COS320: Compiling Techniques

Zak Kincaid

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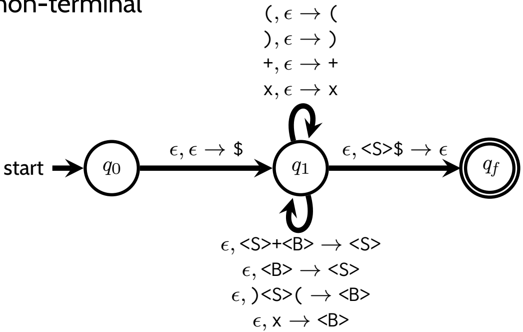
Parsing III: LR parsing

Bottom-up parsing

- Stack holds a word in $(N \cup \Sigma)^*$ such that it is possible to derive the part of the input string that has been consumed from its reverse.
- At any time, may read a letter from input string and push it on top of the stack
- At any time, may non-deterministically choose a rule $A ::= \gamma_1 ... \gamma_n$ and apply it in reverse: pop $\gamma_n ... \gamma_1$ off the top of the stack, and push A.
- Accept when stack just contains start non-terminal

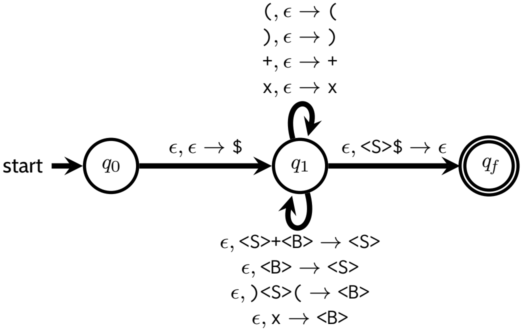
<S> ::= <B>+<S> | <B>

<B> ::= (<S>) | x



<S> ::= <B>+<S> | <B>

<B> ::= (<S>) | x



State	Stack	Input
q_0	ϵ	(x+x)+x
q_1	$	(x+x)+x
q_1	($	x+x)+x
q_1	x($	+x)+x
q_1	<B>($	+x)+x
q_1	+<B>($	x)+x
q_1	x+<B>($	)+x
q_1	<B>+<B>($	)+x
q_1	<S>+<B>($	)+x
q_1	<S>($	)+x
q_1	)<S>($	+x
q_1	<B>$	+x
q_1	+<B>$	x
q_1	x+<B>$	ϵ
q_1	<B>+<B>$	ϵ
q_1	<S>+<B>$	ϵ
q_1	<S>$	ϵ
q_f	ϵ	ϵ

LL vs LR

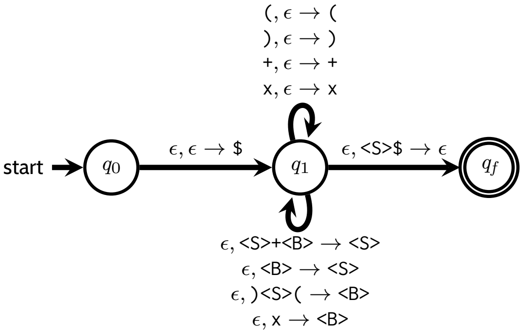
- LL parsers are top-down, LR parsers are bottom-up
- Easier to write LR grammars
  - Every LL(k) grammar is also LR(k), but not vice versa.
  - No need to eliminate left (or right) recursion
  - No need to left-factor
- Harder to write LR parsers
  - But parser generators will do it for us!

Bottom-up PDA has two kinds of actions:

- *Shift*: move lookahead token to the top of the stack
- *Reduce*: remove $\gamma_n, \ldots, \gamma_1$ from the top of the stack, replace with A (where $A ::= \gamma_1 \ldots \gamma_n$ is a rule of the grammar)
- Just as for LL parsing, the trick is to resolve non-determinism.
  - When should the parser shift?
  - When should the parser reduce?

<S> ::= <B>+<S> | <B>

<B> ::= (<S>) | x



Roadmap to LR parsing

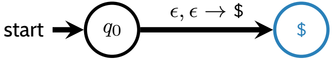
1. "Greedy" determinization: warm-up (not examinable material)
2. LR(0): LR parsing with 0 tokens of lookahead – not used in practice.
3. SLR (Simple LR): LR(0) + lookahead to resolve some nondeterminism
4. LR(1): Add one token of lookahead to LR construction
5. LALR(1): simple, practical optimization of LR(1) (but less powerful!)

Determinizing the bottom-up PDA

- **Intuition**: reduce greedily
  - If any reduce action applies, then apply it
    - Actually, a bit more nuanced: only apply reduction action if it is “relevant” (can eventually lead to the input word being accepted)
  - If no reduce action applies, then shift
- Can use the states of the PDA to implement greedy strategy
  - State tracks top few symbols of the stack – enough to know if a reduction rule applies.

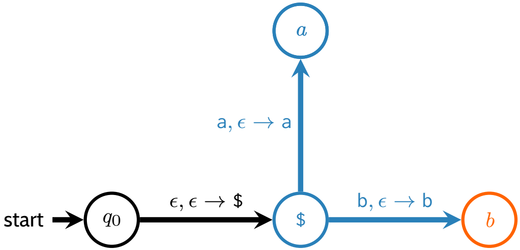
<S'> ::= <S>$

<S> ::= a<S>a | b



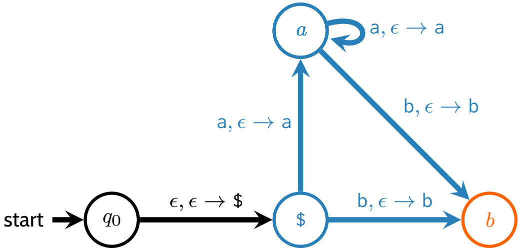
<S’> ::= <S>$

<S> ::= a<S>a | b



<S'> ::= <S>$

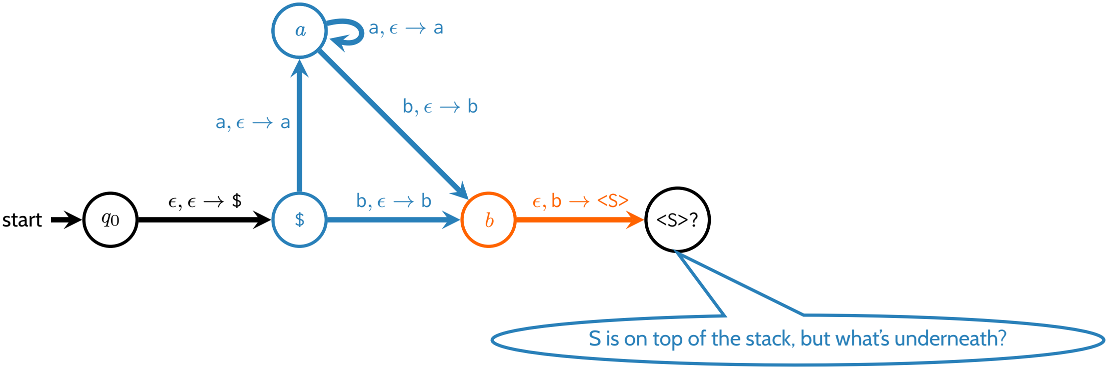
<S> ::= a<S>a | b



start
q_0
$\epsilon, \epsilon \to \$$
$
a
b
a, $\epsilon \to$ a
a, $\epsilon \to$ a
b, $\epsilon \to$ b
b, $\epsilon \to$ b

<S'> ::= <S>$

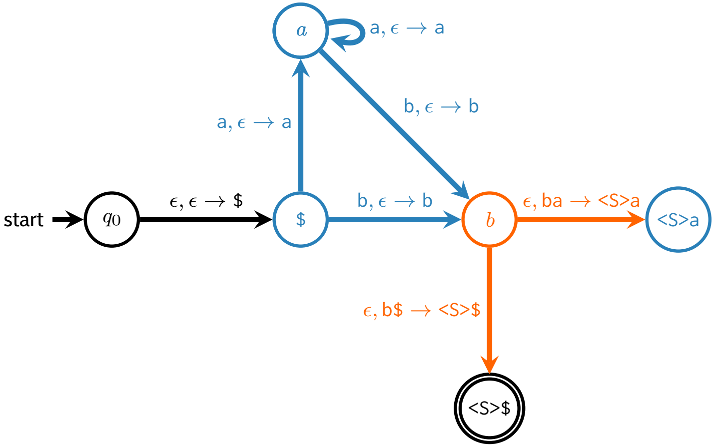
<S> ::= a<S>a | b



start
q_0
$\epsilon, \epsilon \to \$$
$
a
a, ϵ → a
a, ϵ → a
b, ϵ → b
b, ϵ → b
b
ϵ, b → <S>
<S>?
S is on top of the stack, but what's underneath?

<S'> ::= <S>$

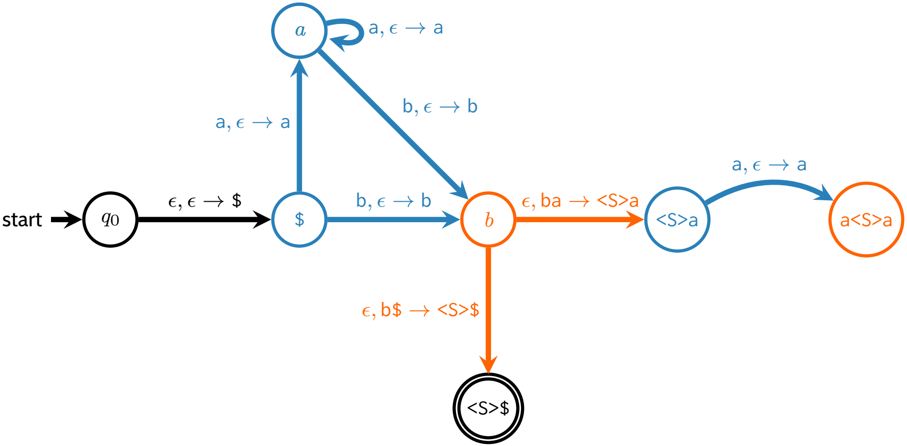
<S> ::= a<S>a | b



start
q_0
$\epsilon, \epsilon \to$ $
$
a
a, $\epsilon \to$ a
a, $\epsilon \to$ a
b, $\epsilon \to$ b
b, $\epsilon \to$ b
b
ϵ, ba $\to$ <S>a
<S>a
ϵ, b$ $\to$ <S>$
<S>$

<S'> ::= <S>$

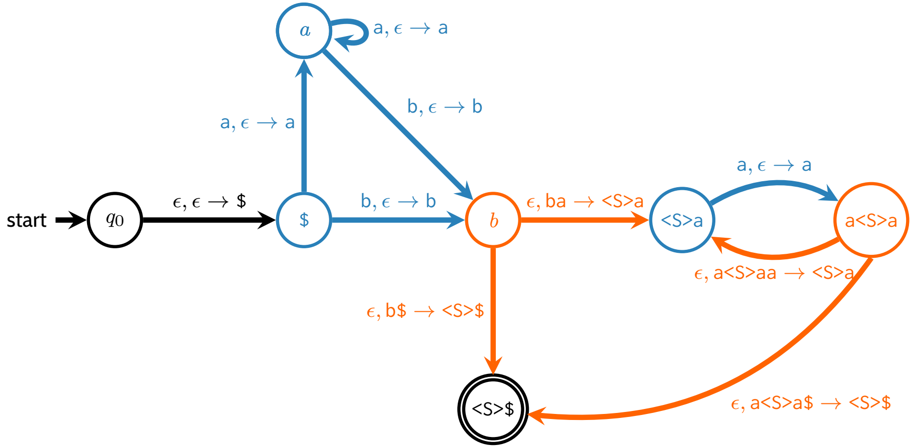
<S> ::= a<S>a | b



start
q_0
$\epsilon, \epsilon \rightarrow$ $
$
a
a, $\epsilon \rightarrow$ a
a, $\epsilon \rightarrow$ a
b, $\epsilon \rightarrow$ b
b, $\epsilon \rightarrow$ b
b
ϵ, ba $\rightarrow$ <S>a
<S>a
a, $\epsilon \rightarrow$ a
a<S>a
ϵ, b$ $\rightarrow$ <S>$
<S>$

<S’> ::= <S>$

<S> ::= a<S>a | b



start
q_0
$\epsilon, \epsilon \to \$$
$
a
a, ϵ → a
a, ϵ → a
b, ϵ → b
b, ϵ → b
b
ϵ, ba → <S>a
<S>a
a, ϵ → a
a<S>a
ϵ, a<S>aa → <S>a
ϵ, b$ → <S>$
<S>$
ϵ, a<S>a$ → <S>$

LR parsing

- Greedy strategy matches right-hand-sides of *all* rules against the top of the stack
  - Consider <S> ::= <A><B>, <A> ::= a, <B> ::= a
  - a on top of stack $\Rightarrow$ conflict between reductions <A> ::= a and <B> ::= a
- LR parsing is *partially* greedy: only apply reduction action if it is "relevant" (can eventually lead to the input word being accepted)
  - E.g., apply <A> ::= a reduction to the *first* a that we push on the stack, but not the *second*.
- $LR(k)$ = LR with k-symbol lookahead

LR(0) parsing

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- An **LR(0) item** of a grammar $G = (N, \Sigma, R, S)$ is of the form $A ::= \gamma_1 \ldots \gamma_i \bullet \gamma_{i+1} \ldots \gamma_n$, where $A ::= \gamma_1 \cdots \gamma_n$ is a rule of G
  - $\gamma_1 \ldots \gamma_i$ derives part of the word that has already been read
  - $\gamma_{i+1} \ldots \gamma_n$ derives part of the word that remains to be read
  - LR(0) items $\sim$ states of an NFA that determines when a reduction applies to the top of the stack
- LR(0) items for the above grammar:
  - `<S> ::= •(<L>)`, `<S> ::= (•<L>)`, `<S> ::= (<L>•)`, `<S> ::= (<L>)•`,
  - `<S> ::= •x`, `<S> ::= x•`,
  - `<L> ::= •<S>`, `<L> ::= <S>•`,
  - `<L> ::= •<L>;<S>`, `<L> ::= <L>•;<S>`, `<L> ::= <L>;•<S>`, `<L> ::= <L>;<S>•`,

closure and goto

- For any set of items I, define **closure**(I) to be the least set of items such that
  - **closure**(I) contains I
  - If **closure**(I) contains an item of the form $A ::= \alpha \bullet B\beta$ where B is a non-terminal, then **closure**(I) contains $B ::= \bullet\gamma$ for all $B ::= \gamma \in R$
- **closure**(I) saturates I with all items that may be relevant to reducing via I
  - E.g., **closure**$(\{$`<S> ::= (•<L>)`$\}) =$ $\{$`<S> ::= (•<L>)`, `<L> ::= •<S>`, `<L> ::= •<L>;<S>`, `<S> ::= •(<L>)<S> ::= •x`$\}$
  - Part of the not-quite greedy strategy: don't try to reduce using all rules all the time, track only a relevant subset

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  - E.g., **closure**$(\{\texttt{<S> ::= (}\bullet\texttt{<L>)}\}) =$
    $\{\texttt{<S> ::= (}\bullet\texttt{<L>)}, \texttt{<L> ::= }\bullet\texttt{<S>}, \texttt{<L> ::= }\bullet\texttt{<L>;<S>}, \texttt{<S> ::= }\bullet\texttt{(<L>)<S> ::= }\bullet\texttt{x}\}$
  - Part of the not-quite greedy strategy: don't try to reduce using all rules all the time, track only a relevant subset
- For any item set I, and (terminal or non-terminal) symbol $\gamma \in N \cup \Sigma$ define **goto**$(I, \gamma) =$ **closure**$(\{A ::= \alpha\gamma \bullet \beta \mid A ::= \alpha \bullet \gamma\beta \in I\})$
  - I.e., **goto**(I, γ) is the result of "moving $\bullet$ across γ"
  - E.g., **goto**(**closure**$(\{\texttt{<S> ::= (}\bullet\texttt{<L>)}\}), \texttt{<L>}) = \{\texttt{<S> ::= (<L>}\bullet\texttt{)}, \texttt{<L> ::= <L>}\bullet\texttt{;<S>}, \}$

Mechanical construction of LR(0) parsers

1. Add a new production $S' ::= S\$$ to the grammar.
   - S' is new start symbol
   - $\$$ marks end of word
2. Stack alphabet = closed item sets, starting with **closure**$(\{S' ::= \bullet S\$\})$
3. Construct transitions as follows: for each closed item set I,
   - For each item of the form $A ::= \gamma_1 ... \gamma_n \bullet$ in I, add *reduce* transition

     $$\epsilon, IJ_1 ... J_{n-1}K \to K'K, \text{ where } K' = \textbf{goto}(K, A)$$

   - For each item of the form $A ::= \gamma \bullet a\beta$ in I with $a \in \Sigma$, add a *shift* transition

     $$a, I \to I'I \text{ where } I' = \textbf{goto}(I, a)$$

Resulting automaton is deterministic $\iff$ grammar is LR(0)

Conflicts

- Recall: Automaton is deterministic $\iff$ grammar is LR(0)
- Two different types of transitions:
  - *Reduce* transitions, from items of the form $A ::= \gamma \bullet$
  - *Shift* transitions, from items of the form $A ::= \gamma \bullet a\beta$, where a is a terminal
  - (No transitions generated by items of the formu $A ::= \gamma \bullet A\beta$ where A is a non-terminal)

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- Reduce/reduce conflict: state has two or more items of the form $A ::= \gamma\bullet$ (choice of reduction is non-deterministic!)

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  - (No transitions generated by items of the formu $A ::= \gamma \bullet A\beta$ where A is a non-terminal)
- Reduce/reduce conflict: state has two or more items of the form $A ::= \gamma \bullet$ (choice of reduction is non-deterministic!)
- Shift/reduce conflict: state has an item of the form $A ::= \gamma \bullet$ *and* one of the form $A ::= \gamma \bullet a\beta$ (choice of whether to shift or reduce is non-deterministic!)

Simple LR (SLR)

- Simple LR is a straight-forward extension of LR(0) with a lookahead token
- **Idea**: proceed exactly as LR(0), but eliminate (some) conflicts using lookahead token
  - For each item of the form $A ::= \gamma_1 ... \gamma_n \bullet$ in I, add *reduce* transition

$$\epsilon, IJ_1 ... J_{n-1} K \rightarrow K' K, \text{ where } K' = \textbf{goto}(K, A)$$

with any lookahead token in **follow**(A)

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  with any lookahead token in **follow**(A)
- Example: the following grammar is SLR, but not LR(0)

$$\text{<S>} ::= \text{<T>b}$$

$$\text{<T>} ::= \text{a<T>} \mid \epsilon$$

  Consider: **closure**$(\{$`<S'> ::= •<S>$`$\})$ contains `<T> ::= •` and `<T> ::= •a<T>`.
- SLR parser generators: Jison

LR(1) parser construction

- LR(1) parser generators: Menhir, Bison
- An **LR(1) item** of a grammar $G = (N, \Sigma, R, S)$ is of the form $(A ::= \gamma_1 ... \gamma_i \bullet \gamma_{i+1} ... \gamma_n, a)$, where $A ::= \gamma_1 \cdots \gamma_n$ is a rule of G and $a \in \Sigma$
  - $\gamma_1 ... \gamma_i$ derives part of the word that has already been read
  - $\gamma_{i+1} ... \gamma_n$ derives part of the word that remains to be read
  - a is a lookahead symbol
- For any set of items I, define **closure**(I) to be the least set of items such that
  - **closure**(I) contains I
  - If **closure**(I) contains an item of the form $(A ::= \alpha \bullet B\beta, a)$ where B is a non-terminal, then **closure**(I) contains $(B ::= \bullet\gamma, b)$ for all $B ::= \gamma \in R$ and all $b \in$ **first**(βa).
- Construct PDA as in LR(0)

LALR(1)

- LR(1) transition tables can be very large
- LALR(1) ("lookahead LR(1)") make transition table smaller by merging states (that is, closed itemsets) that are identical except for lookahead
- Merging states can create reduce/reduce conflicts. Say that a grammar is LALR(1) if this merging *doesn't* create conflicts.
- LALR(1) parser generators: Bison, Yacc, ocamlyacc, Jison

Summary of parsing

- For any k, $LL(k)$ grammars are $LR(k)$
- SLR grammars are $LALR(1)$ are $LR(1)$
- In terms of *language expressivity*, there is an SLR (and therefore LALR(1) and LR(1) grammar for any context-free language that can be accepted by a deterministic pushdown automaton).
- Not every deterministic context free language is LL(k): $\{a^n b^n : n \in \mathbb{N}\} \cup \{a^n c^n : n \in \mathbb{N}\}$ is DCFL but not LL(k) for any k.[1]

[1] John C. Beatty, *Two iteration theorems for the LL(k) Languages*