SMT Theory and DPLL(T)

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Overview of the talk

- Motivation
- SMT
- Theories of Interest
- History of SMT
- Eager approach
- Lazy approach
  - Optimizations
  - Theory propagation and DPLL(T)
  - Propagation and Conflict Analysis in DPLL(T)
  - Combining Theory Solvers

Introduction

- Historically, automated reasoning $\equiv$ uniform proof-search procedures for FO logic
- Limited success: is FO logic the best compromise between expressivity and efficiency?
- Current trend [Sha02] focuses on:
  - addressing only (expressive enough) decidable fragments of a certain logic
  - incorporating domain-specific reasoning, e.g:
    - arithmetic reasoning
    - equality
    - data structures (arrays, lists, stacks, ...)

Introduction (2)

Examples of this recent trend:

- SAT: use propositional logic as the formalization language
  - \+ high degree of efficiency
  - \- expressive (all NP-complete) but involved encodings
- SMT: propositional logic + domain-specific reasoning
  - \+ improves the expressivity
  - \- certain (but acceptable) loss of efficiency

GOAL OF THIS TALK:
introduce SMT, with its main techniques

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The SMT problem

- Some problems are more naturally expressed in other logics than propositional logic, e.g:
  - Software verification needs reasoning about equality, arithmetic, data structures, ...
- SMT consists of deciding the satisfiability of a (ground) FO formula with respect to a background theory
- Example (Equality with Uninterpreted Functions – EUF):

$$g(a) = c \quad \wedge \quad (f(g(a)) \neq f(c) \vee g(a) = d) \quad \wedge \quad c \neq d$$

- Wide range of applications:
  - Predicate abstraction [LNO06]
  - Model checking[AMP06]
  - Scheduling [BNO$^+$08b]
  - Test generation[TdH08]
  - ...

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Theories of Interest - EUF [BD94, NO80, NO07]

- Equality with Uninterpreted Functions, i.e. "$=$" is equality
- If background logic is FO with equality, EUF is empty theory
- Consider formula
$$a * (f(b) + f(c)) = d \quad \wedge \quad b * (f(a) + f(c)) \neq d \quad \wedge \quad a = b$$
- Formula is UNSAT, but no arithmetic resoning is needed
- If we abstract the formula into
$$h(a,\ g(f(b), f(c))\ ) = d \quad \wedge \quad h(b,\ g(f(a), f(c))\ ) \neq d \quad \wedge \quad a = b$$
it is still UNSAT
- EUF is used to abstract non-supported constructions, e.g:
  - Non-linear multiplication
  - ALUs in circuits

Theories of Interest - Arithmetic

- Very useful for obvious reasons
- Restricted fragments support more efficient methods:
  - Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$
  - Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, SM06]
  - UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
  - Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
  - Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM$^+$09, ZM10]
  - Variables are either reals or integers

Th. of Int.- Arrays[SBDL01, BNO+08a, dMB09]

- Used for:
  - Software verification
  - Hardware verification (memories)
- Two interpreted function symbols *read* and *write*
- Theory is axiomatized by:
  - $\forall a \forall i \forall v \; (read(write(a,i,v),i) = v)$
  - $\forall a \forall i \forall j \forall v \; (i \neq j \rightarrow read(write(a,i,v),j) = read(a,j))$
- Sometimes extensionality is added:
  - $\forall a \forall b \; (\, (\forall i (read(a,i) = read(b,i)) \,) \rightarrow a = b$
- Is the following set of literals satisfiable?

$$write(a,i,x) \neq b \quad read(b,i) = y \quad read(write(b,i,x),j) = y$$

$$a = b \qquad\qquad i = j$$

Th. of Interest - Bit vectors [BCF+07, BB09]

- Universe consists of vectors of bits
- Useful both for hardware and software verification
- Different type of operations:
  - String-like operations: concat, extract, ...
  - Logical operations: bit-wise not, or, and, ...
  - Arithmetic operations: add, substract, multiply, ...
- Assume bit-vectors have size 3. Is the formula SAT?

$$a[0:1] \neq b[0:1] \ \wedge \ (a|b) = c \ \wedge \ c[0] = 0 \ \wedge \ a[1] + b[1] = 0$$

Combina. of theories [NO79, Sho84, BBC^{+}05]

- In practice, theories are not isolated
- Software verifications needs arithmetic, arrays, bitvectors, ...
- Formulas of the following form usually arise:

$$a = b + 2 \ \wedge \ A = write(B, a+1, 4) \ \wedge \ (read(A, b+3) = 2 \ \vee \ f(a-1) \neq f(b+1))$$

- The goal is to combine decision procedures for each theory

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SMT Prehistory - Late 70's and 80's

- Pioneers:
  - R. Boyer, J. Moore, G. Nelson, D. Open, R. Shostak
- Influential results:
  - Nelson-Oppen congruence closure procedure [NO80]
  - Nelson-Oppen combination method [NO79]
  - Shostak combination method [Sho84]
- Influential systems:
  - Nqthm prover [BM90] [Boyer, Moore]
  - Simplify [DNS05] [Detlefs, Nelson, Saxe]

Beginnings of SMT - Early 2000s

KEY FACT: SAT solvers improved performance

Two ways of exploiting this fact:

- Eager approach: encode SMT into SAT

  [Bryant, Lahiri, Pnueli, Seshia, Strichman, Velev, ...]

  [PRSS99, SSB02, SLB03, BGV01, BV02]

  First systems: UCLID [LS04]

- Lazy approach: plug SAT solver with a decision procedure

  [Armando, Barrett, Castellini, Cimatti, Dill, Giunchiglia, deMoura, Ruess, Sebastiani, Stump,...]

  [ACG00, dMR02, BDS02a, ABC$^+$02]

  First systems: TSAT [ACG00], ICS [FORS01], CVC [BDS02b], MathSAT [ABC$^+$02]

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Eager approach

- Methodology: translate problem into equisatisfiable propositional formula and use off-the-shelf SAT solver
- Why "eager"?
  Search uses all theory information from the beginning
- Characteristics:
  - + Can use best available SAT solver
  - - Sophisticated encodings are needed for each theory
- Tools: UCLID, Beaver, Boolector, STP, SONOLAR, Spear, SWORD

Eager approach – Example

Let us consider an EUF formula:

- First step: remove function/predicate symbols.
  Assume we have terms $f(a)$, $f(b)$ and $f(c)$.
  - Ackermann reduction:
    - Replace them by fresh constants A, B and C
    - Add clauses:
      $a = b \rightarrow A = B$
      $a = c \rightarrow A = C$
      $b = c \rightarrow B = C$
  - Bryant reduction:
    - Replace $f(a)$ by A
    - Replace $f(b)$ by $ite(b = a, A, B)$
    - Replace $f(c)$ by $ite(c = a, A,\ ite(c = b, B, C)\ )$

Now, atoms are equalities between **constants**

Eager approach – Example (2)

- Second step: encode formula into propositional logic
  - Small-domain encoding:
    - If there are n different constants, there is a model with size at most n
    - $\log n$ bits to encode the value of each constant
    - $a = b$ translated using the bits for a and b
  - Per-constraint encoding:
    - Each atom $a = b$ is replaced by var $P_{a,b}$
    - Transitivity constraints are added (e.g. $P_{a,b} \wedge P_{b,c} \rightarrow P_{a,c}$)

This is a **very rough** overview of an encoding from EUF to SAT.

See [PRSS99, SSB02, SLB03, BGV01, BV02] for details.

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Lazy approach

Methodology:

Example: consider EUF and the CNF

$$\underbrace{g(a)=c}_{1} \;\wedge\; (\underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3}) \;\wedge\; \underbrace{c\neq d}_{\overline{4}}$$

- SAT solver returns model $[1, \overline{2}, \overline{4}]$

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- Send $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4\}$ to SAT solver
- SAT solver returns model $[1, 2, 3, \overline{4}]$
- Theory solver says T-inconsistent
- SAT solver detects $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4, \overline{1} \vee \overline{2} \vee \overline{3} \vee 4\}$
  UNSATISFIABLE

Lazy approach (2)

- Why "lazy"?
  Theory information used lazily when checking T-consistency of propositional models

- Characteristics:
  - \+ Modular and flexible
  - \- Theory information does not guide the search

- Tools:
  Alt-Ergo, ArgoLib, Ario, Barcelogic, CVC, DTP, ICS, MathSAT, OpenSMT, Sateen, SVC, Simplify, tSAT, veriT, Yices, Z3, etc...

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Lazy approach - Optimizations

Several optimizations for enhancing efficiency:

- Check T-consistency only of full propositional models

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- ~~Check T-consistency only of full propositional models~~
- Check T-consistency of partial assignment while being built

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- Given a T-inconsistent assignment M, add $\neg M$ as a clause

Lazy approach - Optimizations

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- ~~Given a T-inconsistent assignment M, add $\neg M$ as a clause~~
- Given a T-inconsistent assignment M, identify a T-inconsistent subset $M_0 \subseteq M$ and add $\neg M_0$ as a clause

Lazy approach - Optimizations

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- Upon a T-inconsistency, add clause and restart

Lazy approach - Optimizations

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- Given a T-inconsistent assignment M, identify a T-inconsistent subset $M_0 \subseteq M$ and add $\neg M_0$ as a clause

- ~~Upon a T-inconsistency, add clause and restart~~
- Upon a T-inconsistency, bactrack to some point where the assignment was still T-consistent

Lazy approach - Important points

Important and benefitial aspects of the lazy approach:
(even with the optimizations)

- Everyone does what he/she is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- Theory solver only receives conjunctions of literals
- Modular approach:
  - SAT solver and T-solver communicate via a simple API
  - SMT for a new theory only requires new T-solver
  - SAT solver can be embedded in a lazy SMT system with very few new lines of code (tens)

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Lazy approach - T-propagation

- As pointed out the lazy approach has one drawback:
  - Theory information does not guide the search (too lazy)
- How can we improve that?

T-Propagate :

$$M \parallel F \quad\Rightarrow\quad M\, l \parallel F \quad \textbf{if} \begin{cases} M \models_T l \\ l \text{ or } \neg l \text{ occurs in } F \text{ and not in } M \end{cases}$$

- Search guided by T-Solver by finding T-consequences, instead of only validating it as in basic lazy approach.
- Naive implementation::
  Add $\neg l$. If T-inconsistent then infer l [ACG00]
  But for efficient Theory Propagation we need:

  -T-Solvers specialized and fast in it.
  -fully exploited in conflict analysis
- This approach has been named DPLL(T) [NOT06]

DPLL(T)

In a nutshell:

$$\text{DPLL}(T) = \text{DPLL}(X) + T\text{-Solver}$$

- DPLL(X):
  - Very similar to a SAT solver, enumerates Boolean models
  - Not allowed: pure literal, blocked literal detection, ...
  - Desirable: partial model detection
- T-Solver:
  - Checks consistency of conjunctions of literals
  - Computes theory propagations
  - Produces explanations of inconsistency/T-propagation
  - Should be incremental and backtrackable

DPLL(T) - Example

Consider again EUF and the formula:

$$\underbrace{g(a)=c}_{1} \;\wedge\; (\underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3}) \;\wedge\; \underbrace{c\neq d}_{\overline{4}}$$

$$\emptyset \parallel 1,\ \overline{2}\vee 3,\ \overline{4} \quad\Rightarrow\quad (\text{UnitPropagate})$$

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$$\begin{array}{lllll} \emptyset \;\|\; & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{UnitPropagate}) \\ 1 \;\|\; & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{UnitPropagate}) \end{array}$$

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$$\begin{array}{rcllcl}
\emptyset & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{UnitPropagate}) \\
1 & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{UnitPropagate}) \\
1\,\overline{4} & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{T-Propagate})
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DPLL(T) - Example

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1\,\overline{4}\,2 & \| & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{T-Propagate})
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1\,\overline{4}\,2\,\overline{3} & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{Fail})
\end{array}$$

DPLL(T) - Example

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\end{array}$$

UNSAT

DPLL(T) - Overall algorithm

High-levew view gives the same algorithm as a CDCL SAT solver:

```
while(true){

    while (propagate_gives_conflict()){
        if (decision_level==0) return UNSAT;
        else analyze_conflict();
    }

    restart_if_applicable();
    remove_lemmas_if_applicable();

    if (!decide()) returns SAT; // All vars assigned
}
```

Differences are in:

- `propagate_gives_conflict`
- `analyze_conflict`

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DPLL(T) - Propagation

```
propagate_gives_conflict( ) returns Bool

    do {

      // unit propagate
      if ( unit_prop_gives_conflict() ) then return  true

      // check T-consistency of the model
      if ( solver.is_model_inconsistent() ) then return true

      // theory propagate
      solver.theory_propagate()

    } while  (someTheoryPropagation)

    return false
```

DPLL(T) - Propagation (2)

- Three operations:
  - Unit propagation (SAT solver)
  - Consistency checks (T-solver)
  - Theory propagation (T-solver)
- Cheap operations are computed first
- If theory is expensive, calls to T-solver are sometimes skipped
- For completeness, only necessary to call T-solver at the leaves
  (i.e. when we have a full propositional model)
- Theory propagation is not necessary for completeness

DPLL(T) - Conflict Analysis

Remember conflict analysis in SAT solvers:

C`:= conflicting clause`

while C `contains more than one lit of last DL`

  l`:=last literal assigned in` C
  C`:=Resolution(`C`,reason(`l`))`

end while

```
// let C = C′ v l where l is UIP
backjump(maxDL(C′))
```
`add l to the model with reason` C
```
learn(C)
```

DPLL(T) - Conflict Analysis (2)

Conflict analysis in DPLL(T):

if `boolean conflict` **then** C`:= conflicting clause`
else C`:=`$\neg$`( solver.explain_inconsistency() )`

while C `contains more than one lit of last DL`

l`:=last literal assigned in` C
C`:=Resolution(`C`,reason(`l`))`

end while

```
// let C = C′ v l where l is UIP
backjump(maxDL(C′))
add l to the model with reason C
learn(C)
```

DPLL(T) - Conflict Analysis (3)

What does `explain_inconsistency` return?

- A (small) conjuntion of literals $l_1 \wedge \ldots \wedge l_n$ such that:
  - They were in the model when T-inconsistency was found
  - It is T-inconsistent

What is now $reason(l)$?

- If l was unit propagated, reason is the clause that propagated it
- If l was T-propagated?
  - T-solver has to provide an explanation for l, i.e. a (small) set of literals $l_1, \ldots, l_n$ such that:
    - They were in the model when l was T-propagated
    - $l_1 \wedge \ldots \wedge l_n \models_T l$
  - Then $reason(l)$ is $\neg l_1 \vee \ldots \vee \neg l_n \vee l$

DPLL(T) - Conflict Analysis (4)

Let M be of the form $\ldots, c = b, \ldots$ and let F contain

$$h(a) = h(c) \ \vee\ p \qquad a = b \ \vee\ \neg p \ \vee\ a = d \qquad a \neq d \ \vee\ a = b$$

Take the following sequence:

1. Decide $h(a) \neq h(c)$
2. UnitPropagate p (due to clause $h(a) = h(c) \ \vee\ p$)
3. T-Propagate $a \neq b$ (since $h(a) \neq h(c)$ and $c = b$)
4. UnitPropagate $a = d$ (due to clause $a = b \ \vee\ \neg p \ \vee\ a = d$)
5. Conflicting clause $a \neq d \ \vee\ a = b$

Explain($a \neq b$) is $\{h(a) \neq h(c), c = b\}$

$$\dfrac{h(a) = h(c) \vee \mathbf{p} \qquad \dfrac{h(a) = h(c) \vee c \neq b \vee \mathbf{a} \neq \mathbf{b} \qquad \dfrac{a = b \vee \neg p \vee \mathbf{a} = \mathbf{d} \qquad \mathbf{a} \neq \mathbf{d} \vee a = b}{\mathbf{a} = \mathbf{b} \vee \neg p}}{h(a) = h(c) \vee c \neq b \vee \neg \mathbf{p}}}{h(a) = h(c) \ \vee\ c \neq b}$$

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Need for combination

- In software verification, formulas like the following one arise:

$$a = b + 2 \land A = write(B, a+1, 4) \land (read(A, b+3) = 2 \lor f(a-1) \neq f(b+1))$$

- Here reasoning is needed over
  - The theory of linear arithmetic ($\mathbb{T}_{LA}$)
  - The theory of arrays ($\mathbb{T}_{A}$)
  - The theory of uninterpreted functions ($\mathbb{T}_{EUF}$)
- Remember that T-solvers only deal with conjunctions of lits.
- Given T-solvers for the three individual theories, can we combine them to obtain one for ($\mathbb{T}_{LA} \cup \mathbb{T}_{A} \cup \mathbb{T}_{EUF}$)?
- Under certain conditions the **Nelson-Oppen** combination method gives a positive answer

Motivating example - Convex case

Consider the following set of literals:

$$\begin{aligned} f(f(x)-f(y)) &= a \\ f(0) &= a+2 \\ x &= y \end{aligned}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{R})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(f(x)-f(y)) = a \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= f(x)-f(y) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= e_2 - e_3 \\ e_2 &= f(x) \\ e_3 &= f(y) \end{aligned}$$

Motivating example - Convex case

Consider the following set of literals:

$$\begin{array}{rcl} f(f(x)-f(y)) & = & a \\ f(0) & = & a+2 \\ x & = & y \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{R})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(0) = a+2 \quad \Longrightarrow \quad \begin{array}{rcl} f(e_4) & = & a+2 \\ e_4 & = & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} f(e_4) & = & e_5 \\ e_4 & = & 0 \\ e_5 & = & a+2 \end{array}$$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{lcr}
\multicolumn{3}{c}{EUF} \\
f(e_1) & = & a \\
f(x) & = & e_2 \\
f(y) & = & e_3 \\
f(e_4) & = & e_5 \\
x & = & y
\end{array}
\qquad\qquad
\begin{array}{lcr}
\multicolumn{3}{c}{Arithmetic} \\
e_2 - e_3 & = & e_1 \\
e_4 & = & 0 \\
e_5 & = & a+2
\end{array}
$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

To merge the two models into a single one, the solvers have to agree on equalities between shared constants (interface equalities)

This can be done by exchanging entailed interface equalities

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{rcl}
\multicolumn{3}{c}{EUF} \\
f(e_1) & = & a \\
f(x) & = & e_2 \\
f(y) & = & e_3 \\
f(e_4) & = & e_5 \\
x & = & y
\end{array}
\qquad
\begin{array}{rcl}
\multicolumn{3}{c}{Arithmetic} \\
e_2 - e_3 & = & e_1 \\
e_4 & = & 0 \\
e_5 & = & a+2 \\
e_2 & = & e_3
\end{array}
$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $EUF \models e_2 = e_3$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

EUF			$Arithmetic$		
$f(e_1)$	$=$	a	$e_2 - e_3$	$=$	e_1
$f(x)$	$=$	e_2	e_4	$=$	0
$f(y)$	$=$	e_3	e_5	$=$	$a+2$
$f(e_4)$	$=$	e_5	e_2	$=$	e_3
x	$=$	y			
e_1	$=$	e_4			

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $Ari \models e_1 = e_4$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{lcr}
\multicolumn{3}{c}{EUF} \\
f(e_1) & = & a \\
f(x) & = & e_2 \\
f(y) & = & e_3 \\
f(e_4) & = & e_5 \\
x & = & y \\
e_1 & = & e_4
\end{array}
\qquad\qquad
\begin{array}{lcr}
\multicolumn{3}{c}{Arithmetic} \\
e_2 - e_3 & = & e_1 \\
e_4 & = & 0 \\
e_5 & = & a+2 \\
e_2 & = & e_3 \\
a & = & e_5
\end{array}
$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $EUF \models a = e_5$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{lcl}
\multicolumn{3}{c}{EUF} \\
f(e_1) & = & a \\
f(x) & = & e_2 \\
f(y) & = & e_3 \\
f(e_4) & = & e_5 \\
x & = & y \\
e_1 & = & e_4
\end{array}
\qquad
\begin{array}{lcr}
\multicolumn{3}{c}{Arithmetic} \\
e_2 - e_3 & = & e_1 \\
e_4 & = & 0 \\
e_5 & = & a+2 \\
e_2 & = & e_3 \\
a & = & e_5
\end{array}
$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- *EUF*-Solver says SAT
- *Ari*-Solver says **UNSAT**
- Hence the original set of lits was **UNSAT**

Nelson-Oppen – The convex case

- A theory T is stably-infinite iff every T-satisfiable quantifier-free formula has an infinite model
- A theory T is convex iff, given a set of lits S
  $S \models_T a_1 = b_1 \vee \ldots \vee a_n = b_n \implies S \models_T a_i = b_i$ for some i

Deterministic Nelson-Oppen: [NO79, TH96, MZ02]

- Given two signature-disjoint, stably-infinite and convex theories T_1 and T_2
- Given a set of literals S over the signature of $T_1 \cup T_2$
- The $(T_1 \cup T_2)$-satisfiability of S can be checked with the following algorithm:

Nelson-Oppen – The convex case (2)

Deterministic Nelson-Oppen

1. Purify S and split it into $S_1 \cup S_2$.
   Let $\mathcal{E}$ the set of interface equalities between S_1 and S_2
2. If S_1 is T_1-unsatisfiable then **UNSAT**
3. If S_2 is T_2-unsatisfiable then **UNSAT**
4. If $S_1 \models_{T_1} x = y$ with $x = y \in \mathcal{E} \setminus S_2$ **then**
   $S_2 := S_2 \cup \{x = y\}$ and **goto** 3
5. If $S_2 \models_{T_2} x = y$ with $x = y \in \mathcal{E} \setminus S_1$ **then**
   $S_1 := S_1 \cup \{x = y\}$ and **goto** 2
6. Report **SAT**

Motivating example – Non-convex case

Consider the following **UNSATISFIABLE** set of literals:

$$
\begin{array}{rcl}
1 \leq & x & \leq 2 \\
f(1) & = & a \\
f(x) & = & b \\
a & = & b+2 \\
f(2) & = & f(1)+3
\end{array}
$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{Z})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$
f(1) = a \implies \begin{array}{rcl} f(e_1) & = & a \\ e_1 & = & 1 \end{array}
$$

Motivating example – Non-convex case

Consider the following **UNSATISFIABLE** set of literals:

$$\begin{array}{rcl} 1 \leq & x & \leq 2 \\ f(1) & = & a \\ f(x) & = & b \\ a & = & b+2 \\ f(2) & = & f(1)+3 \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{Z})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(2) = f(1)+3 \quad \Longrightarrow \quad \begin{array}{rcl} e_2 & = & 2 \\ f(e_2) & = & e_3 \\ f(e_1) & = & e_4 \\ e_3 & = & e_4+3 \end{array}$$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{ccc}
 & \mathit{Arithmetic} & \\
1 & \leq & x \\
x & \leq & 2 \\
e_1 & = & 1 \\
a & = & b+2 \\
e_2 & = & 2 \\
e_3 & = & e_4+3 \\
a & = & e_4
\end{array}
\qquad
\begin{array}{ccc}
 & \mathit{EUF} & \\
f(e_1) & = & a \\
f(x) & = & b \\
f(e_2) & = & e_3 \\
f(e_1) & = & e_4
\end{array}
$$

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- *Ari*-Solver says SAT
- *EUF*-Solver says SAT
- $EUF \models a = e_4$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

Arithmetic			*EUF*		
1	$\leq$	x	$f(e_1)$	$=$	a
x	$\leq$	2	$f(x)$	$=$	b
e_1	$=$	1	$f(e_2)$	$=$	e_3
a	$=$	$b+2$	$f(e_1)$	$=$	e_4
e_2	$=$	2			
e_3	$=$	e_4+3			
a	$=$	e_4			

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- *Ari*-Solver says SAT
- *EUF*-Solver says SAT
- No theory entails any other interface equality, but...

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

Arithmetic	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- *Ari*-Solver says SAT
- *EUF*-Solver says SAT
- $Ari \models_T x = e_1 \vee x = e_2$. Let's consider both cases.

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

Arithmetic	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b+2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_1$	

- *Ari*-Solver says SAT
- *EUF*-Solver says SAT
- $EUF \models_T a = b$, that when sent to *Ari* makes it **UNSAT**

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{rcr}
\multicolumn{3}{c}{\mathit{Arithmetic}} \\
1 & \leq & x \\
x & \leq & 2 \\
e_1 & = & 1 \\
a & = & b+2 \\
e_2 & = & 2 \\
e_3 & = & e_4+3 \\
a & = & e_4
\end{array}
\qquad
\begin{array}{rcr}
\multicolumn{3}{c}{\mathit{EUF}} \\
f(e_1) & = & a \\
f(x) & = & b \\
f(e_2) & = & e_3 \\
f(e_1) & = & e_4
\end{array}
$$

Let's try now with $x = e_2$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

Arithmetic	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_2$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_2$	

- *Ari*-Solver says SAT
- *EUF*-Solver says SAT
- $EUF \models_T e_3 = b$, that when sent to *Ari* makes it **UNSAT**
  (since we had $e_3 = e_4 + 3 = a + 3 = b + 5$)

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

Arithmetic	*EUF*
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b+2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_2$
$e_3 = e_4+3$	
$a = e_4$	
$x = e_2$	

Since both $x = e_1$ and $x = e_2$ are **UNSAT**, the set of literals is **UNSAT**

Nelson-Oppen - The non-convex case

- In the previous example Deterministic NO does not work
- This was because $T_{LA(\mathbb{Z})}$ is not convex:

  $S_{LA(\mathbb{Z})} \models_{T_{LA(\mathbb{Z})}} x = e_1 \vee x = e_2$, but

  $S_{LA(\mathbb{Z})} \not\models_{T_{LA(\mathbb{Z})}} x = e_1$ and

  $S_{LA(\mathbb{Z})} \not\models_{T_{LA(\mathbb{Z})}} x = e_2$
- However, there is a version of NO for non-convex theories
- Given a set constants $\mathcal{C}$, an arrangement $\mathcal{A}$ over $\mathcal{C}$ is:
  - A set of equalities and disequalites between constants in $\mathcal{C}$
  - For each $x, y \in \mathcal{C}$ either $x = y \in \mathcal{A}$ or $x \neq y \in \mathcal{A}$

Nelson-Oppen – The non-convex case (2)

Non-deterministic Nelson-Oppen: [NO79, TH96, MZ02]

- Given two signature-disjoint, stably-infinite theories T_1 and T_2
- Given a set of literals S over the signature of $T_1 \cup T_2$
- The $(T_1 \cup T_2)$-satisfiability of S can be checked via:

1. Purify S and split it into $S_1 \cup S_2$
   Let $\mathcal{C}$ be the set of shared constants
2. **For every** arrangement $\mathcal{A}$ over $\mathcal{C}$ **do**
   If $(S_1 \cup \mathcal{A})$ is T_1-satisfiable and $(S_2 \cup \mathcal{A})$ is T_2-satisfiable
   report **SAT**
3. Report **UNSAT**

Conclusions

- SMT incorporates domain-specific reasoning into SAT...
- ...but SMT is much more than that
- Lots of applications and a lot more to appear
- See references for more depth

Bibliography - Some further reading

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