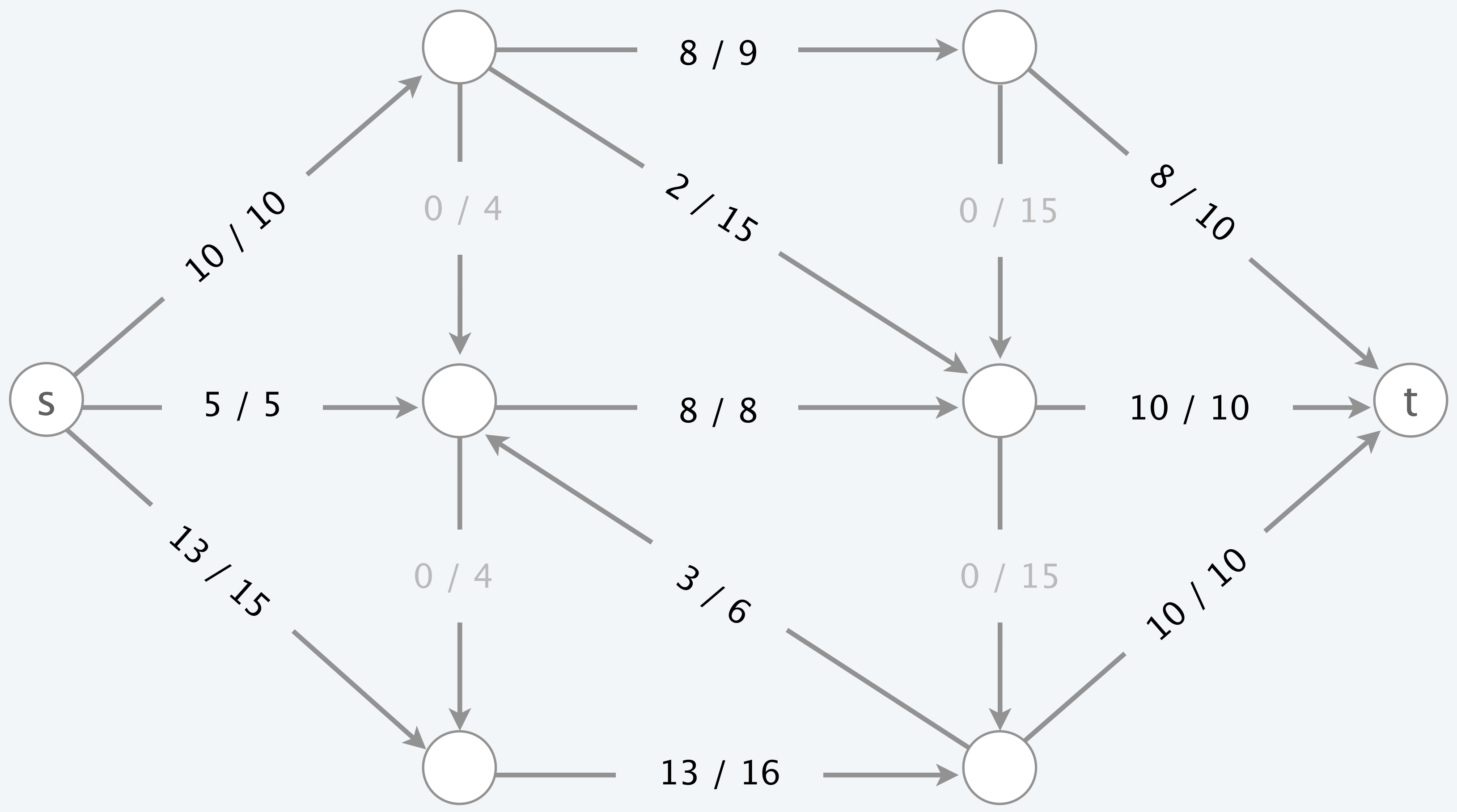
Mincut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.

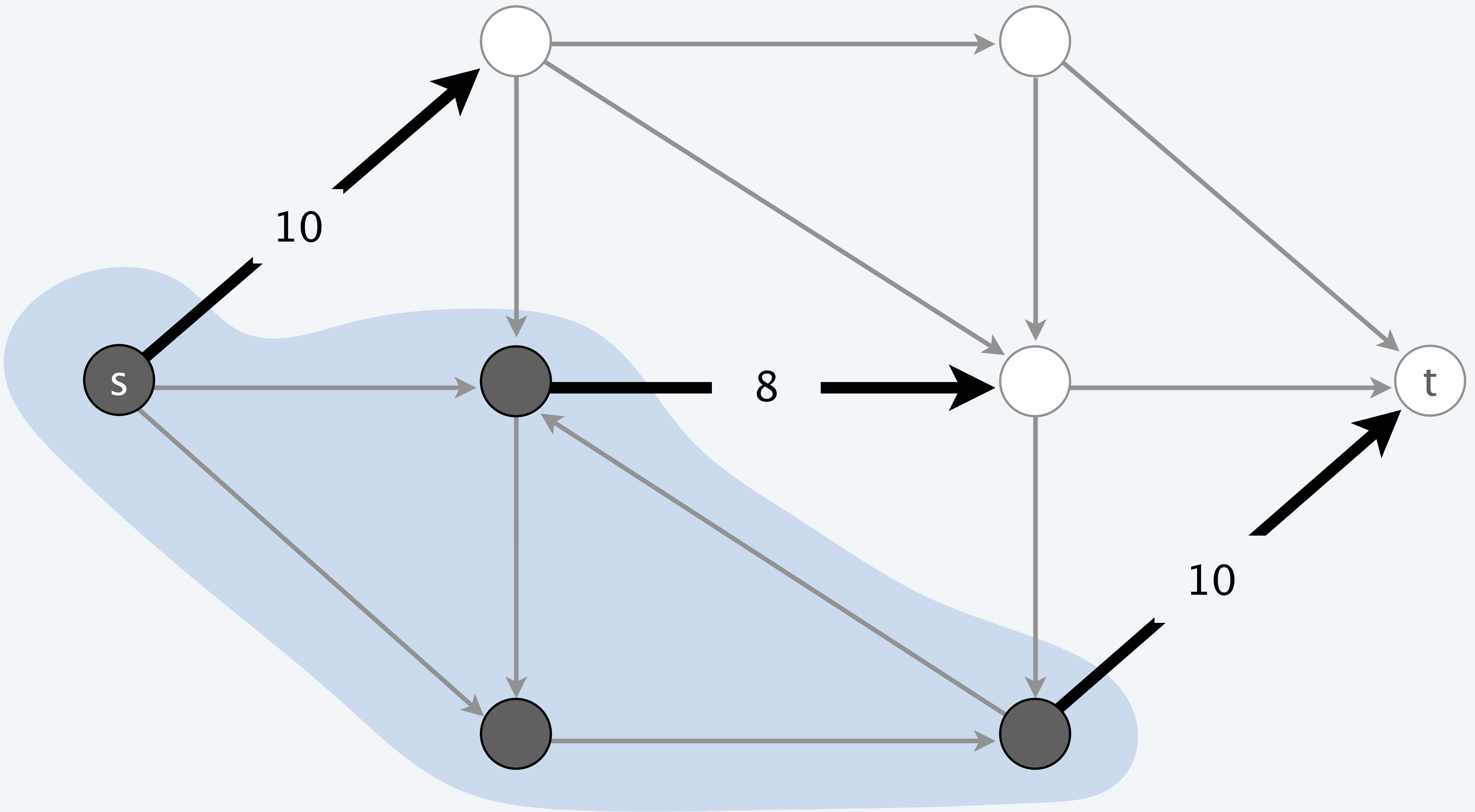
Duality. Value of the maxflow = capacity of mincut.

Proven successful approaches.

- Ford-Fulkerson (various augmenting-path strategies).
- Preflow-push (various versions).



value of flow = 28



capacity of cut = 28

Credits

image	source	license
Warsaw Pact Rail Network	RAND Corporation	
Efficient Max Flow Algorithms	Communications of the ACM	
Liver Segmentation	S. Esneault, T. Pham, K. Torres	
Workers	pictofigo.com	Pictofigo
Todo List	Adobe Stock	education license
Question Marks	pikpng.com	non-commercial use
Theory vs. Practice	Ela Sjolie	