https://algs4.cs.princeton.edu

4.3 Minimum Spanning Trees

- *introduction*
- *cut property*
- *edge-weighted graph API*
- *Kruskal's algorithm*
- *Prim's algorithm*

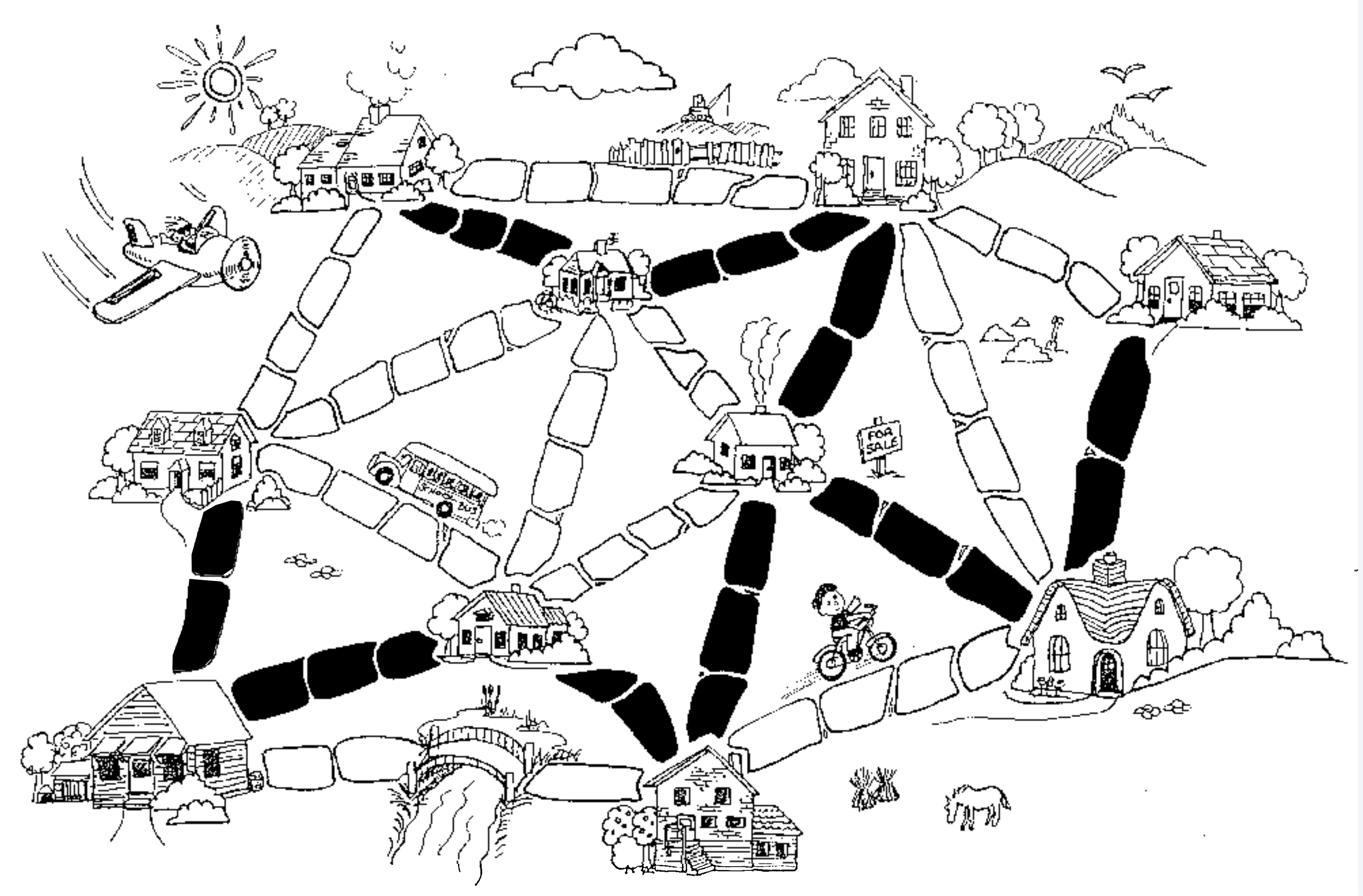
https://algs4.cs.princeton.edu

4.3 MINIMUM SPANNING TREES

- *introduction*
- *cut property*
- *edge-weighted graph API*
- *Kruskal's algorithm*
- *Prim's algorithm*

A motivating example

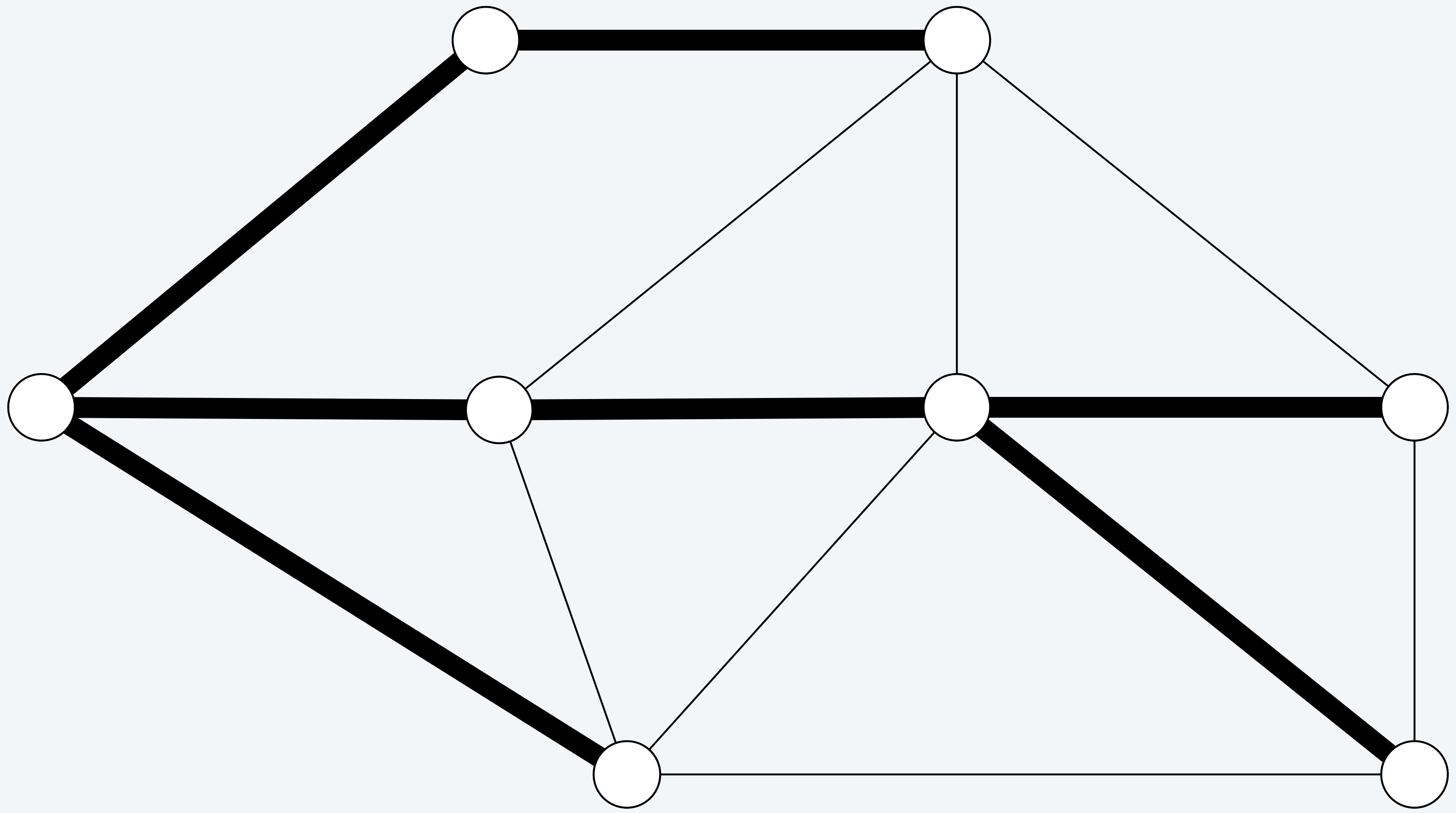
Install minimum number of paving stones to connect all of the houses.



Spanning tree

Def. A spanning tree of G is a subgraph T that is:

- A tree: connected and acyclic.
- Spanning: includes all of the vertices.



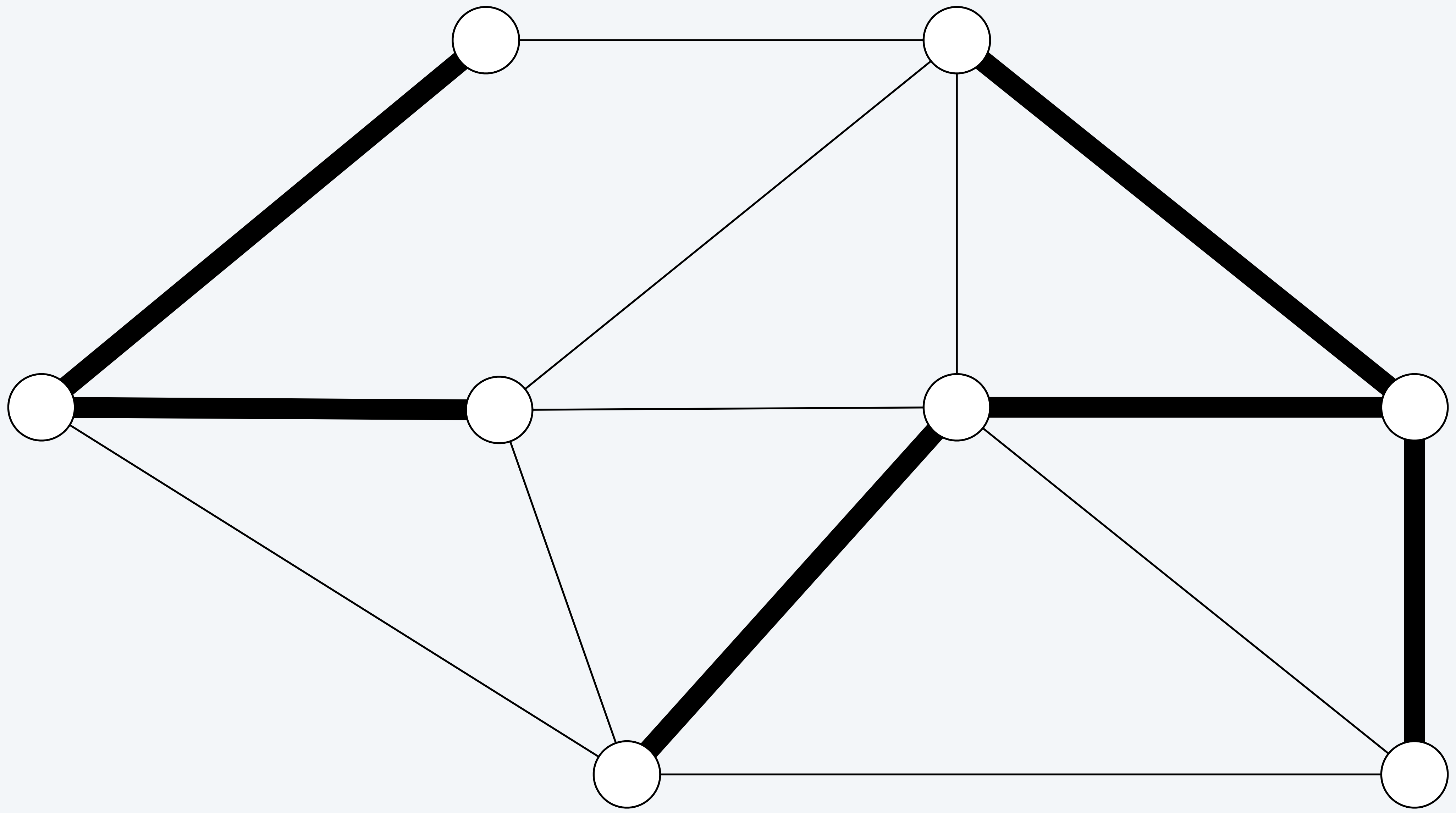
graph G

spanning tree T

Spanning tree

Def. A spanning tree of G is a subgraph T that is:

- A tree: connected and acyclic.
- Spanning: includes all of the vertices.

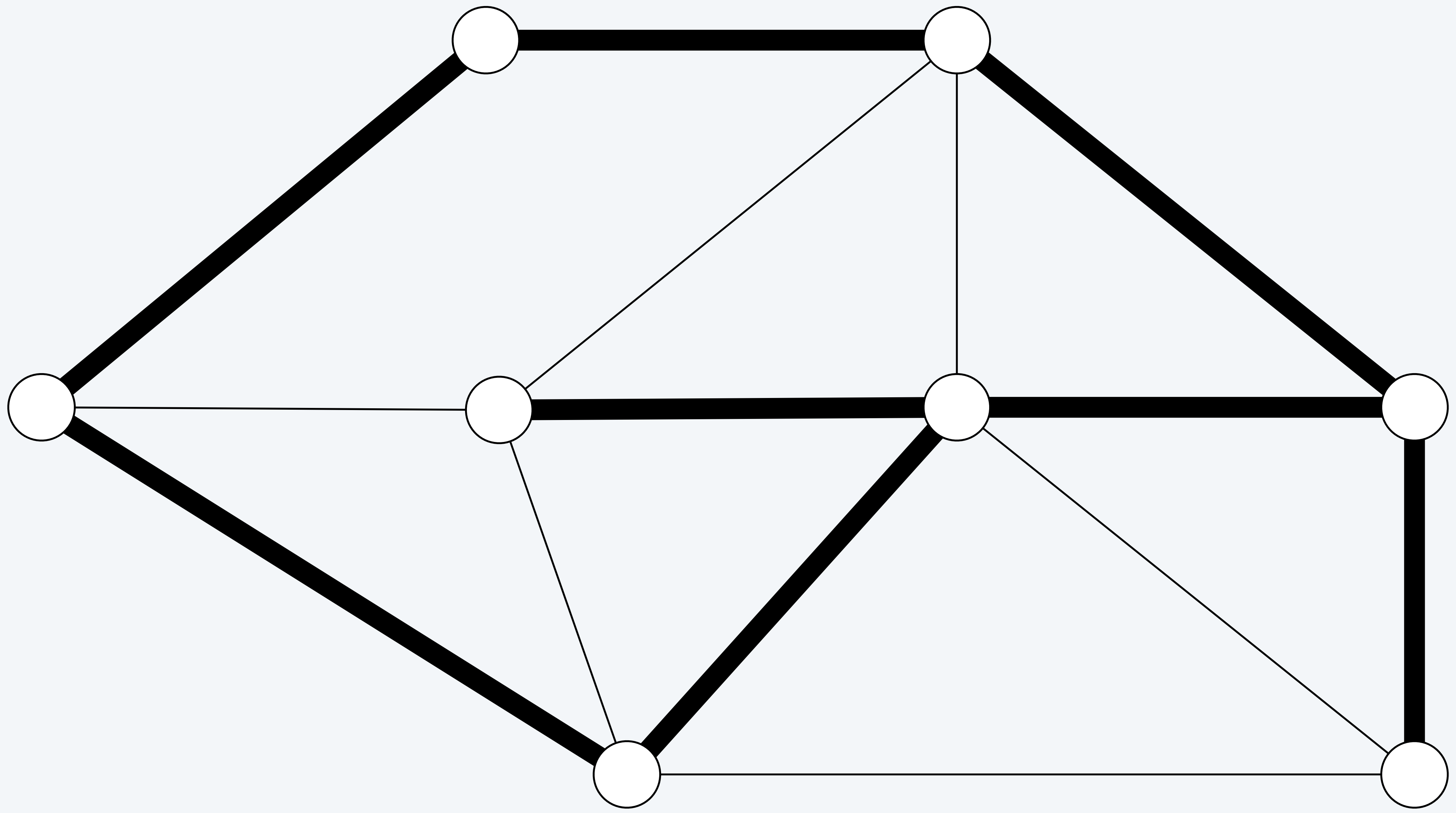


not connected

Spanning tree

Def. A spanning tree of G is a subgraph T that is:

- A tree: connected and acyclic.
- Spanning: includes all of the vertices.

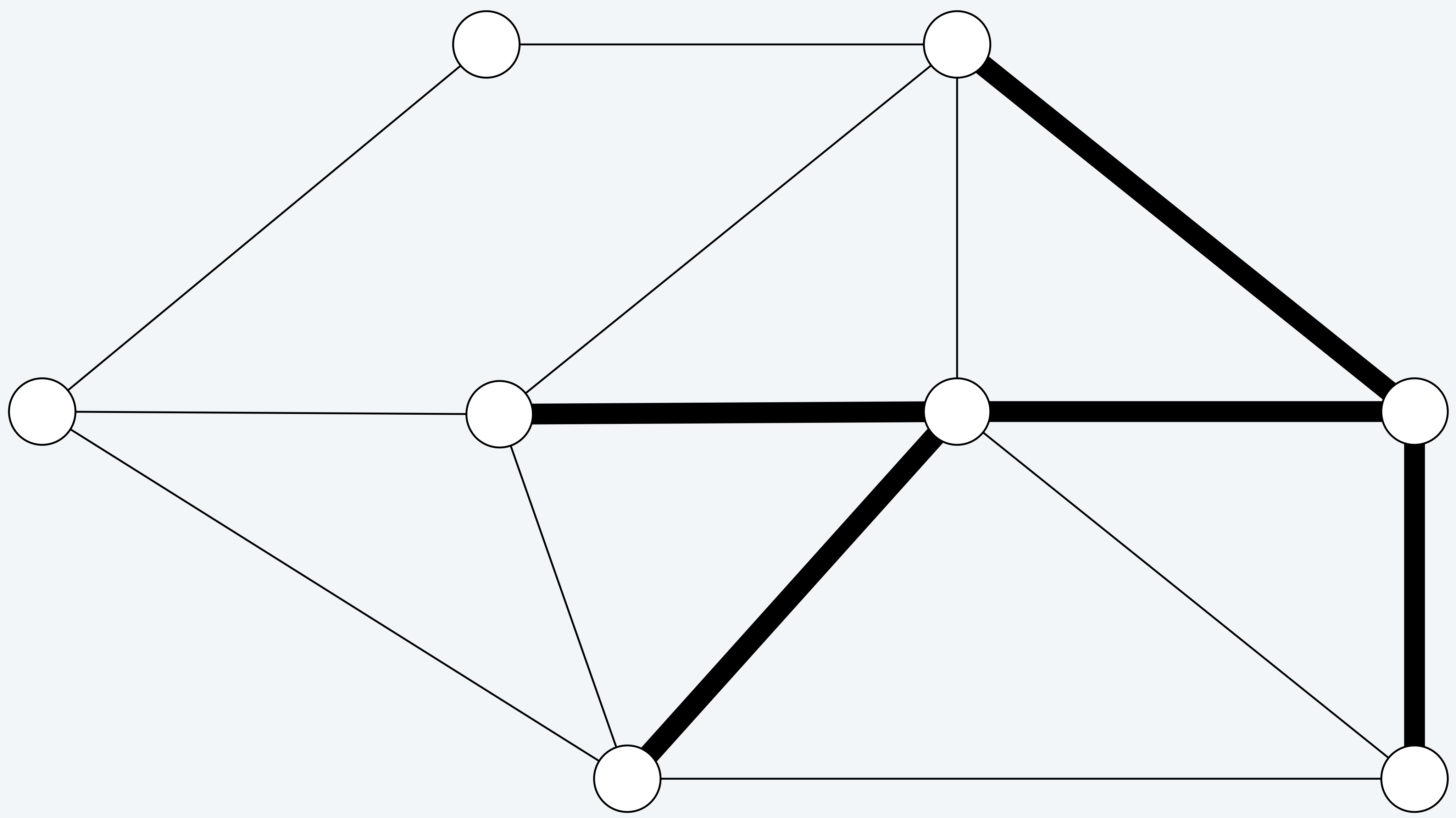


not acyclic

Spanning tree

Def. A spanning tree of G is a subgraph T that is:

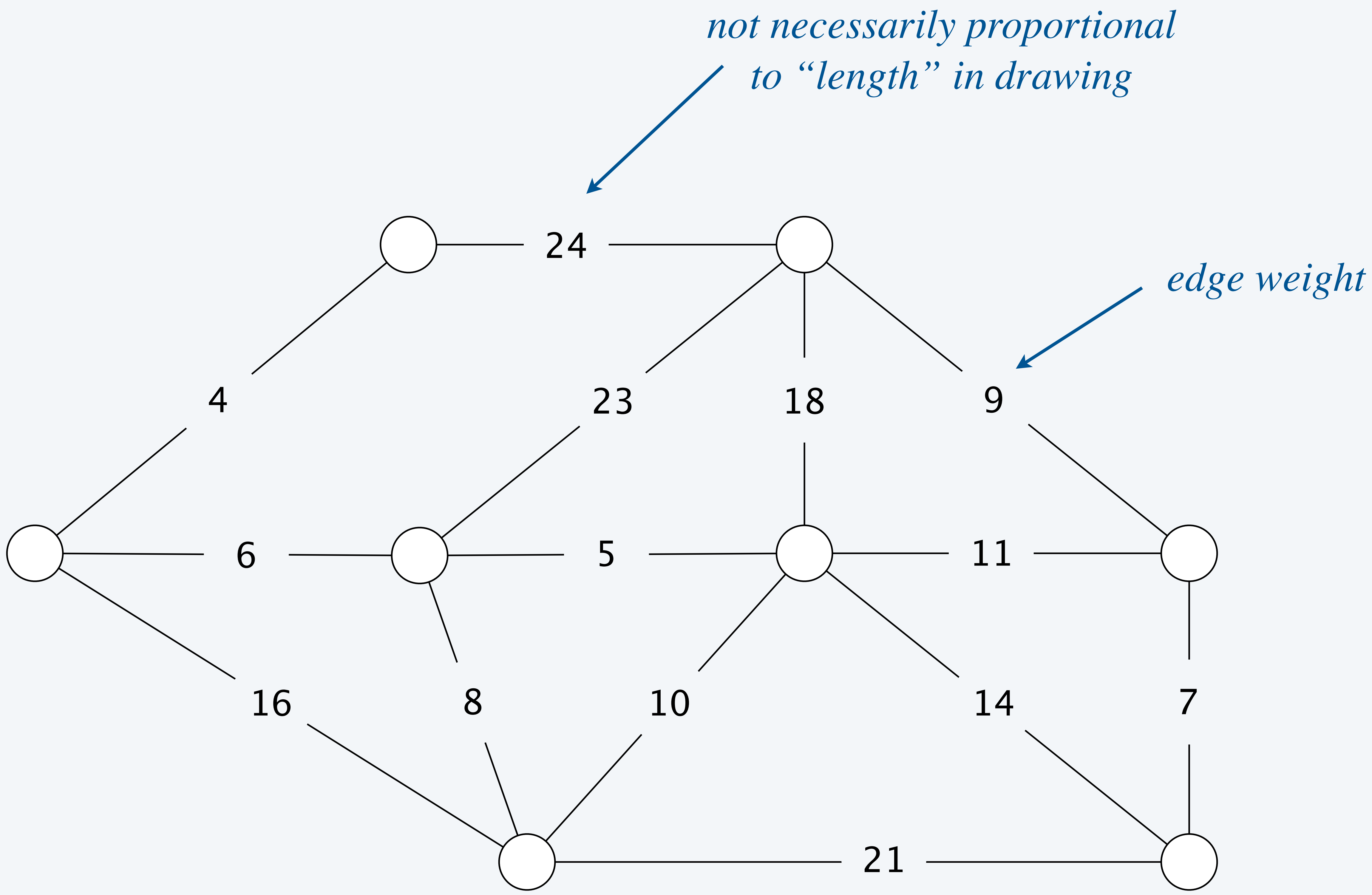
- A tree: connected and acyclic.
- Spanning: includes all of the vertices.



not spanning

Minimum spanning tree problem

Input. Connected, undirected graph G with positive edge weights.

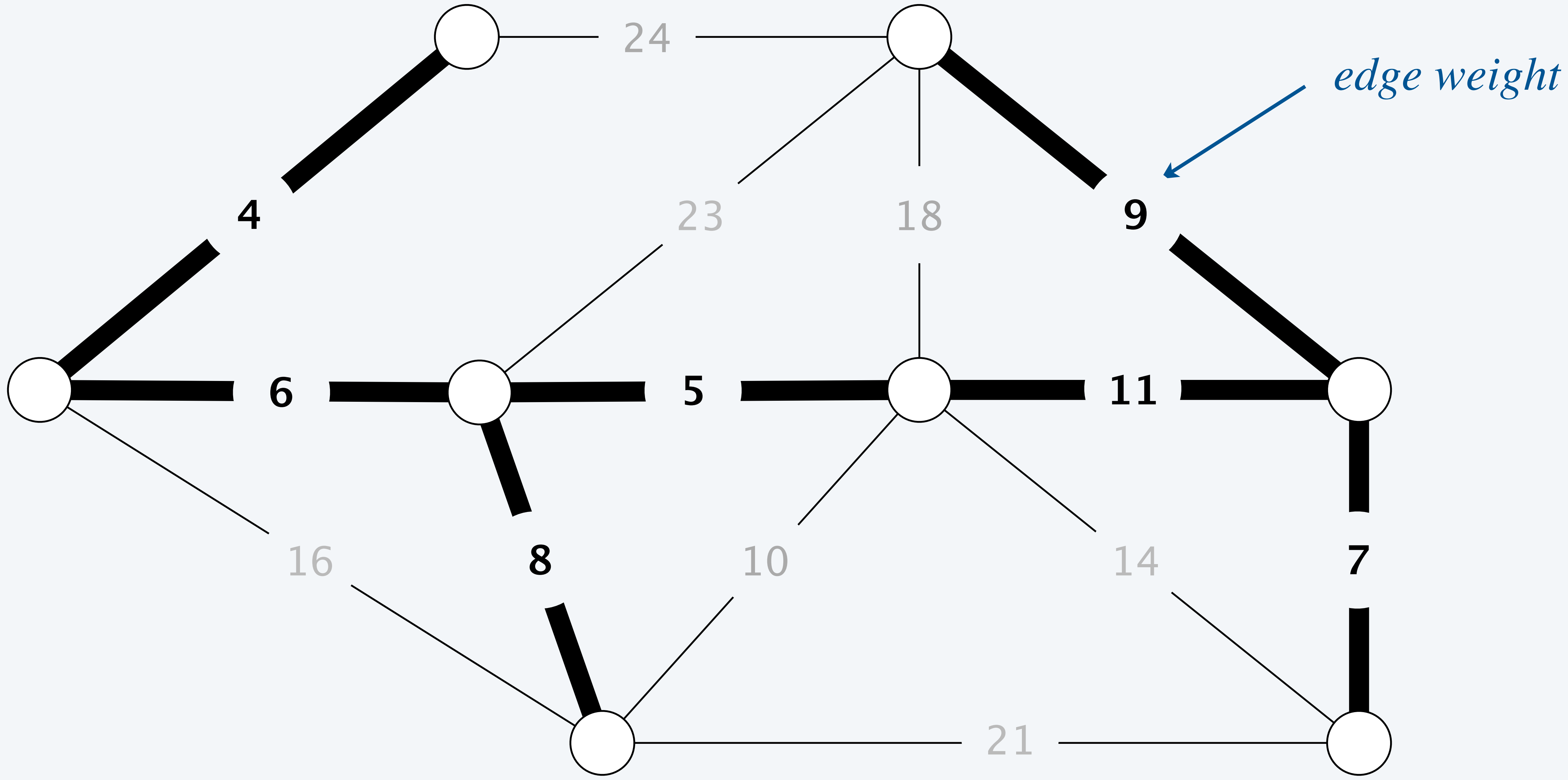


edge-weighted graph G

Minimum spanning tree problem

Input. Connected, undirected graph G with positive edge weights.

Output. A spanning tree of minimum weight.



minimum spanning tree T
(weight = 50 = 4 + 6 + 5 + 8 + 9 + 11 + 7)

Brute force. Try all spanning trees?

Minimum spanning trees: quiz 1

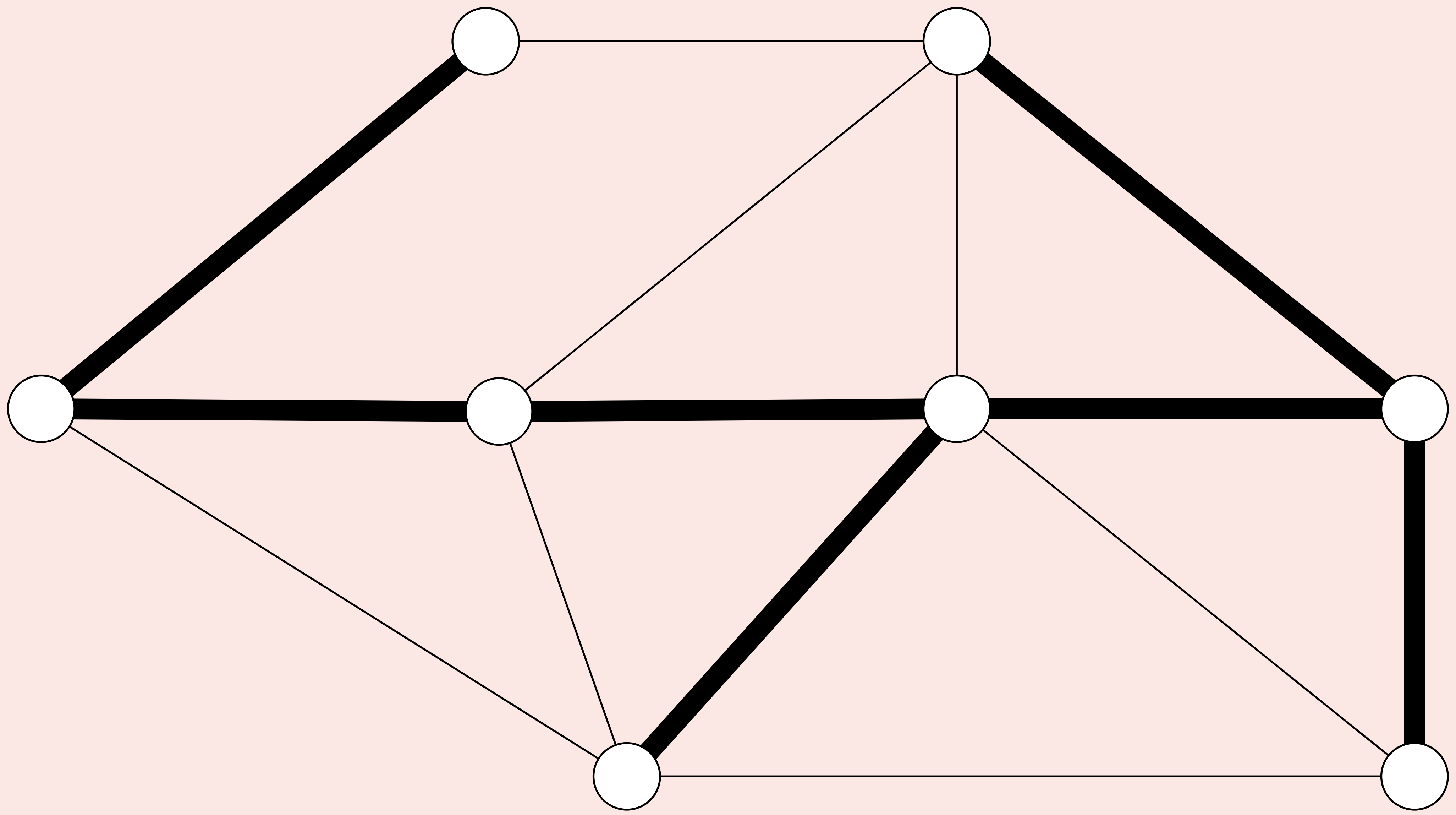
Let T be any spanning tree of a connected graph G with V vertices.
Which of the following properties must hold?

A. Removing any edge from T disconnects it.

B. Adding any edge to T creates a cycle.

C. T contains exactly $V - 1$ edges.

D. All of the above.

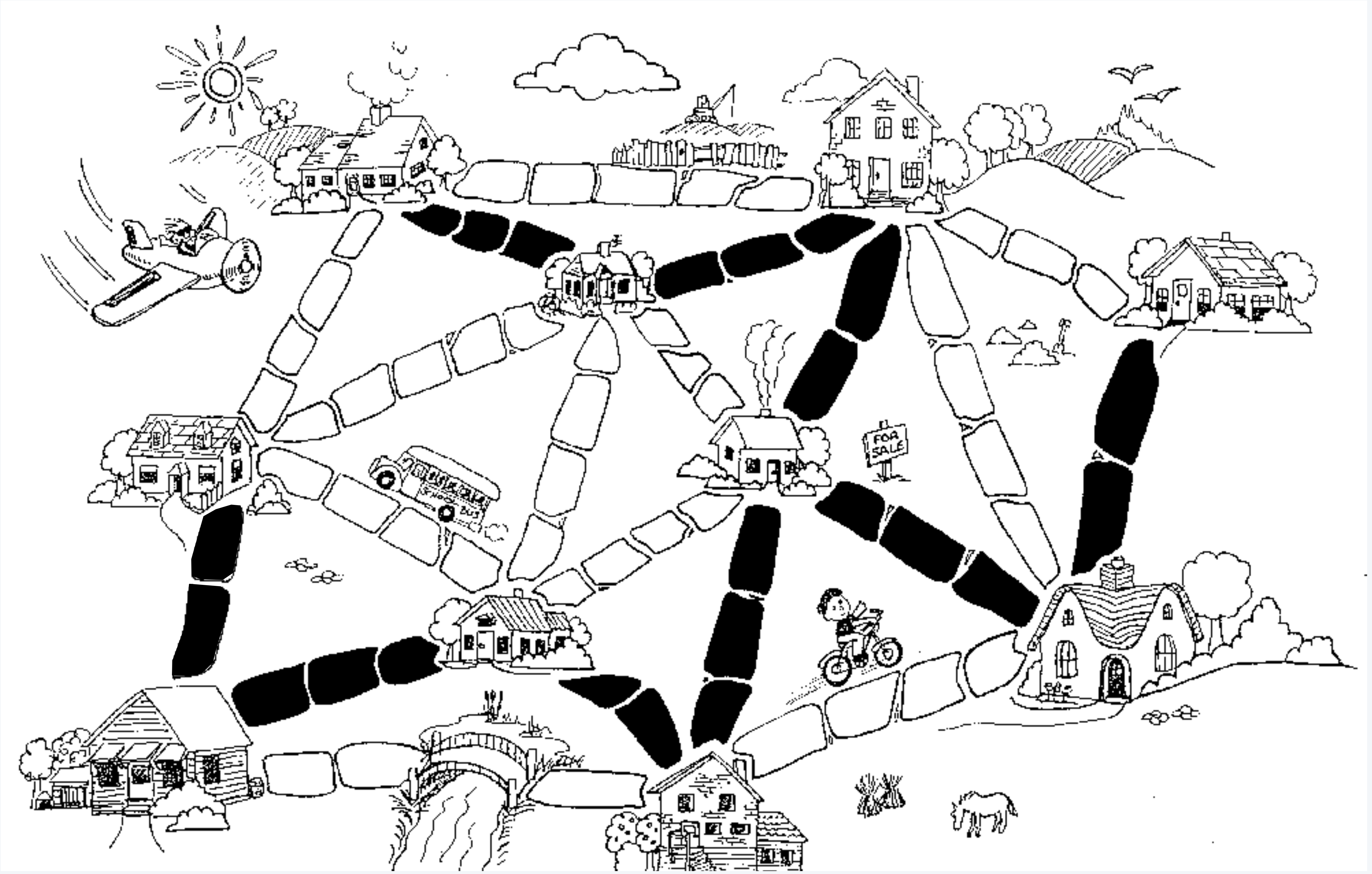


spanning tree T of graph G

Network design

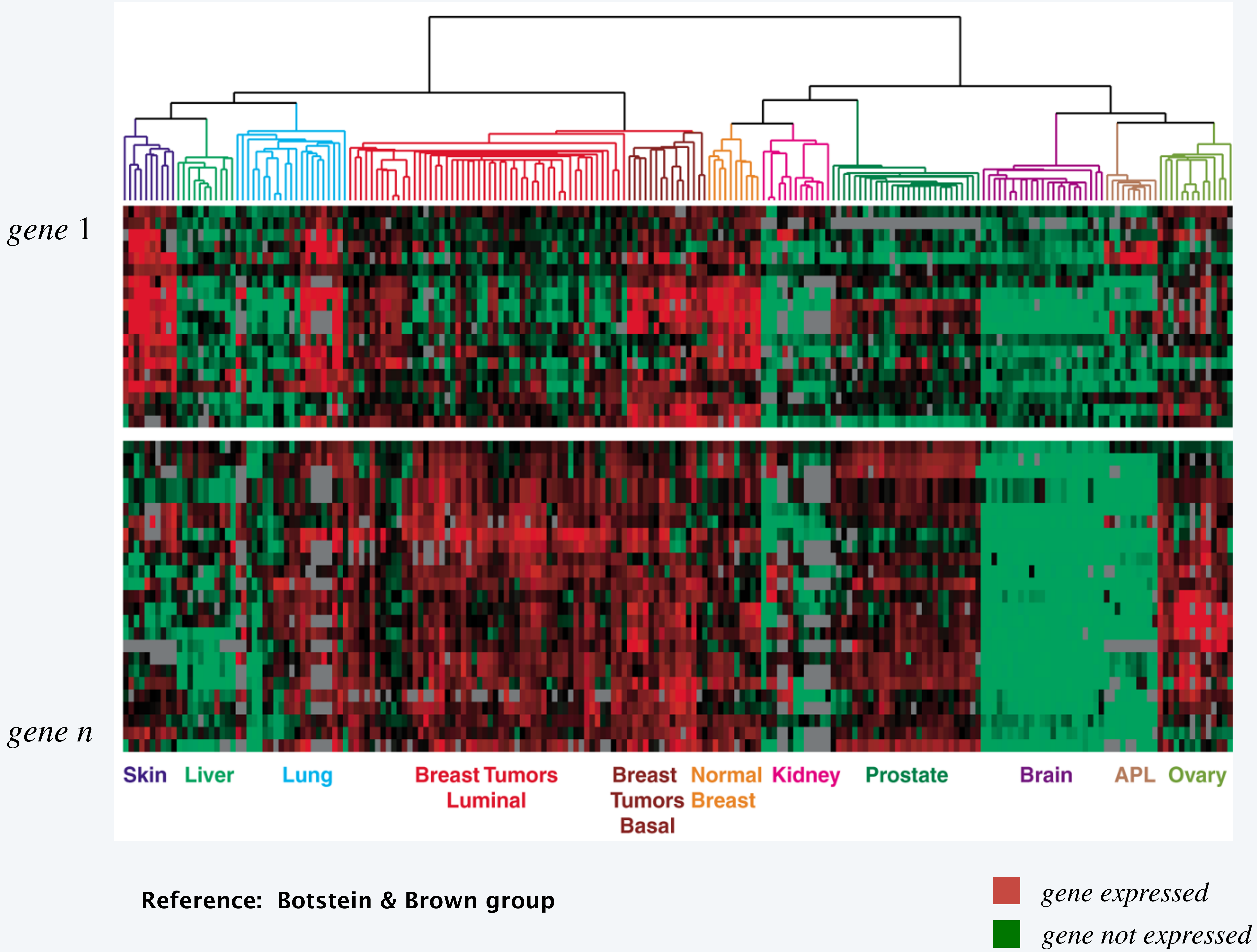
Network. Vertex = network component; edge = potential connection; edge weight = cost.

electrical, computer, telecommunication, transportation



Hierarchical clustering

Microarray graph. Vertex = cancer tissue; edge = all pairs; edge weight = dissimilarity.



Reference: Botstein & Brown group

More MST applications

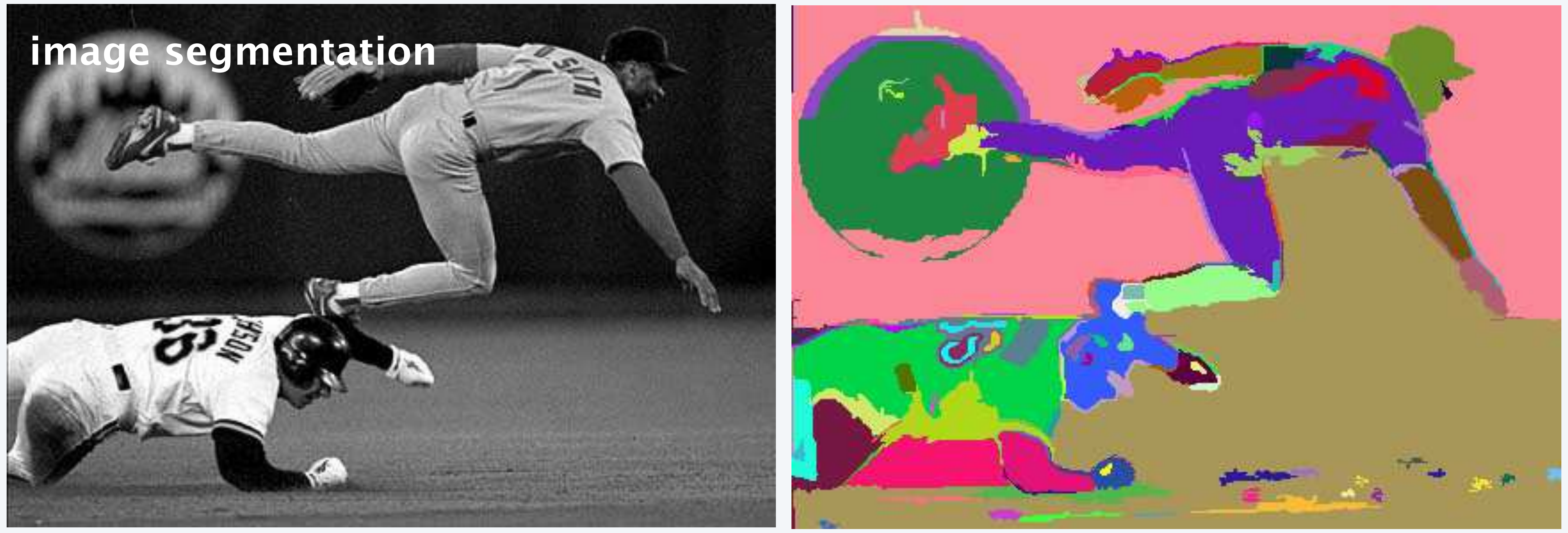
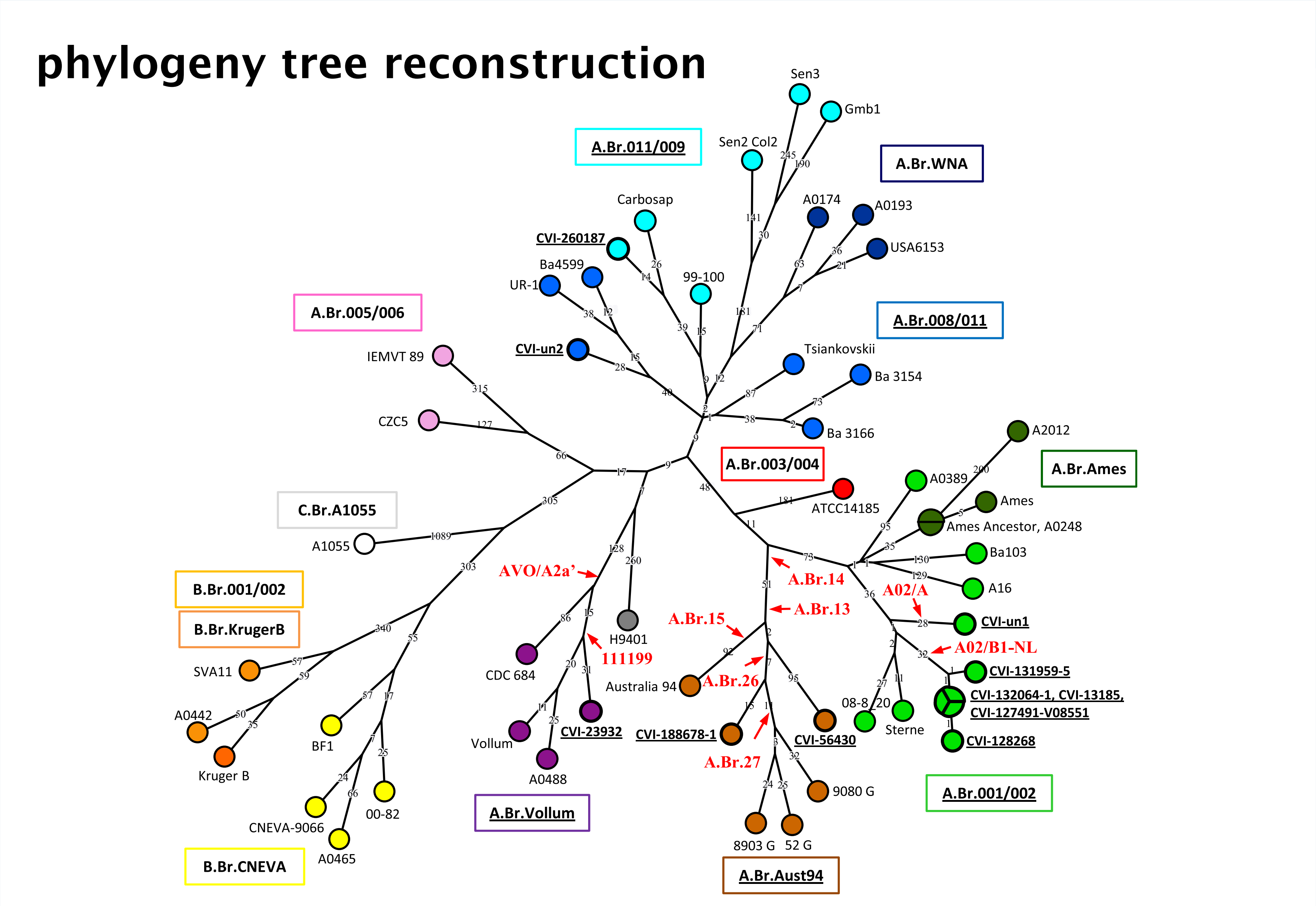
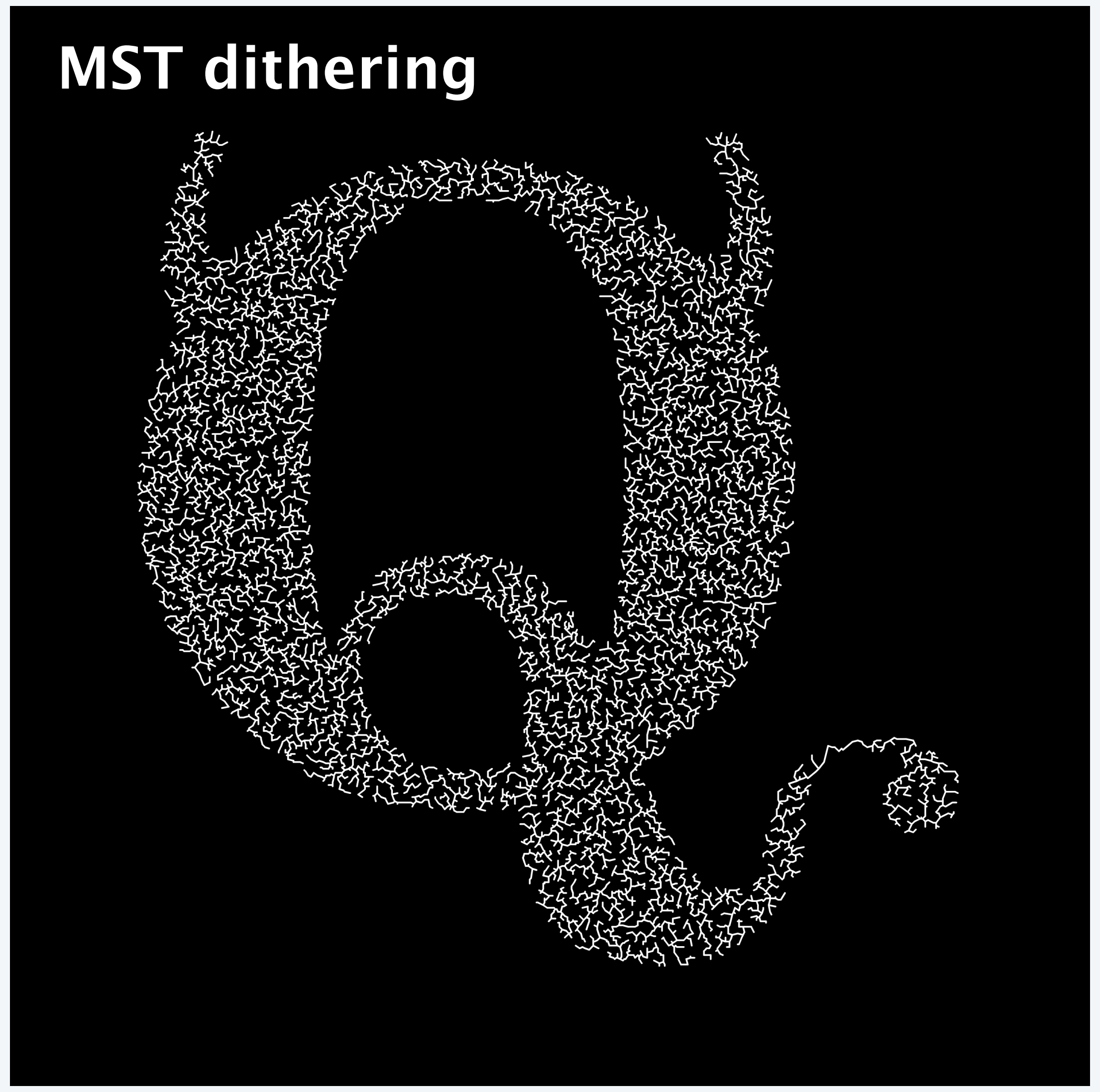
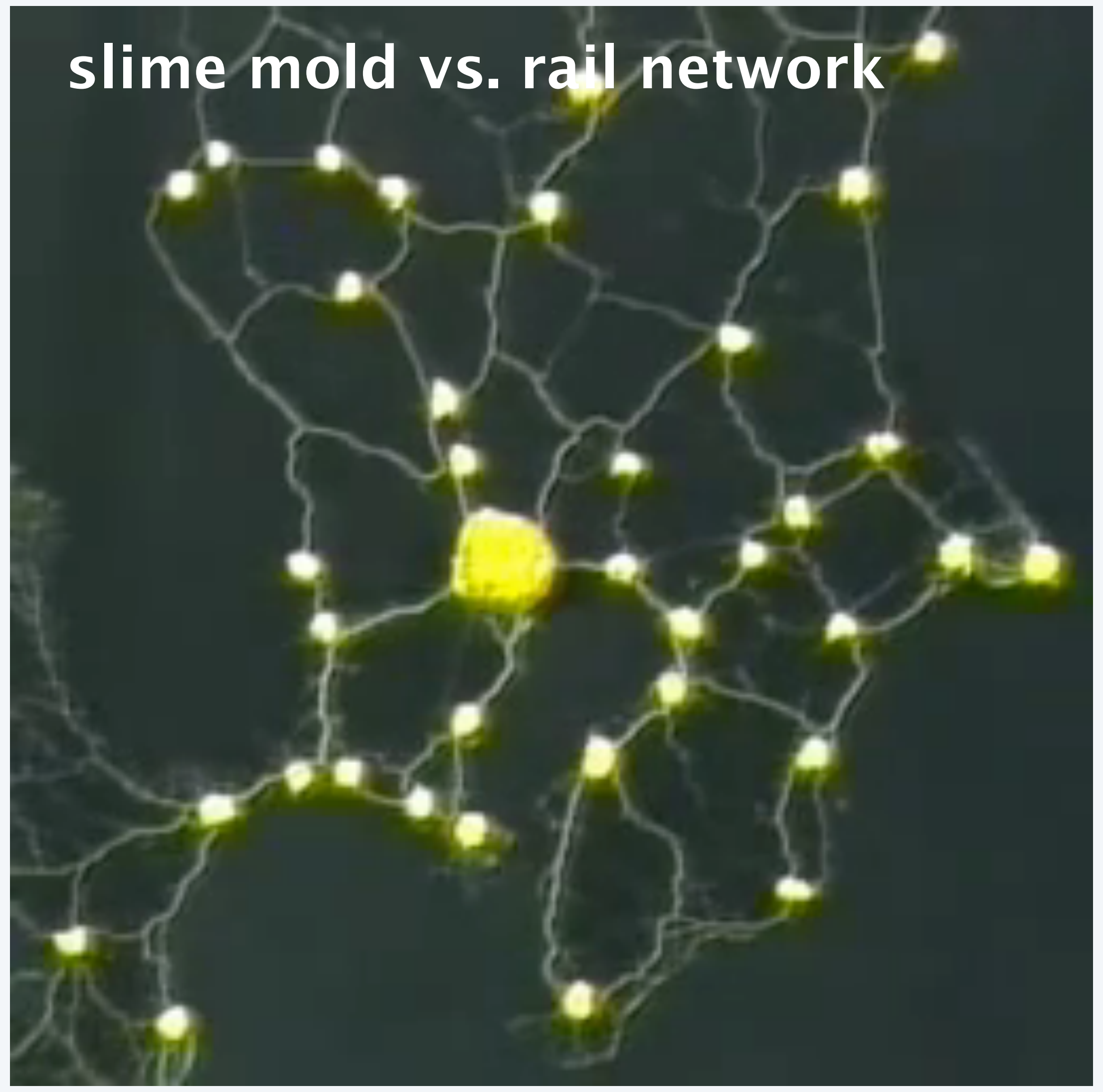


image segmentation


phylogeny tree reconstruction


MST dithering


slime mold vs. rail network

https://algs4.cs.princeton.edu

4.3 Minimum Spanning Trees

- introduction
- cut property
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm

Simplifying assumptions

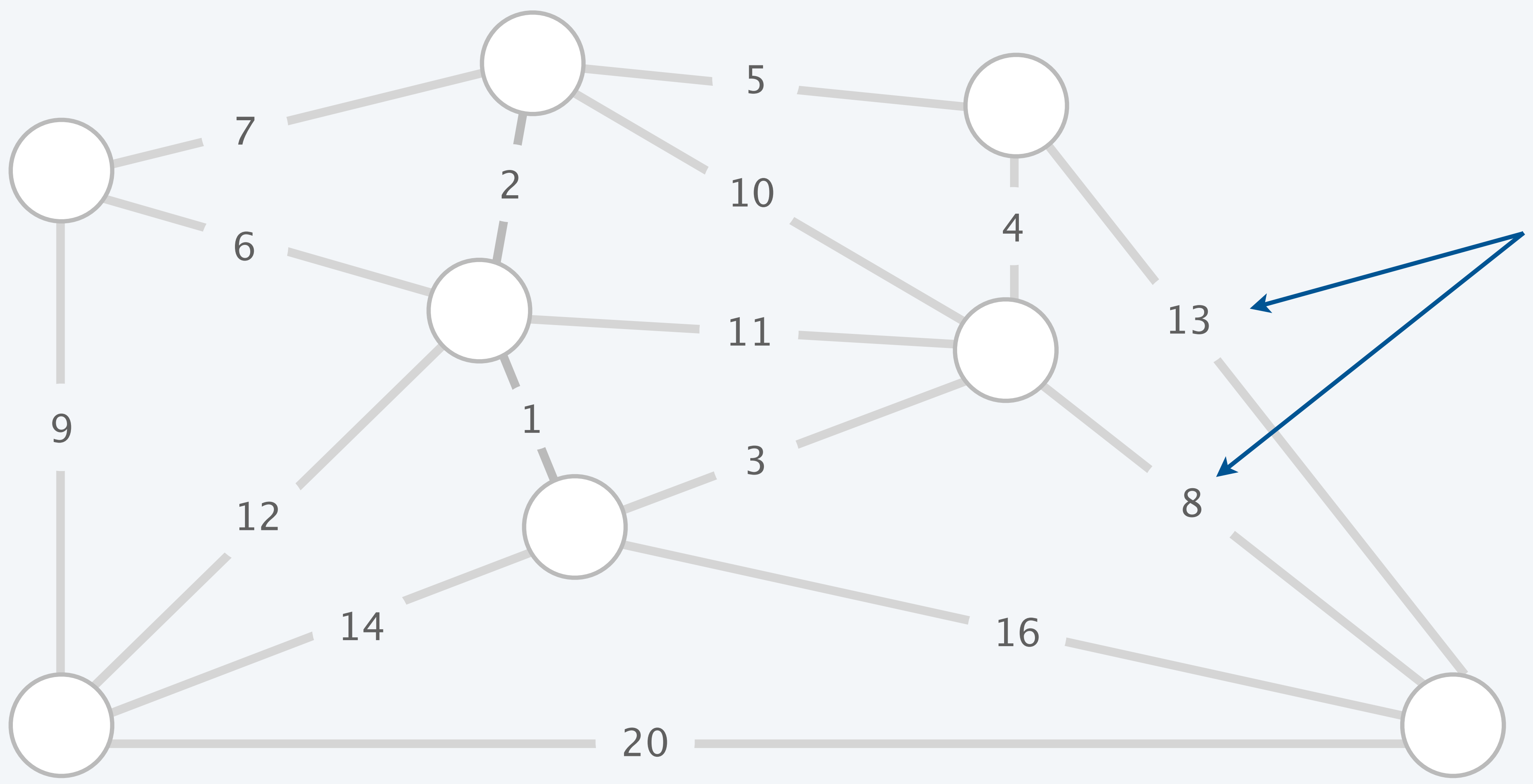
For simplicity, we assume:

- The graph is connected. $\Rightarrow$ MST exists.
- The edge weights are distinct. $\Rightarrow$ MST is unique.

see Exercise 4.3.3 (solved on booksite)

Note. Today's algorithms all work with duplicate edge weights.

but assumption simplifies the analysis



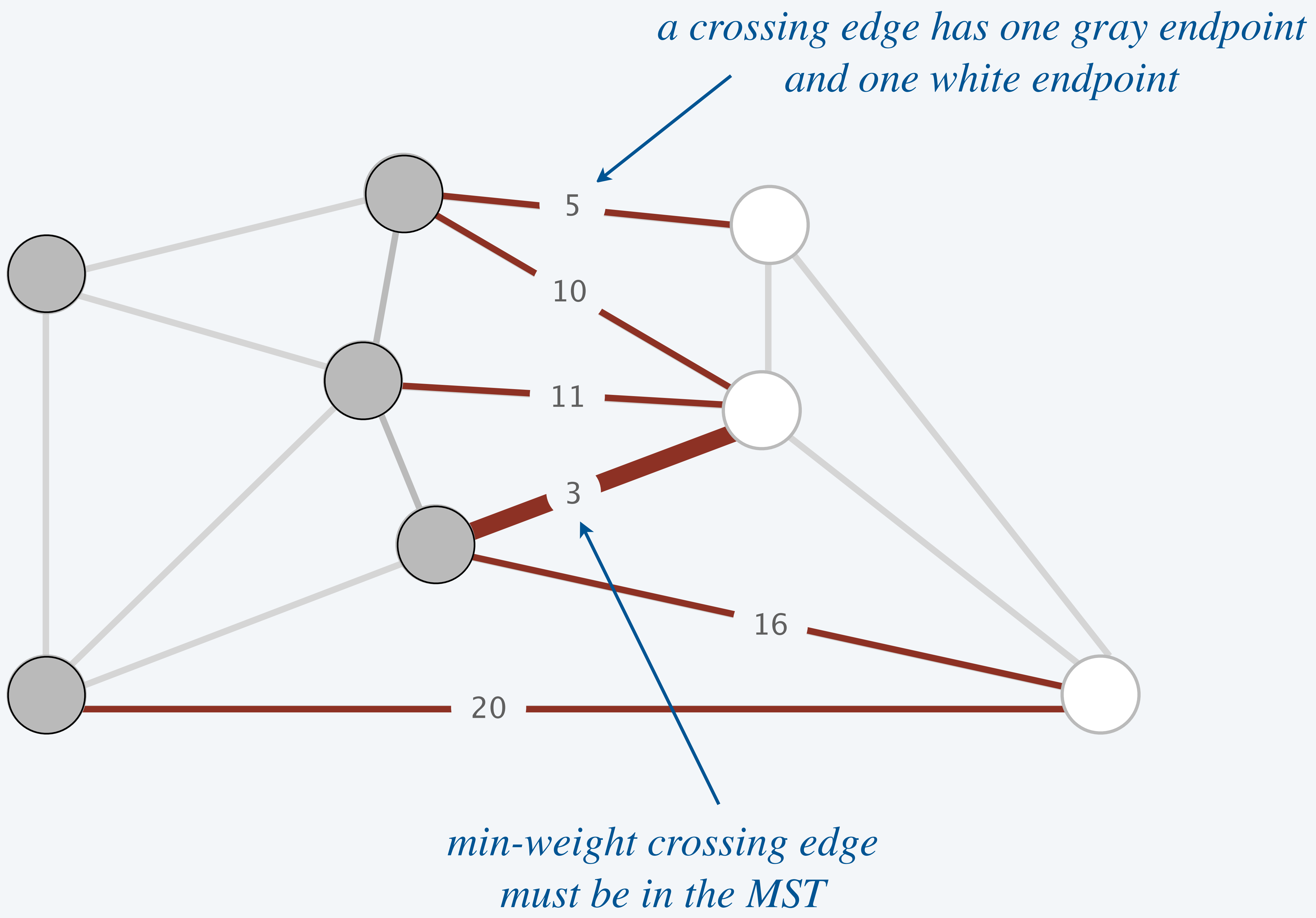
no two edge weights are equal

Cut property

Def. A cut in a graph is a partition of its vertices into two nonempty sets.

Def. A crossing edge of a cut is an edge that has one endpoint in each set.

Cut property. For any cut, its min-weight crossing edge is in the MST.



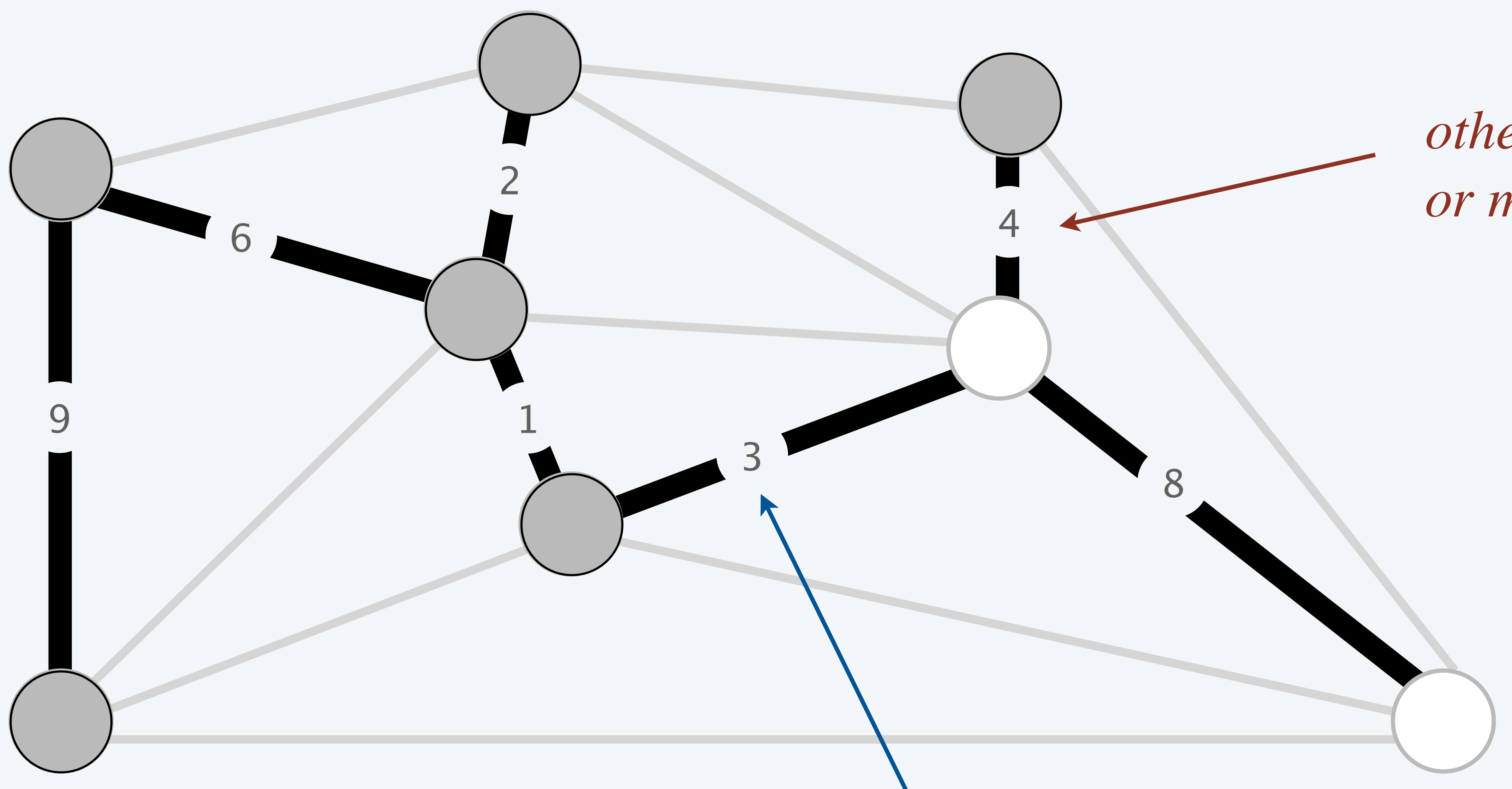
Cut property

Def. A cut in a graph is a partition of its vertices into two nonempty sets.

Def. A crossing edge of a cut is an edge that has one endpoint in each set.

Cut property. For any cut, its min-weight crossing edge is in the MST.

Note. A cut may have multiple edges in the MST.



Minimum spanning trees: quiz 2

Which is the min-weight edge crossing the cut { 2, 3, 5, 6 } ?

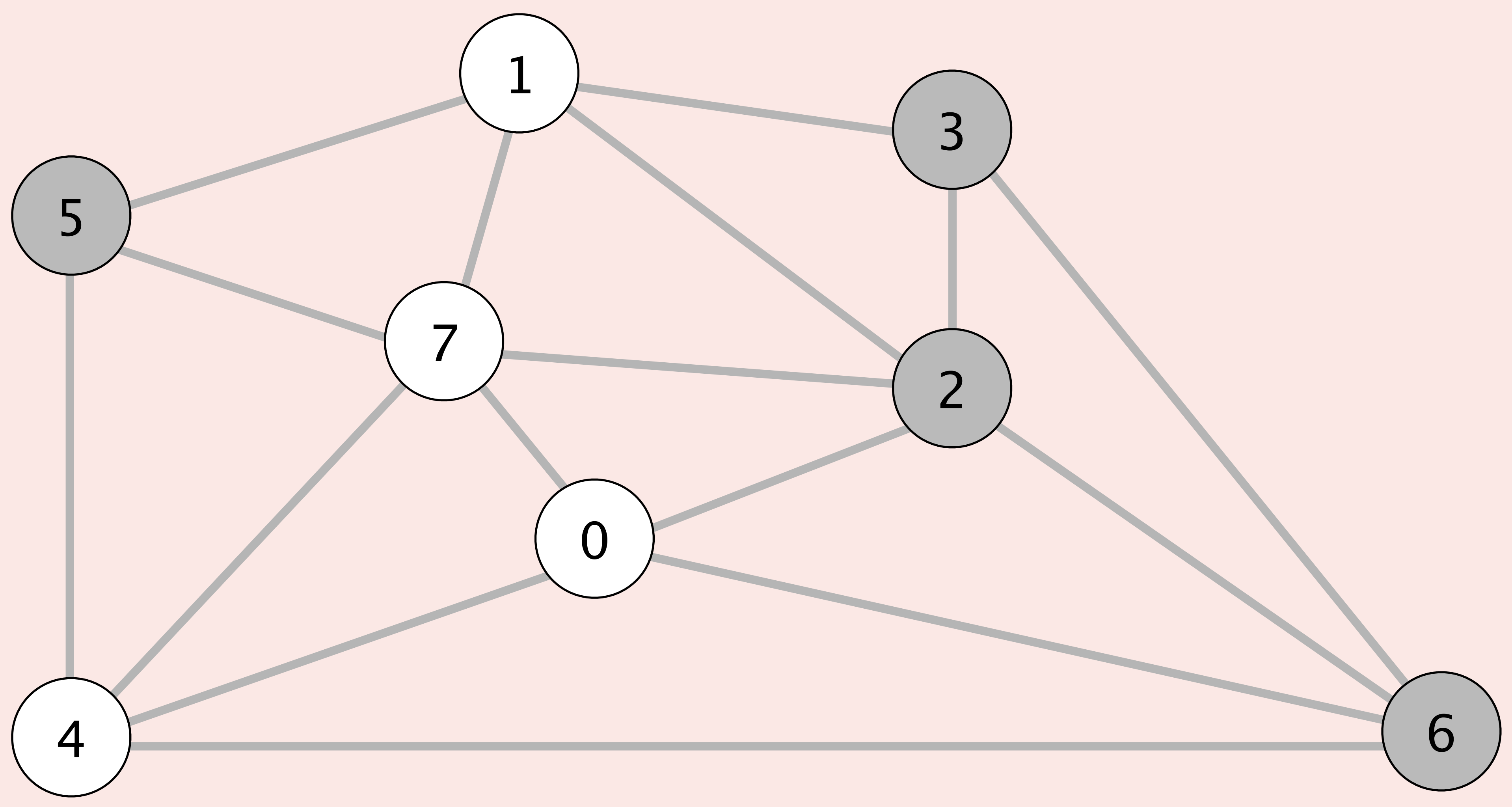
A. 0–7 (0.16)

B. 2–3 (0.17)

C. 0–2 (0.26)

D. 5–7 (0.28)

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



Cut property: correctness proof

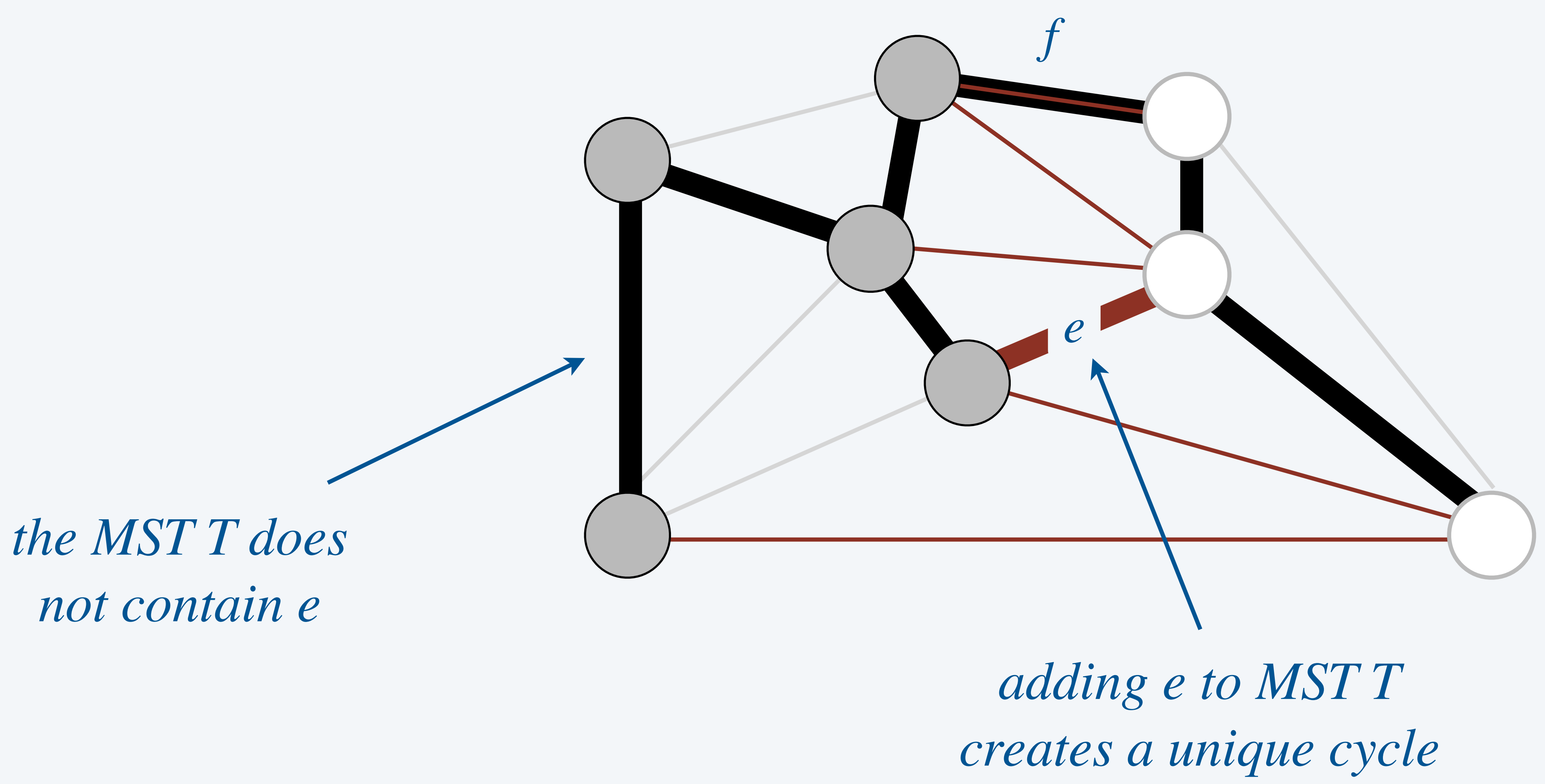
Def. A cut in a graph is a partition of its vertices into two nonempty sets.

Def. A crossing edge of a cut is an edge that has one endpoint in each set.

Cut property. For any cut, its min-weight crossing edge e is in the MST.

Pf. [by contradiction] Suppose e is not in the MST T.

- Adding e to T creates a unique cycle.
- Some other edge f in cycle must also be a crossing edge.
- Removing f and adding e to T yields a different spanning tree T'.
- Since $weight(e) < weight(f)$, we have $weight(T') < weight(T)$.
- Contradiction. ▪



Framework for minimum spanning tree algorithms

Generic algorithm (to compute MST in G)

T = ∅.

Repeat until T is a spanning tree: ⟵ *$V-1$ edges*

- **Find a cut in G.**
- **e ← min-weight crossing edge.**
- **T ← T ∪ { e }.**

Efficient implementations.

- Which cut? ⟵ *2^{V-2} distinct cuts*
- How to compute min-weight crossing edge?

Ex 1. Kruskal's algorithm.

Ex 2. Prim's algorithm.

Ex 3. Borüvka's algorithm.

https://algs4.cs.princeton.edu

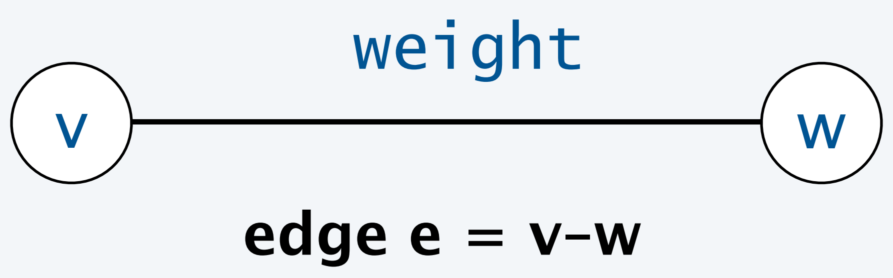
4.3 Minimum Spanning Trees

- introduction
- cut property
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm

Weighted edge API

API. Edge abstraction for weighted edges.

public class Edge implements Comparable<Edge>		
	`Edge(int v, int w, double weight)`	*create a weighted edge v–w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`int`	`compareTo(Edge that)`	*compare edges by weight*
	⋮	⋮



Idiom for processing an edge e. `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge> {
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight) {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either() {
     return v;
   }

   public int other(int vertex) {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that) {
      return Double.compare(this.weight, that.weight);
   }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

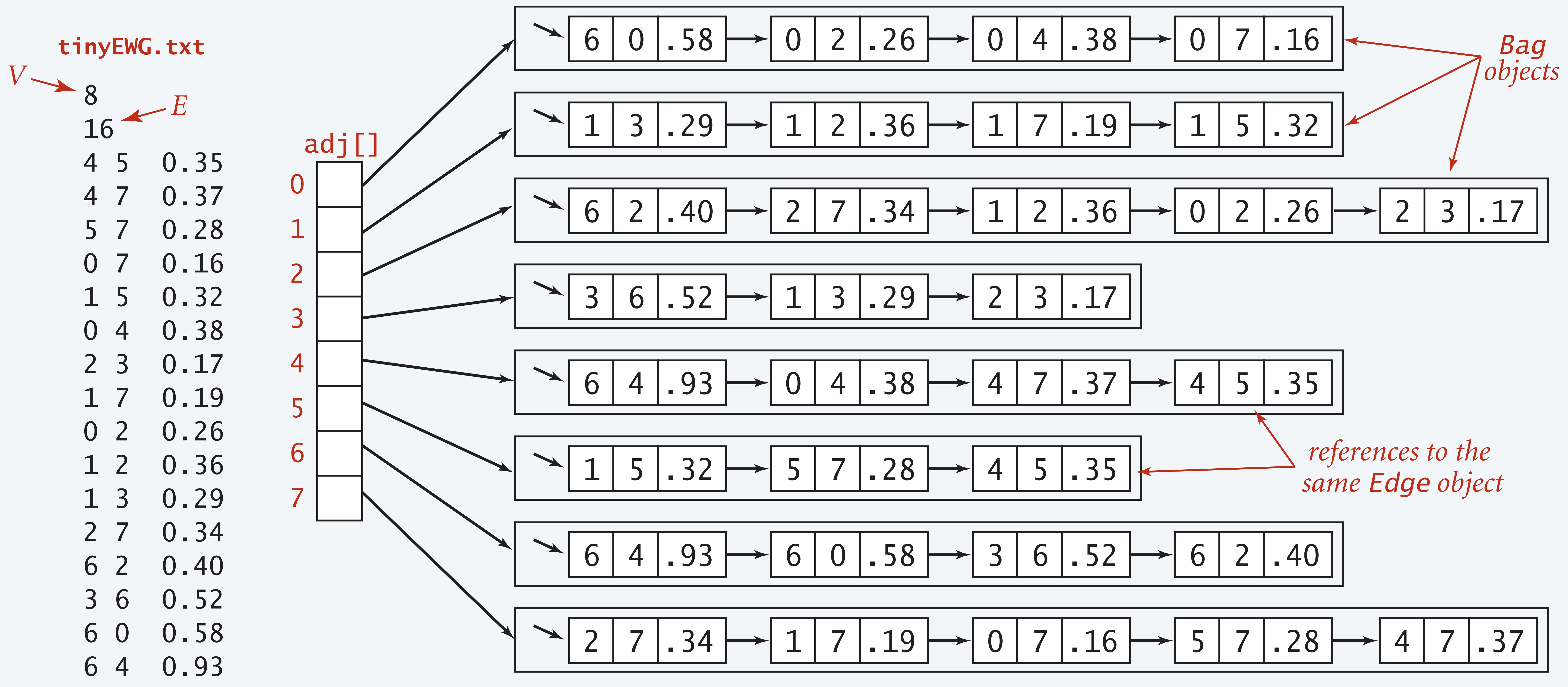
Edge-weighted graph API

API. Same as `Graph` and `Digraph`, except with explicit `Edge` objects.

public class EdgeWeightedGraph		
	`EdgeWeightedGraph(int V)`	*edge-weighted graph with V vertices (and no edges)*
`void`	`addEdge(Edge e)`	*add weighted edge e to this graph*
`Iterable<Edge>`	`adj(int v)`	*edges incident to v*
	⋮	⋮

Edge-weighted graph: adjacency-lists representation

Representation. Maintain vertex-indexed array of Edge lists.



Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph {
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V) {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
        adj[v] = new Bag<>();
   }

   public void addEdge(Edge e) {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v) {
      return adj[v];
   }

}
```

same as Graph *(but adjacency lists of* Edge *objects)*

constructor

add same Edge *object to both adjacency lists*

Minimum spanning tree API

Q. How to represent the MST?

A. Technically, an MST is an edge-weighted graph.

For convenience, we represent it as a set of edges.

```
public class MST
```

	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*
	⋮	⋮

https://algs4.cs.princeton.edu

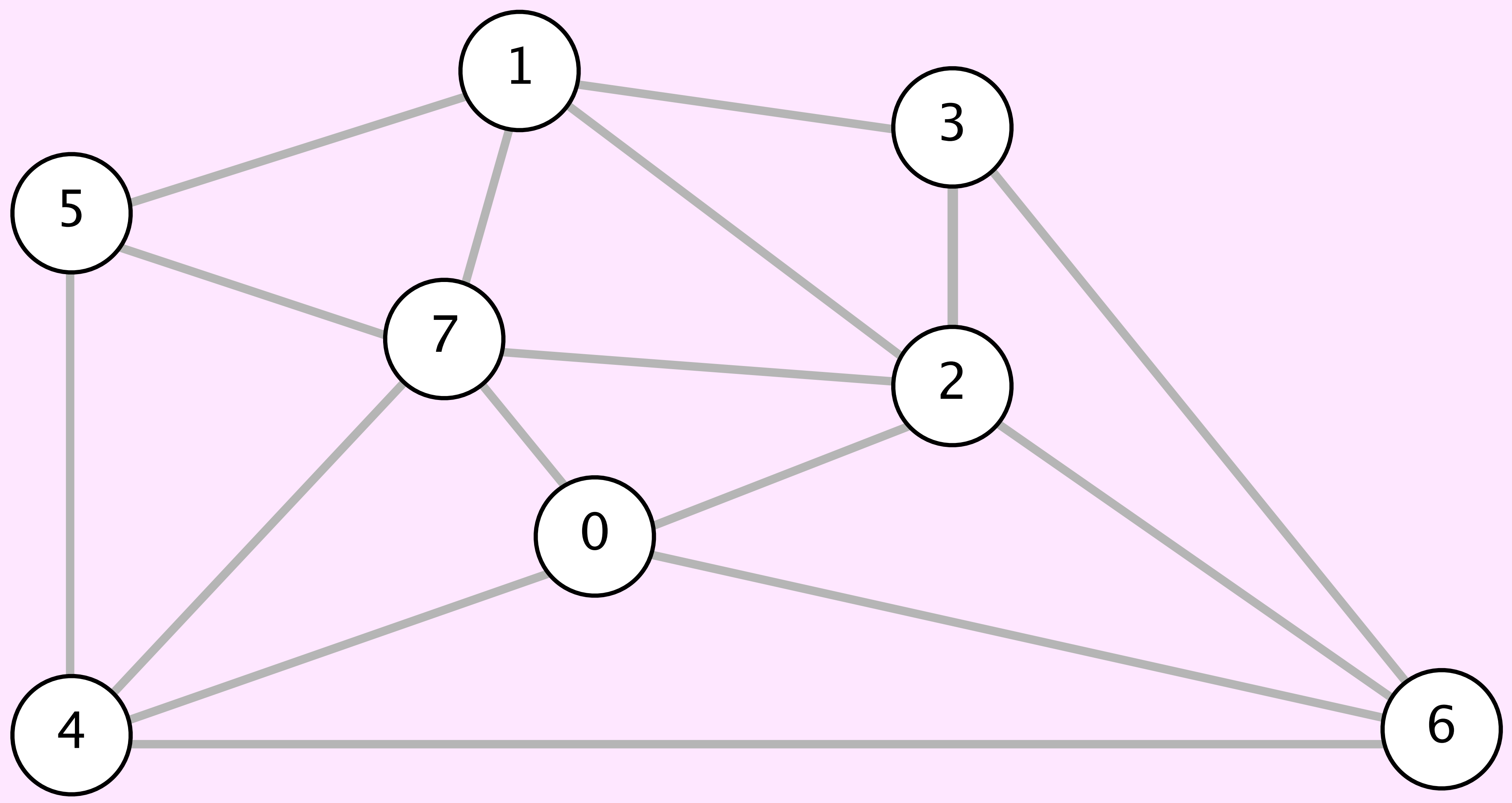
4.3 Minimum Spanning Trees

- introduction
- cut property
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to T unless doing so would create a cycle.



an edge-weighted graph

graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

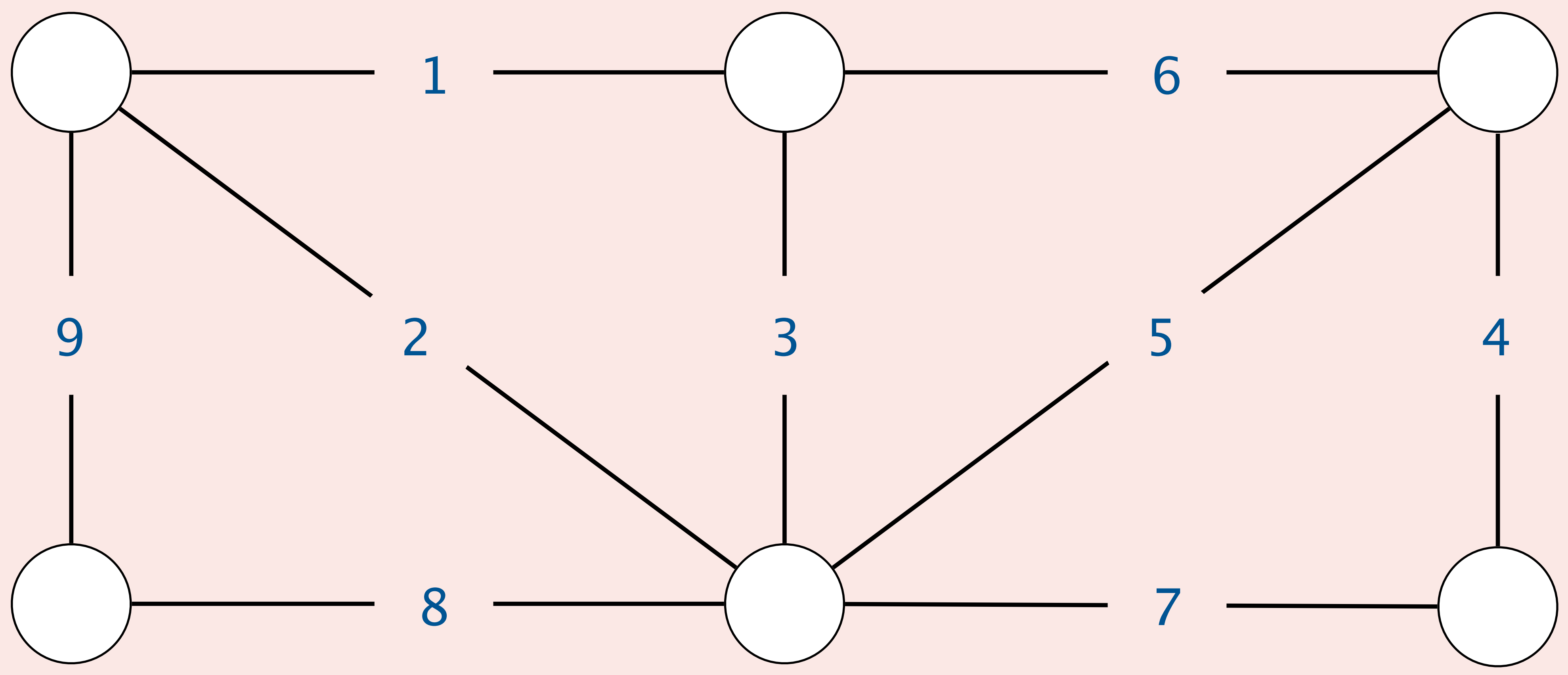
In which order does Kruskal's algorithm select edges in MST?

A. 1, 2, 4, 5, 6

B. 1, 2, 4, 5, 8

C. 1, 2, 5, 4, 8

D. 8, 2, 1, 5, 4



Kruskal's algorithm: correctness proof

Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

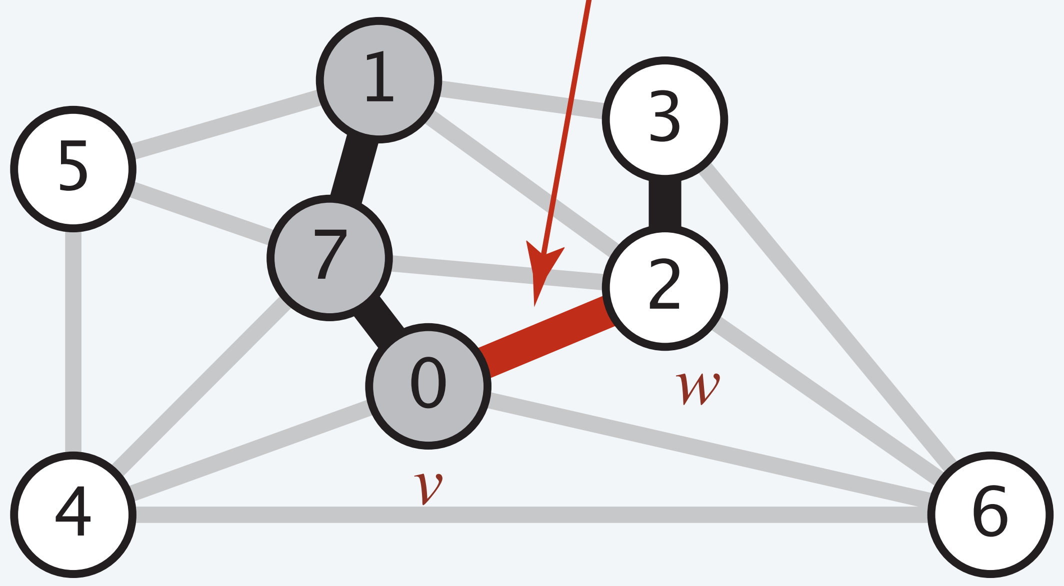
Pf. Kruskal's algorithm adds edge e to T if and only if e is in the MST.

[Case 1 $\Rightarrow$] Kruskal's algorithm adds edge $e = v\text{–}w$ to T.

- Vertices v and w are in different connected components of T.
- Cut = set of vertices connected to v in T.
- By construction of cut, e is a crossing edge and no crossing edge
  - is currently in T
  - was considered by Kruskal before e
- Thus, e is a min weight crossing edge.
- Cut property $\Rightarrow$ e is in the MST.

Kruskal considers edges in ascending order by weight

add edge to tree



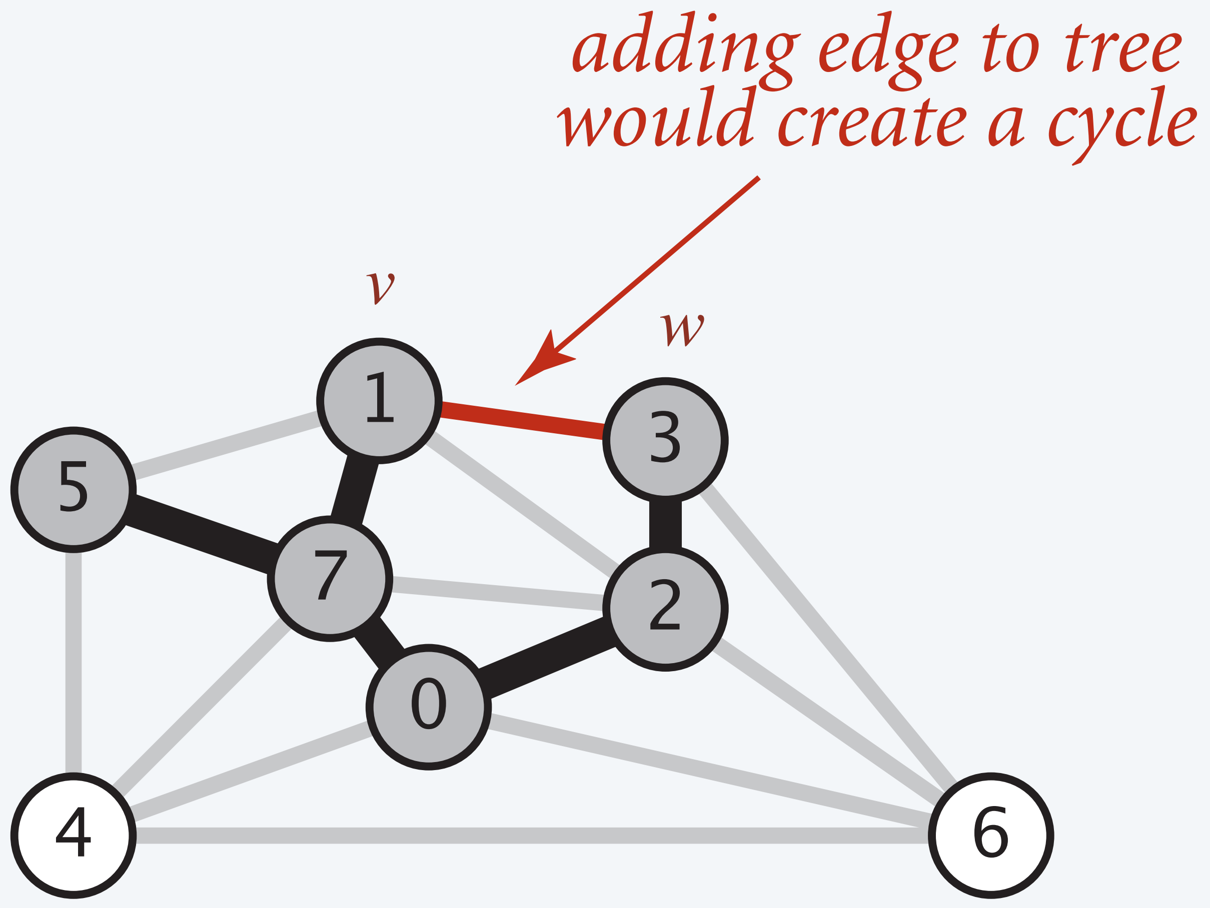
Kruskal's algorithm: correctness proof

Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm adds edge e to T if and only if e is in the MST.

[Case 2 $\Leftarrow$] Kruskal's algorithm discards edge $e = v\text{–}w$.

- From Case 1, all edges currently in T are in the MST.
- The MST can't contain a cycle, so it can't also contain e. ▪

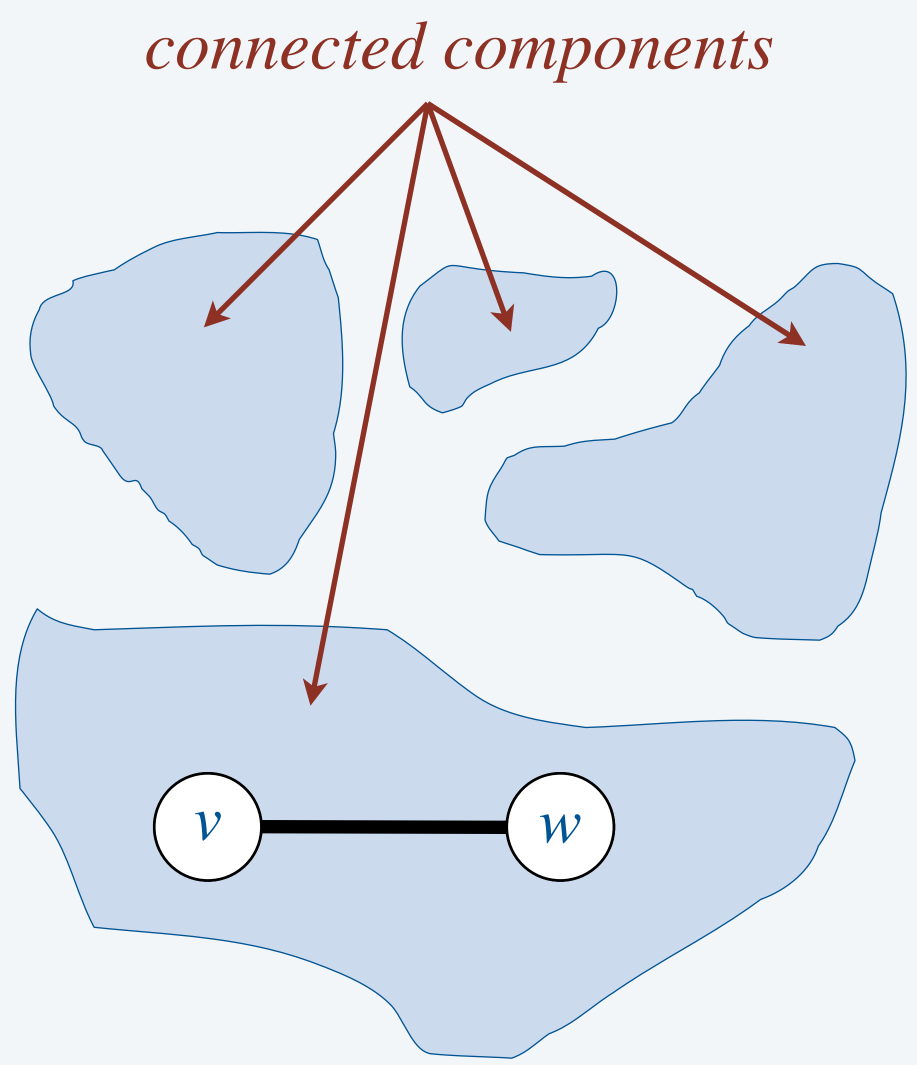


Kruskal's algorithm: implementation challenge

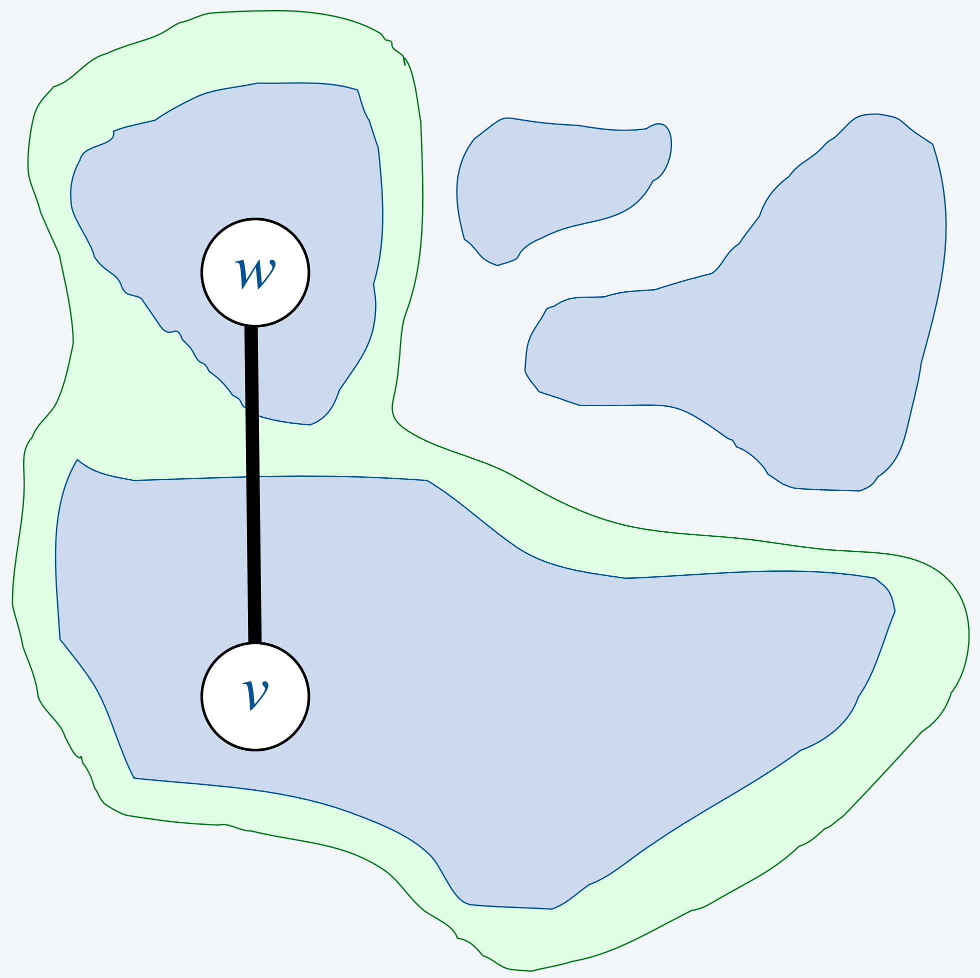
Challenge. Would adding edge $v–w$ to T create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T, initially each vertex in its own set.
- If v and w are in same set, then adding $v–w$ to T would create a cycle. [Case 2]
- Otherwise, add $v–w$ to T and merge sets containing v and w. [Case 1]



Case 2: adding v–w creates a cycle



Case 1: add v–w to T and merge sets containing v and w

Kruskal's algorithm: Java implementation

```
public class KruskalMST {
   private Queue<Edge> mst = new Queue<>();

   public KruskalMST(EdgeWeightedGraph G) {
      Edge[] edges = G.edges();
      Arrays.sort(edges);
      UF uf = new UF(G.V());

      for (int i = 0; i < G.E(); i++) {
         Edge e = edges[i];
         int v = e.either(), w = e.other(v);
         if (uf.find(v) != uf.find(w)) {
            mst.enqueue(e);
            uf.union(v, w);
         }
      }
   }

   public Iterable<Edge> edges() {
      return mst;
   }
}
```

edges in the MST

sort edges by weight

maintain connected components

optimization: stop as soon as V–1 edges in T

greedily add edges to MST

edge v–w does not create cycle

add edge e to MST

merge connected components

Kruskal's algorithm: running time

Proposition. In the worst case, Kruskal's algorithm computes the MST in an edge-weighted graph in $\Theta(E \log E)$ time and $\Theta(E)$ extra space.

Pf.

- Bottlenecks are sorting and union–find operations.

operation	frequency	time per op
SORT	1	$E \log E$
UNION	$V - 1$	$\log V$ †
FIND	$2\,E$	$\log V$ †

† *using weighted quick union*

- Total. $\Theta(V \log V) \ + \ \Theta(E \log V) \ + \ \Theta(E \log E)$.

dominated by $\Theta(E \log E)$
since graph is connected

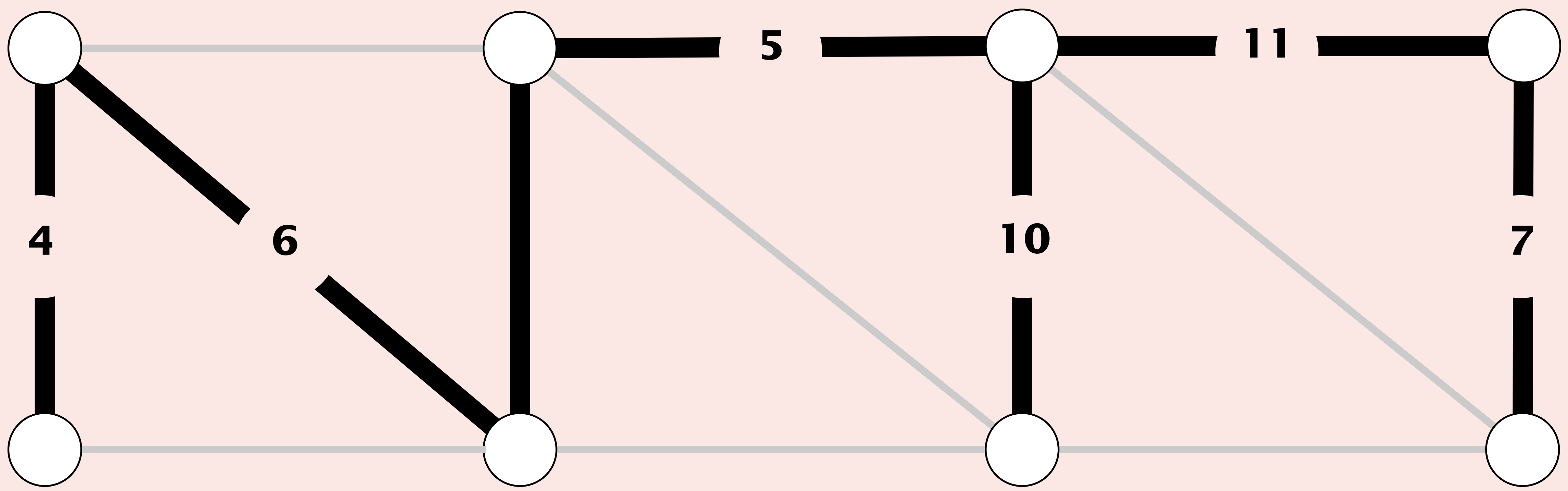
Given a graph with positive edge weights, how to find a spanning tree that minimizes the sum of the squares of the edge weights?

A. Run Kruskal's algorithm using the original edge weights.

B. Run Kruskal's algorithm using the squares of the edge weights.

C. Run Kruskal's algorithm using the square roots of the edge weights.

D. All of the above.

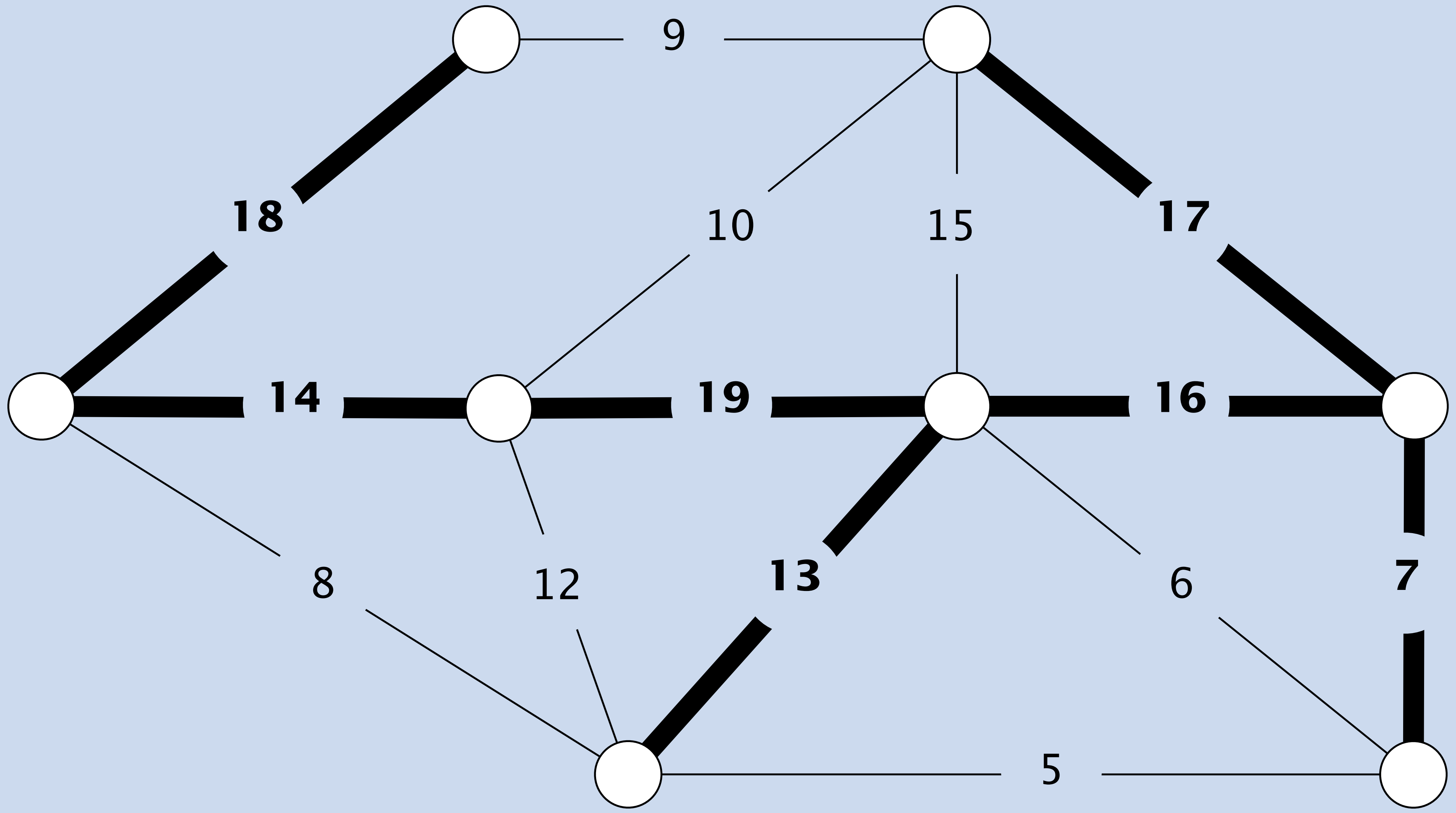


sum of squares $= 4^2 + 6^2 + 5^2 + 10^2 + 11^2 + 7^2 = 347$

Maximum Spanning Tree

Problem. Given an undirected graph G with positive edge weights, find a spanning tree that maximizes the sum of the edge weights.

Goal. Design algorithm that takes $\Theta(E \log E)$ time in the worst case.



maximum spanning tree T (weight = 104)

Algorithms

Robert Sedgewick | Kevin Wayne

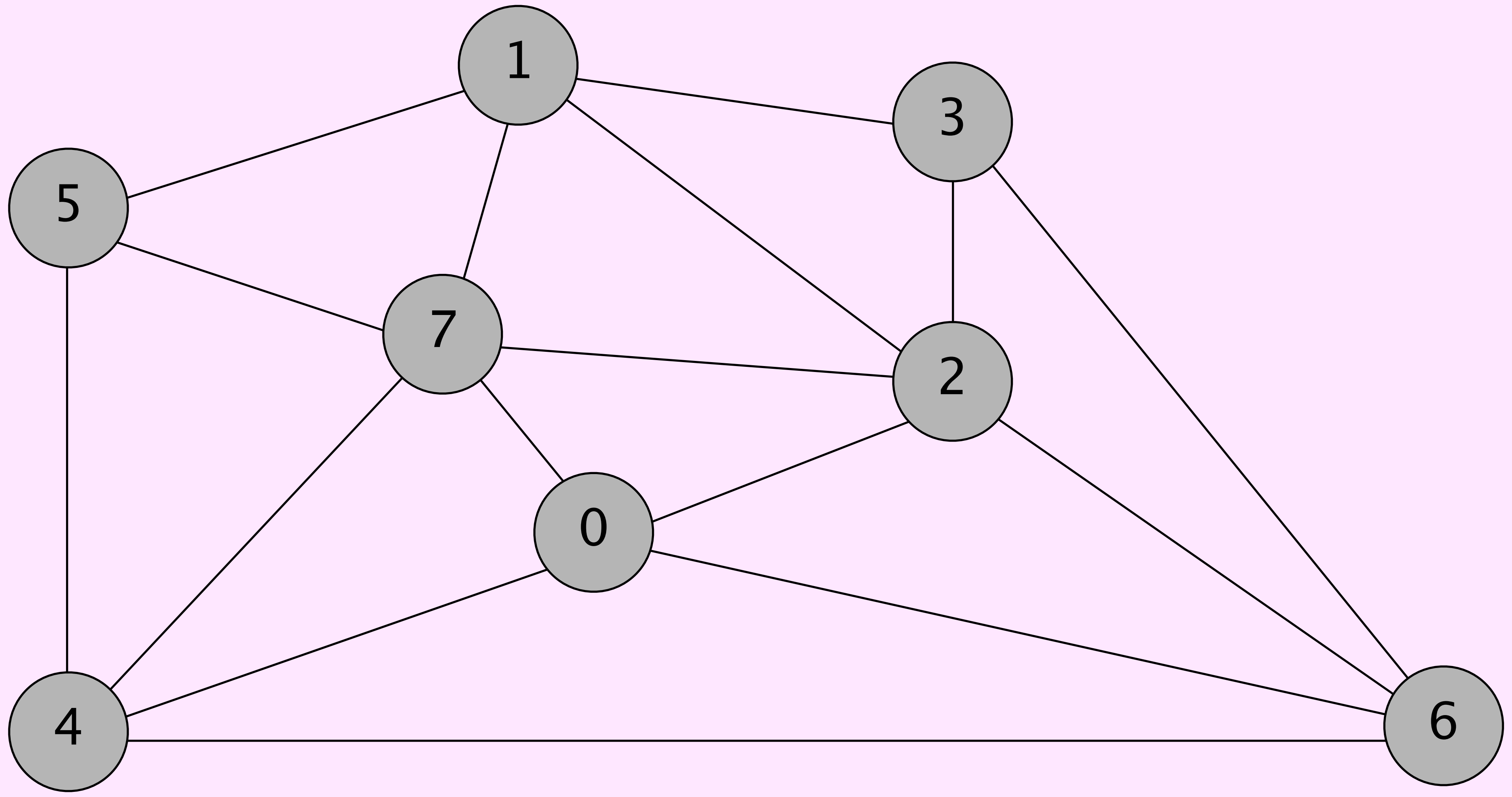
https://algs4.cs.princeton.edu

4.3 Minimum Spanning Trees

- introduction
- cut property
- edge-weighted graph API
- Kruskal's algorithm
- *Prim's algorithm*

Prim's algorithm demo

- Start with vertex 0 and grow tree T.
- Repeat until $V - 1$ edges:
  - add to T the min-weight edge with exactly one endpoint in T



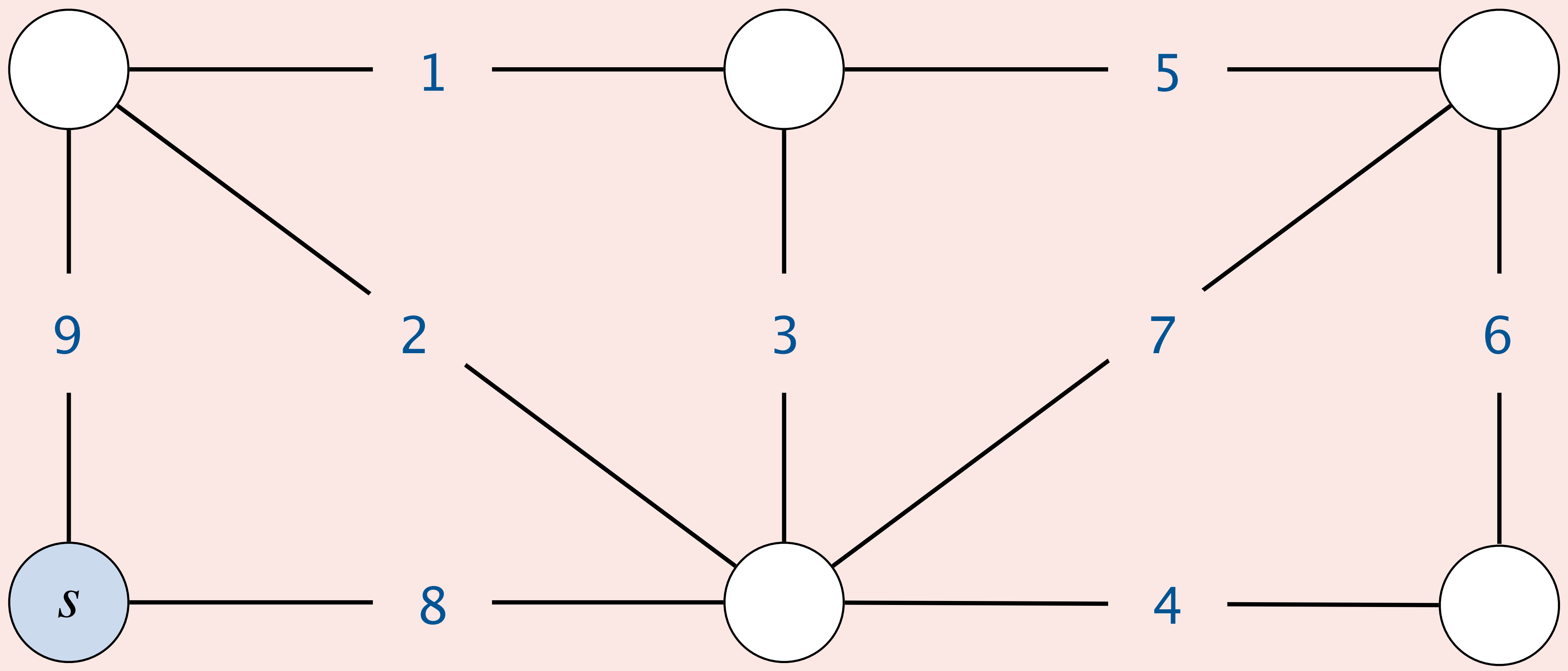
an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

In which order does Prim's algorithm select edges in the MST?

Assume it starts from vertex s.

- **A.** 8, 2, 1, 4, 5
- **B.** 8, 2, 1, 5, 4
- **C.** 8, 2, 1, 5, 6
- **D.** 8, 2, 3, 4, 5



Prim's algorithm: proof of correctness

Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]

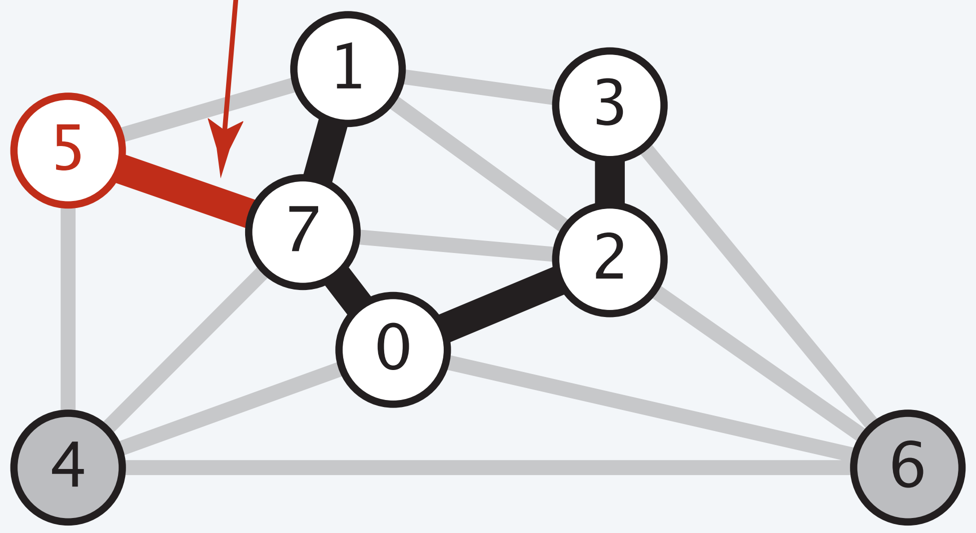
Prim's algorithm computes the MST.

Pf. Let e = min-weight edge with exactly one endpoint in T.

- Cut = set of vertices in T.
- Cut property $\Rightarrow$ edge e is in the MST. ▪

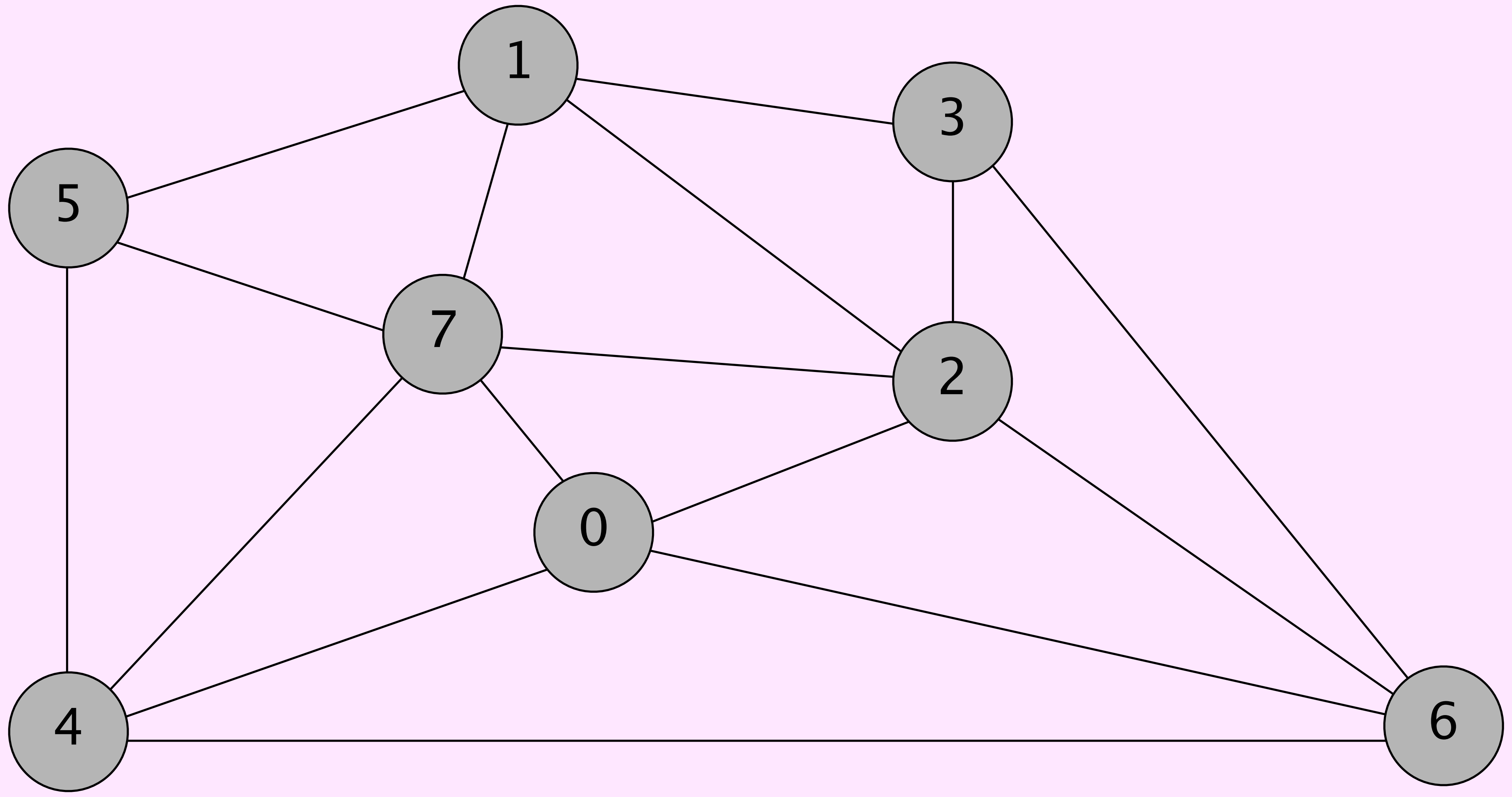
Challenge. How to efficiently find min-weight edge with exactly one endpoint in T?

edge e = 7-5 added to tree



Prim's algorithm: lazy implementation demo

- Start with vertex 0 and grow tree T.
- Repeat until $V - 1$ edges:
  - add to T the min-weight edge with exactly one endpoint in T



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm: lazy implementation

Challenge. How to efficiently find min-weight edge with exactly one endpoint in T?

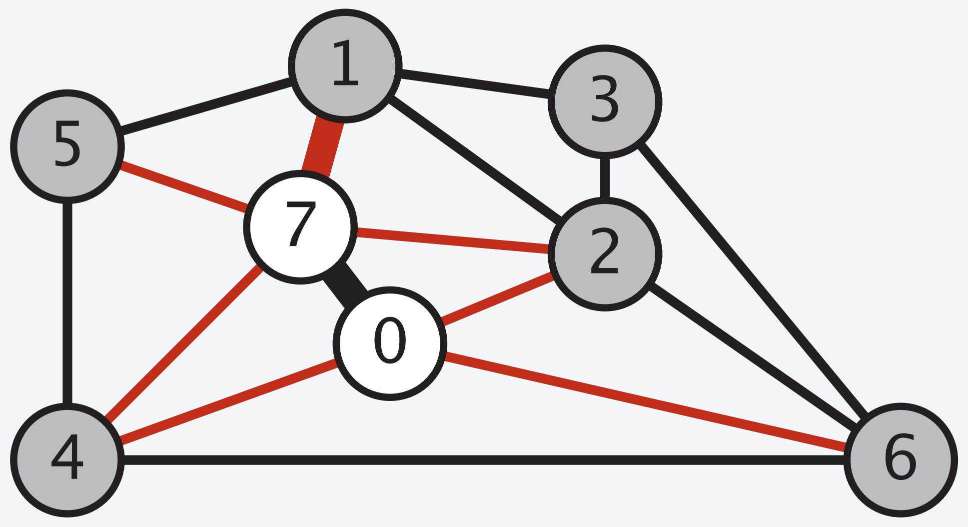
Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.

- Key = edge; priority = weight of edge.
- DELETE-MIN to determine next edge $e = v–w$ to add to T.
- If both endpoints v and w are marked (both in T), disregard.
- Otherwise, let w be the unmarked vertex (not in T):
  - add e to T and mark w
  - add to PQ any edge incident to w ← *but don't bother if other endpoint is already in T*

1-7 is min weight edge with exactly one endpoint in T

priority queue of crossing edges

```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```



Prim's algorithm: lazy implementation

```
public class LazyPrimMST {
   private boolean[] marked;   // MST vertices
   private Queue<Edge> mst;    // MST edges
   private MinPQ<Edge> pq;     // PQ of edges

    public LazyPrimMST(WeightedGraph G) {
        pq = new MinPQ<>();
        mst = new Queue<>();
        marked = new boolean[G.V()];
        visit(G, 0);  ←  assume graph G is connected

        while (mst.size() < G.V() - 1) {
           Edge e = pq.delMin();
           int v = e.either(), w = e.other(v);
           if (marked[v] && marked[w]) continue;
           mst.enqueue(e);
           if (!marked[v]) visit(G, v);
           if (!marked[w]) visit(G, w);
        }
   }
   ...
}
```

repeatedly delete the min-weight edge e = v–w from PQ

ignore if both endpoints in tree T

add edge e to tree T

add either v or w to tree T

```
private void visit(WeightedGraph G, int v) {
  marked[v] = true;  ←  add v to tree T
  for (Edge e : G.adj(v))
     if (!marked[e.other(v)])
        pq.insert(e);
}

public Iterable<Edge> mst() {
   return mst;
}
```

for each edge e = v–w: add e to PQ if w not already in T

Lazy Prim's algorithm: running time

Proposition. In the worst case, lazy Prim's algorithm computes the MST in $\Theta(E \log E)$ time and $\Theta(E)$ extra space.

Pf.

- Bottlenecks are PQ operations.
- Each edge is added to PQ at most once.
- Each edge is deleted from PQ at most once.

operation	frequency	time per op
INSERT	E	$\log E$ †
DELETE-MIN	E	$\log E$ †

† *using binary heap*

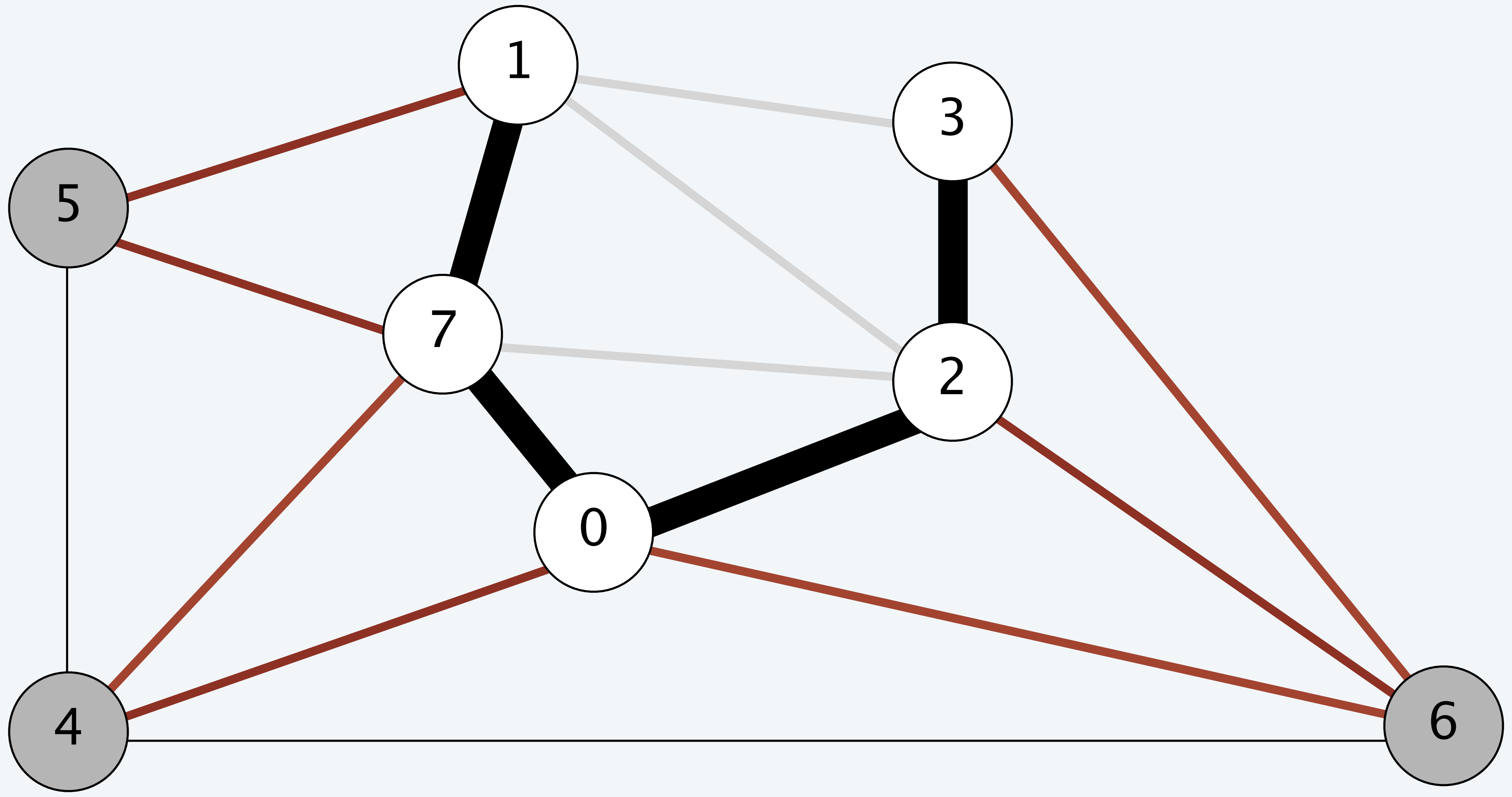
Prim's algorithm: eager implementation

Challenge. Find min-weight edge with exactly one endpoint in T.

Observation. For each vertex v, need only min-weight edge connecting v to T.

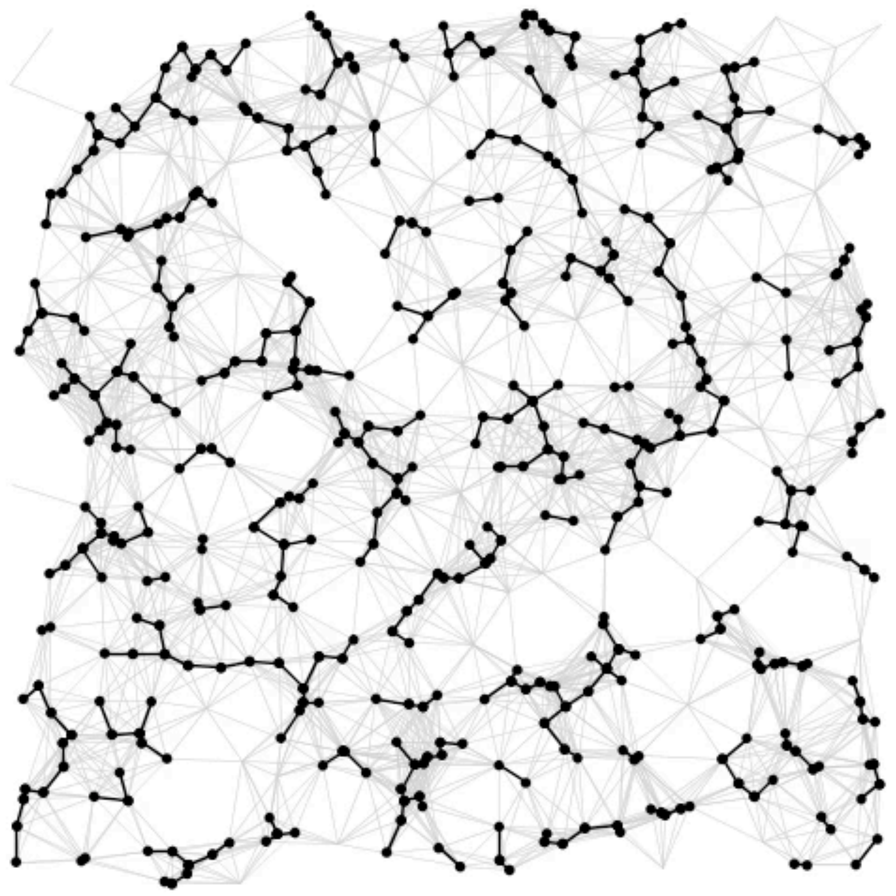
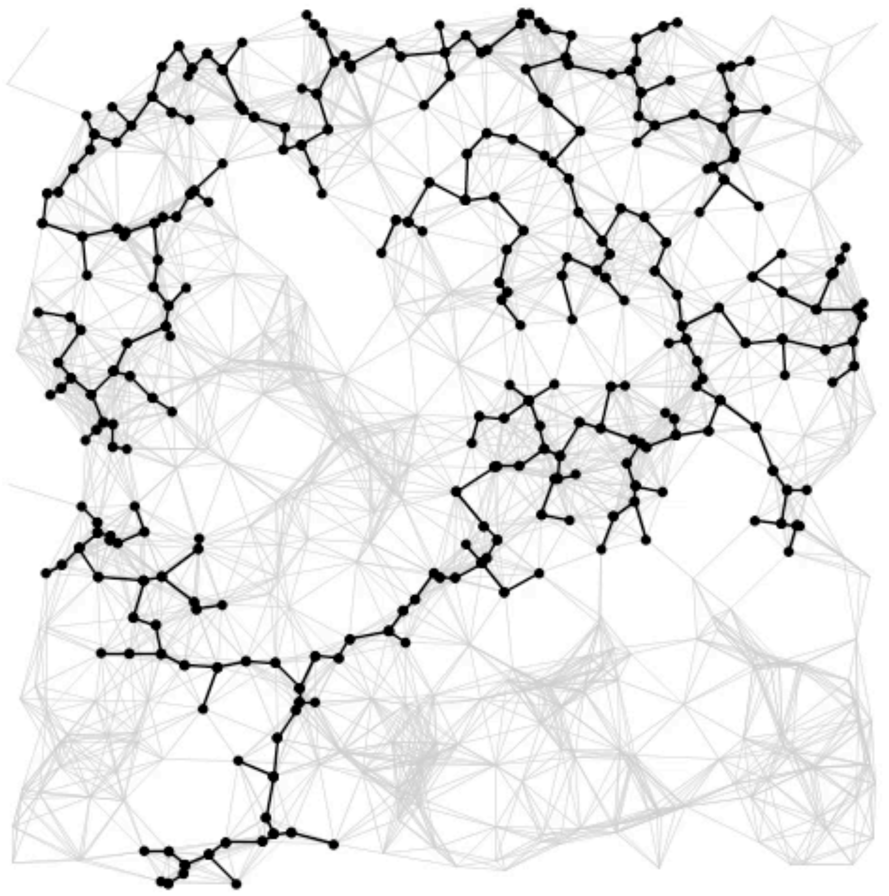
- MST includes at most one edge connecting v to T. Why?
- If MST includes such an edge, it must take lightest such edge. Why?

Impact. PQ of vertices; $\Theta(V)$ extra space; $\Theta(E \log V)$ running time in worst case.



see textbook for details

MST: algorithms of the day

algorithm	visualization	bottleneck	running time
Kruskal		*sorting* *union–find*	$E \log E$
Prim		*priority queue*	$E \log V$

Credits

image	source	license
Muddy City Problem	CS Unplugged	CC BY-NC-SA 4.0
Microarrays and Clustering	Botstein and Brown	by author
Image Segmentation	Felzenszwalb and Huttenlocher	
Phylogeny Tree	Derzelle et al.	
MST Dithering	Mario Klingemann	CC BY-NC 2.0
Slime Mold vs. Rail Network	Harvard Magazine	
Mona Singh	Princeton University	

A final thought

“ *The algorithms we write are only as good as the questions we ask. And the best questions come from collaboration and creative thinking.* ” — Mona Singh