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# 2.4 Priority Queues

- *APIs*
- *elementary implementations*
- *binary heaps*
- *heapsort*


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# 2.4 Priority Queues

- *APIs*
- *elementary implementations*
- *binary heaps*
- *heapsort*

# Collections

A collection is a data type that stores a group of items.

| data type | core operations | data structure |
| --- | --- | --- |
| **stack** | PUSH, POP | *singly linked list*<br>*resizable array* |
| **queue** | ENQUEUE, DEQUEUE | |
| **deque** | ADD-FIRST, REMOVE-FIRST,<br>ADD-LAST, REMOVE-LAST | *doubly linked list*<br>*resizable array* |
| **priority queue** | INSERT, DELETE-MAX | *binary heap* |
| **symbol table** | PUT, GET, DELETE | *binary search tree*<br>*hash table* |
| **set** | ADD, CONTAINS, DELETE | |

# Priority queue

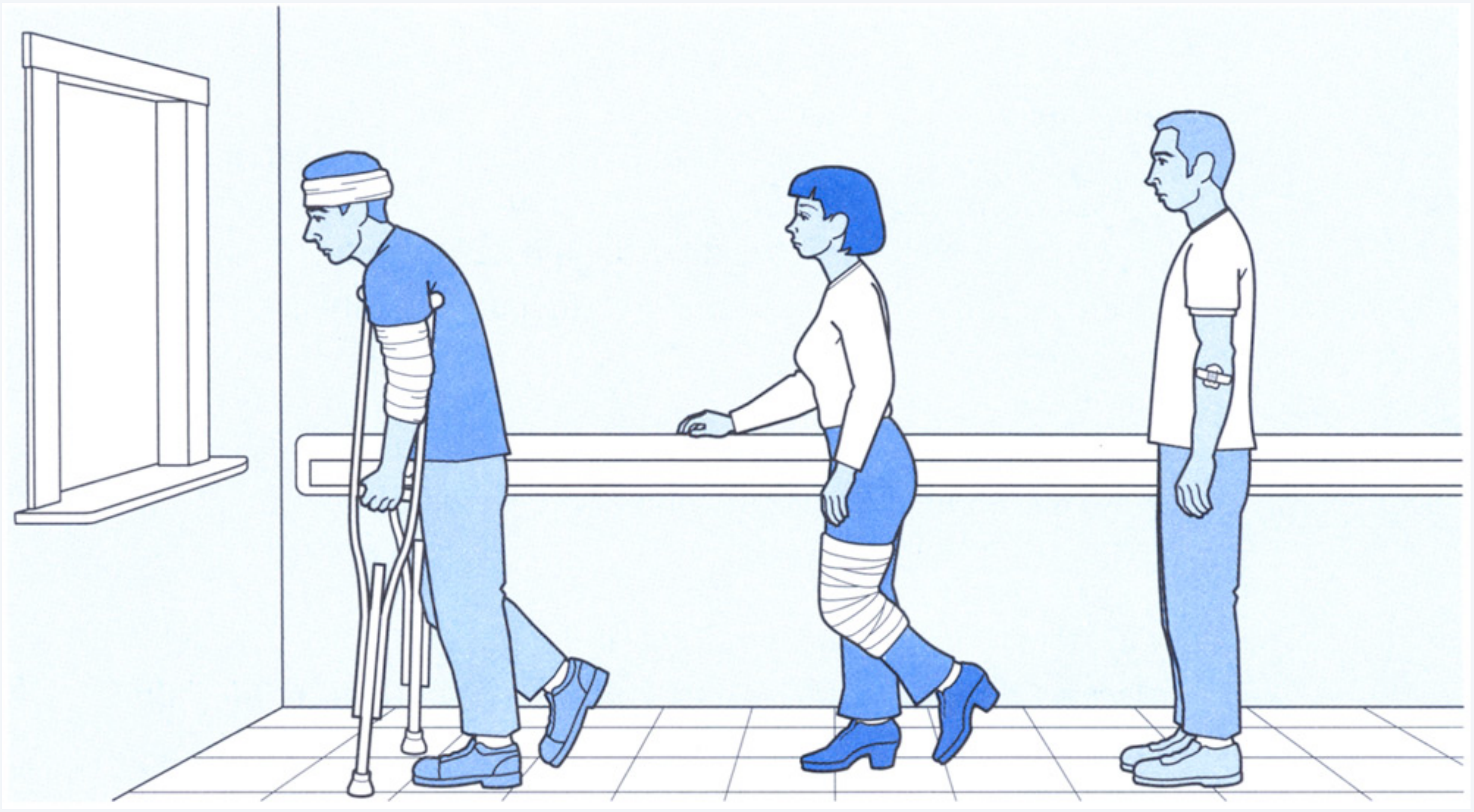
**Collections.** Insert and remove items. Which item to remove?

**Stack.** Remove the item most recently added.

**Queue.** Remove the item least recently added.

**Randomized queue.** Remove a random item.

**Priority queue.** Remove the largest (or smallest) item.



triage in an emergency room
(priority = urgency of wound/illness)

| *operation* | *argument* | *return value* |
|---|---|---|
| *insert* | P | |
| *insert* | Q | |
| *insert* | E | |
| *remove max* | | Q |
| *insert* | X | |
| *insert* | A | |
| *insert* | M | |
| *remove max* | | X |
| *insert* | P | |
| *insert* | L | |
| *insert* | E | |
| *remove max* | | P |

# Max-oriented priority queue API

*"bounded type parameter"*

```
public class MaxPQ<Key extends Comparable<Key>>
```

| | | |
|---|---|---|
| | `MaxPQ()` | *create an empty priority queue* |
| `void` | `insert(Key key)` | *insert a key* |
| `Key` | `delMax()` | *return and remove a largest key* |
| `Key` | `max()` | *return a largest key* |
| `boolean` | `isEmpty()` | *is the priority queue empty?* |
| `int` | `size()` | *number of keys in the priority queue* |

Note 1. Keys are generic, but must be `Comparable`.

Note 2. Duplicate keys allowed; `delMax()` removes and returns any maximum key.

# Min-oriented priority queue API

Analogous to MaxPQ.

public class MinPQ<Key extends Comparable<Key>>

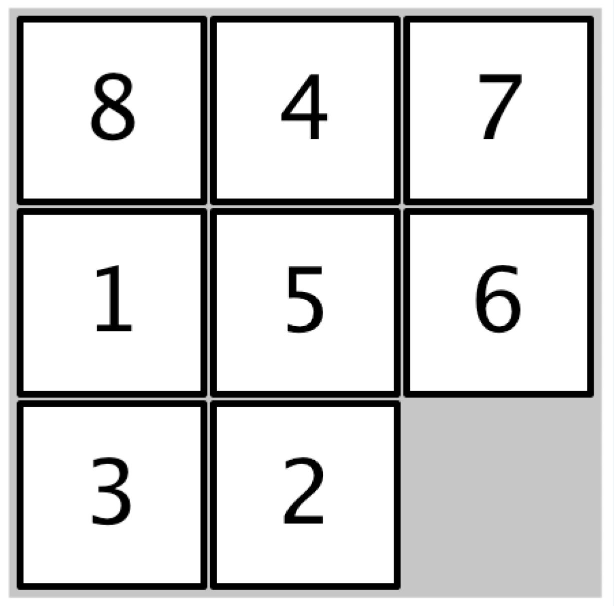
| | | |
|---|---|---|
| | MinPQ() | *create an empty priority queue* |
| void | insert(Key key) | *insert a key* |
| Key | delMin() | *return and remove a smallest key* |
| Key | min() | *return a smallest key* |
| boolean | isEmpty() | *is the priority queue empty?* |
| int | size() | *number of keys in the priority queue* |

Warmup client. Sort a stream of integers from standard input.

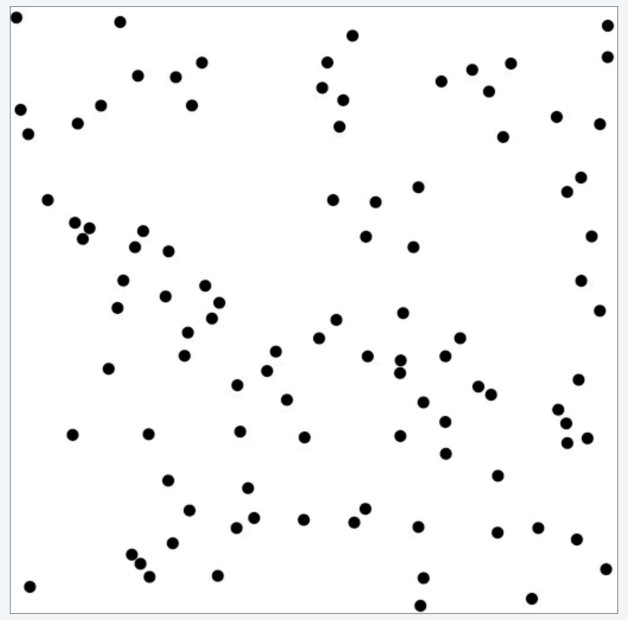
# Priority queue: applications

- Event-driven simulation. [ customers in a line, colliding particles ]
- Discrete optimization. [ bin packing, scheduling ]
- Artificial intelligence. [ A* search ]
- Computer networks. [ web cache ]
- Data compression. [ Huffman codes ]
- Operating systems. [ load balancing, interrupt handling ]
- Graph searching. [ Dijkstra's algorithm, Prim's algorithm ]
- Number theory. [ sum of powers ]
- Spam filtering. [ Bayesian spam filter ]
- Statistics. [ online median in data stream ]


**priority = length of best known path**




**priority = "distance" to goal board**



**priority = event time**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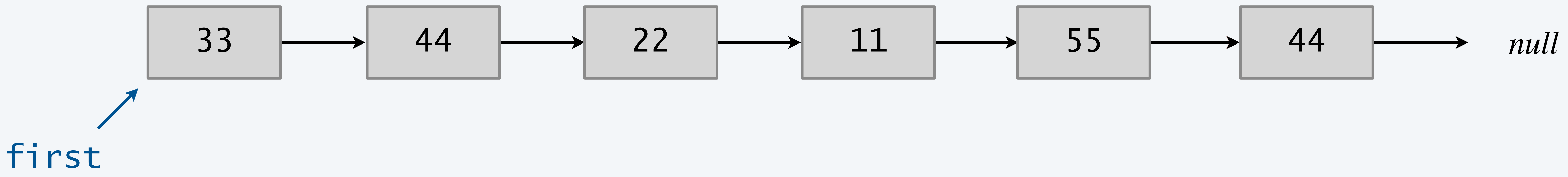
# 2.4 Priority Queues

- *APIs*
- ***elementary implementations***
- *binary heaps*
- *heapsort*

# Priority queue: elementary implementations

Unordered list. Store keys in a singly linked list.




Performance. INSERT takes $\Theta(1)$ time; DELETE-MAX takes $\Theta(n)$ time.

# Priority queue: elementary implementations

Ordered array. Store keys in an array in ascending (or descending) order.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a[] | 11 | 22 | 33 | 44 | 44 | 55 | 99 | | | | |
| | | | | | | | | n | | | |

**ordered array implementation of a MaxPQ**

# Priority queues: quiz 1

**What are the worst-case running times for INSERT and DELETE-MAX, respectively, in a MaxPQ implemented with an ordered array ?**

*ignore array resizing*

**A.** $\Theta(1)$ and $\Theta(n)$

**B.** $\Theta(1)$ and $\Theta(\log n)$

**C.** $\Theta(\log n)$ and $\Theta(1)$

**D.** $\Theta(n)$ and $\Theta(1)$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a[] | 11 | 22 | 33 | 44 | 44 | 55 | 99 | | | | |
| | | | | | | | | n | | | |

**ordered array implementation of a MaxPQ**

# Priority queue: implementations cost summary

Elementary implementations. Either INSERT or DELETE-MAX takes $\Theta(n)$ time.

| implementation | INSERT | DELETE-MAX |
|---|---|---|
| unordered list | $1$ | $n$ |
| ordered array | $n$ | $1$ |
| goal | $\log n$ | $\log n$ |

order of growth of running time for priority queue with n items

Challenge. Implement both INSERT and DELETE-MAX efficiently.

Solution. "Somewhat-ordered" array.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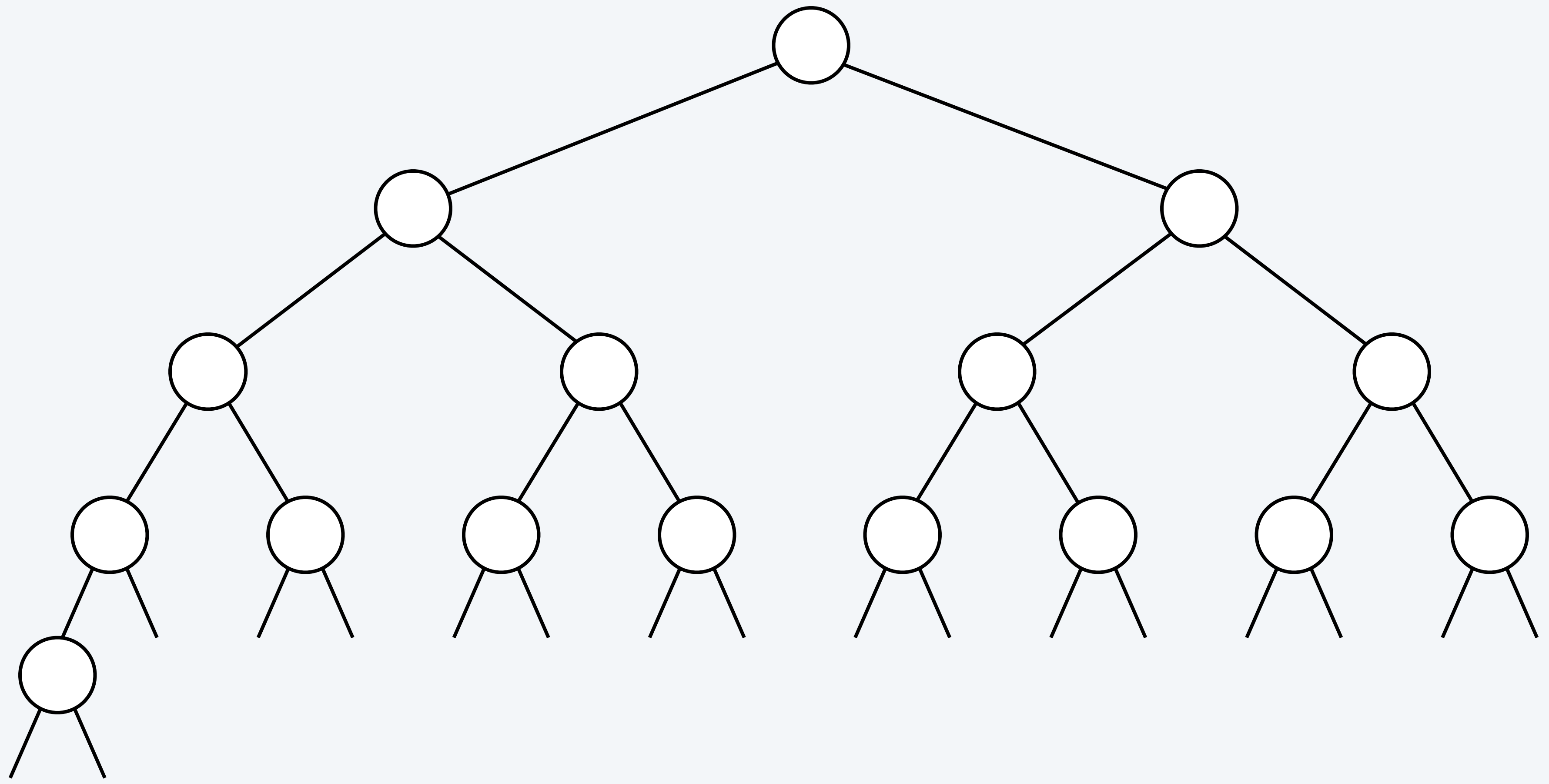
# 2.4 Priority Queues

- APIs
- elementary implementations
- binary heaps
- heapsort

# Complete binary tree

Binary tree. Empty or node with links to two disjoint binary trees (left and right subtrees).

Complete tree. Every level (except possibly the last) is completely filled;
the last level is filled from left to right.



complete binary tree
(n = 16 nodes, height = 4)

Property. Height of complete binary tree with $n$ nodes is $\lfloor \log_2 n \rfloor$.

Pf. As you successively add nodes, height increases (by $1$) only when $n$ is a power of $2$.

# A complete binary tree in nature (of height 4)

# Binary heap: representation

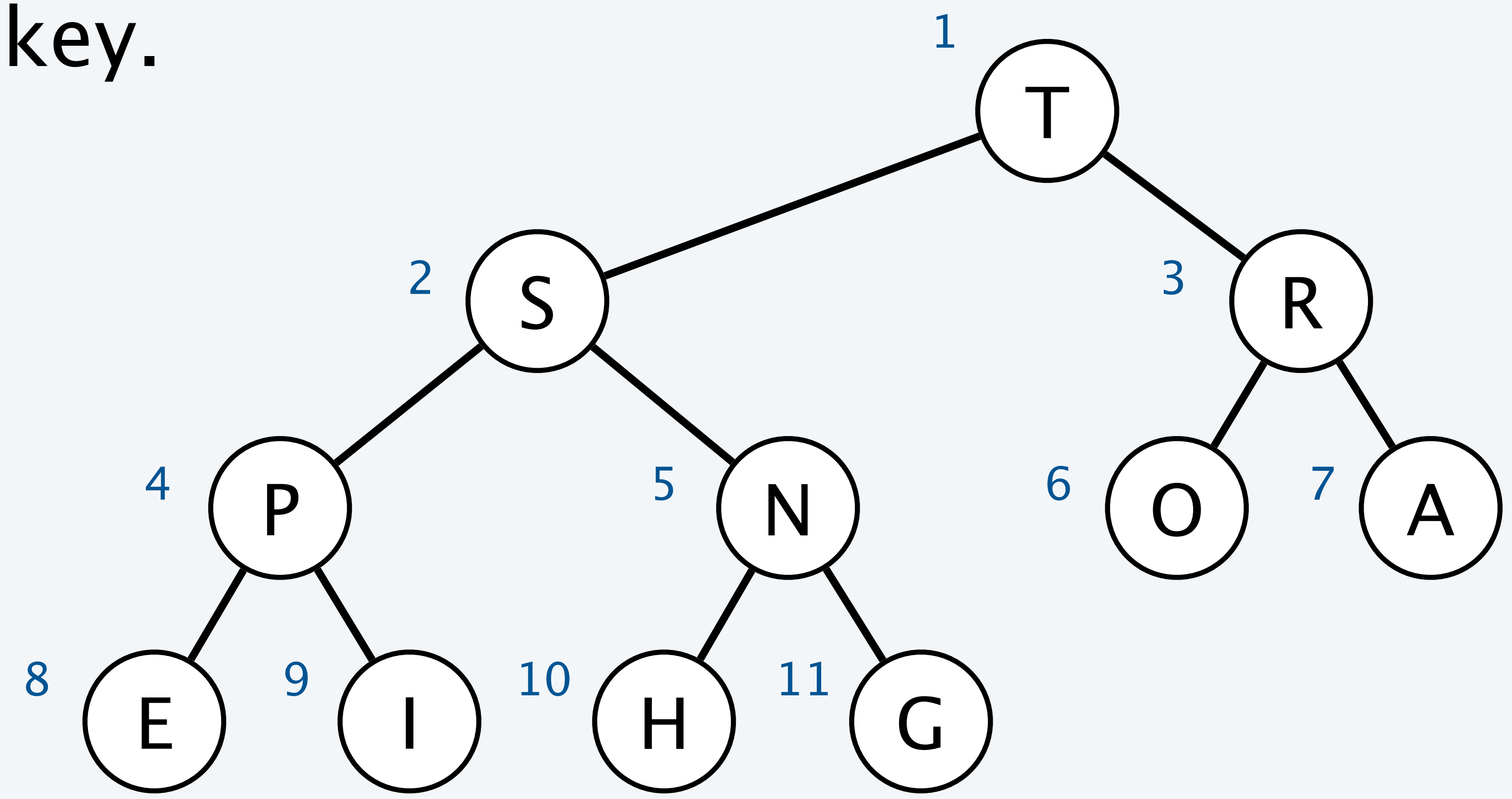
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered tree.

- Keys in nodes.
- Child's key no larger than parent's key.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links!




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a[] | - | T | S | R | P | N | O | A | E | I | H | G |

# Priority queues: quiz 2

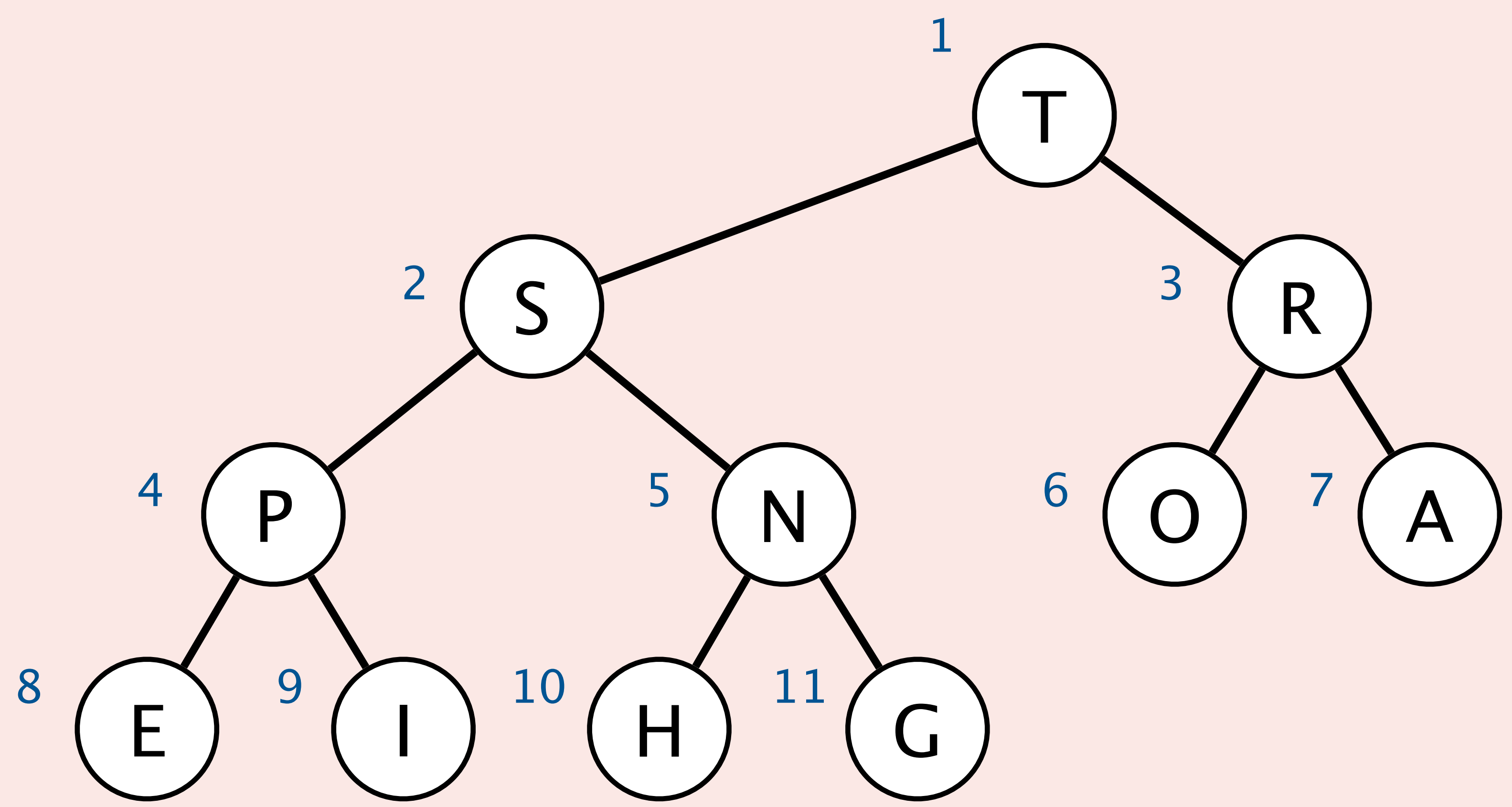
**Consider the node at index k in a binary heap. Which Java expression produces the index of its parent?**

**A.** `(k - 1) / 2`

**B.** `k / 2`

**C.** `(k + 1) / 2`

**D.** `2 * k`



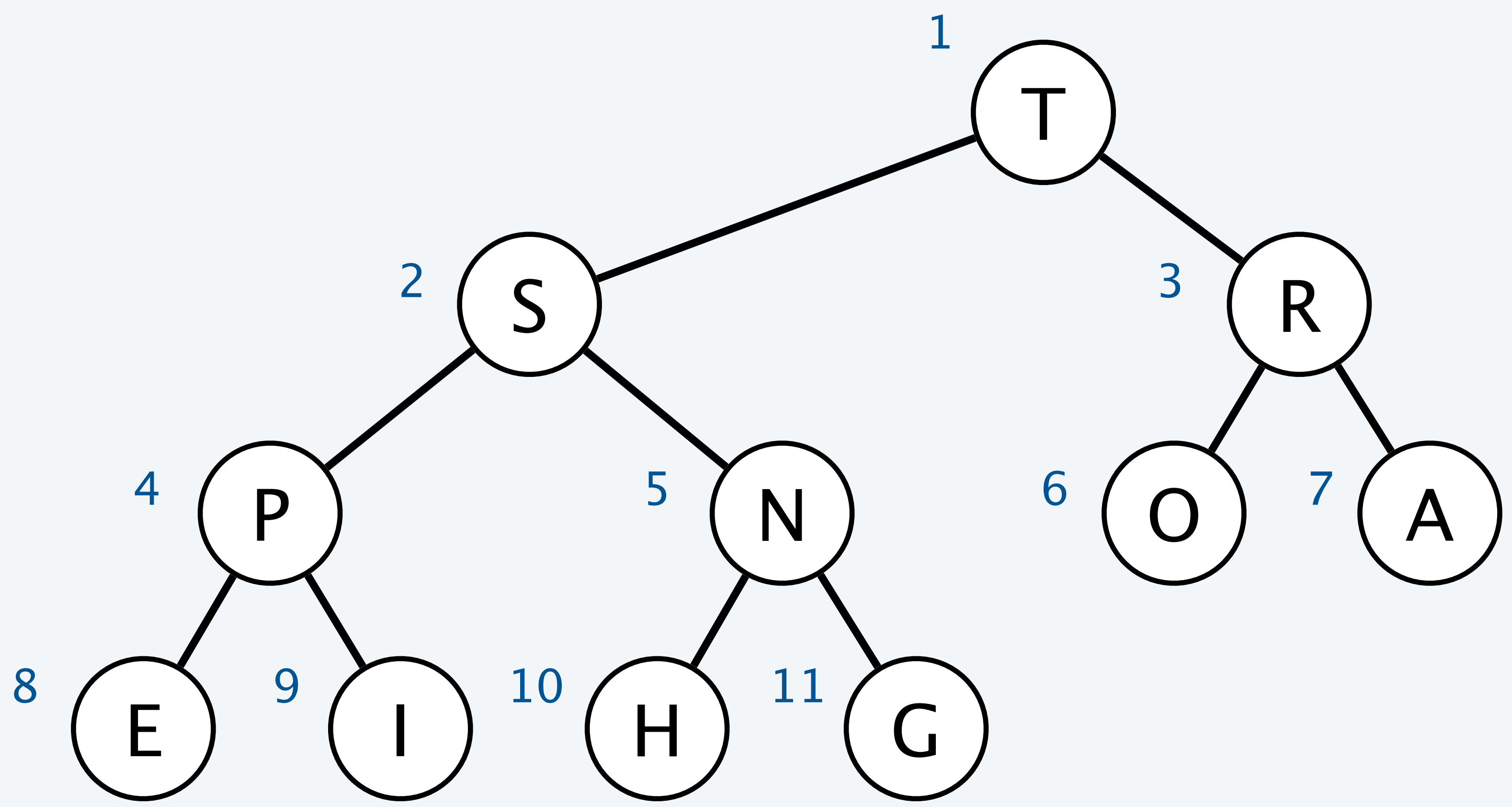

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a[] | - | T | S | R | P | N | O | A | E | I | H | G |

# Binary heap: properties

Proposition. Largest key is at index 1, which is root of binary tree.

Proposition. Can use array indices to move up or down tree.

- Parent of key at index k is at index k/2.
- Children of key at index k are at indices 2*k and 2*k + 1.



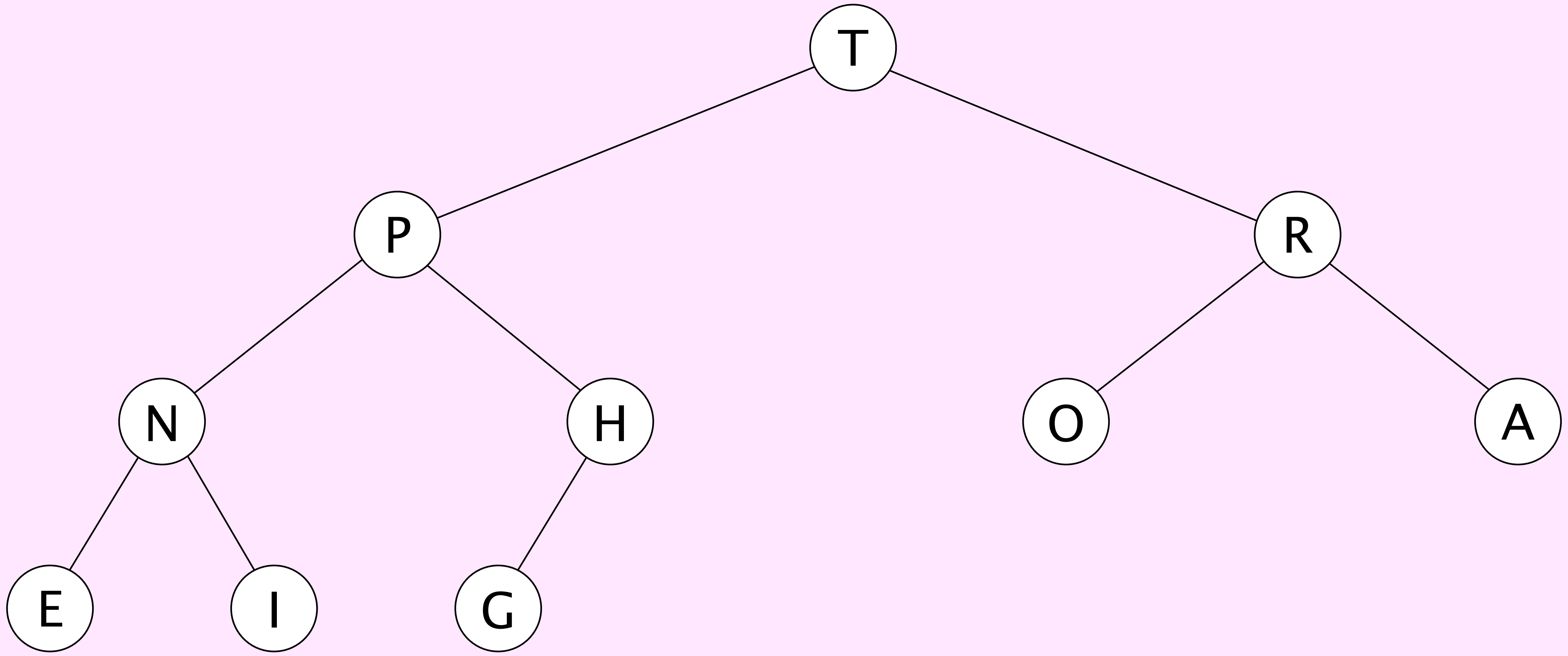

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a[] | - | T | S | R | P | N | O | A | E | I | H | G |

# Binary heap demo

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | P | R | N | H | O | A | E | I | G | |

# Binary heap: promotion

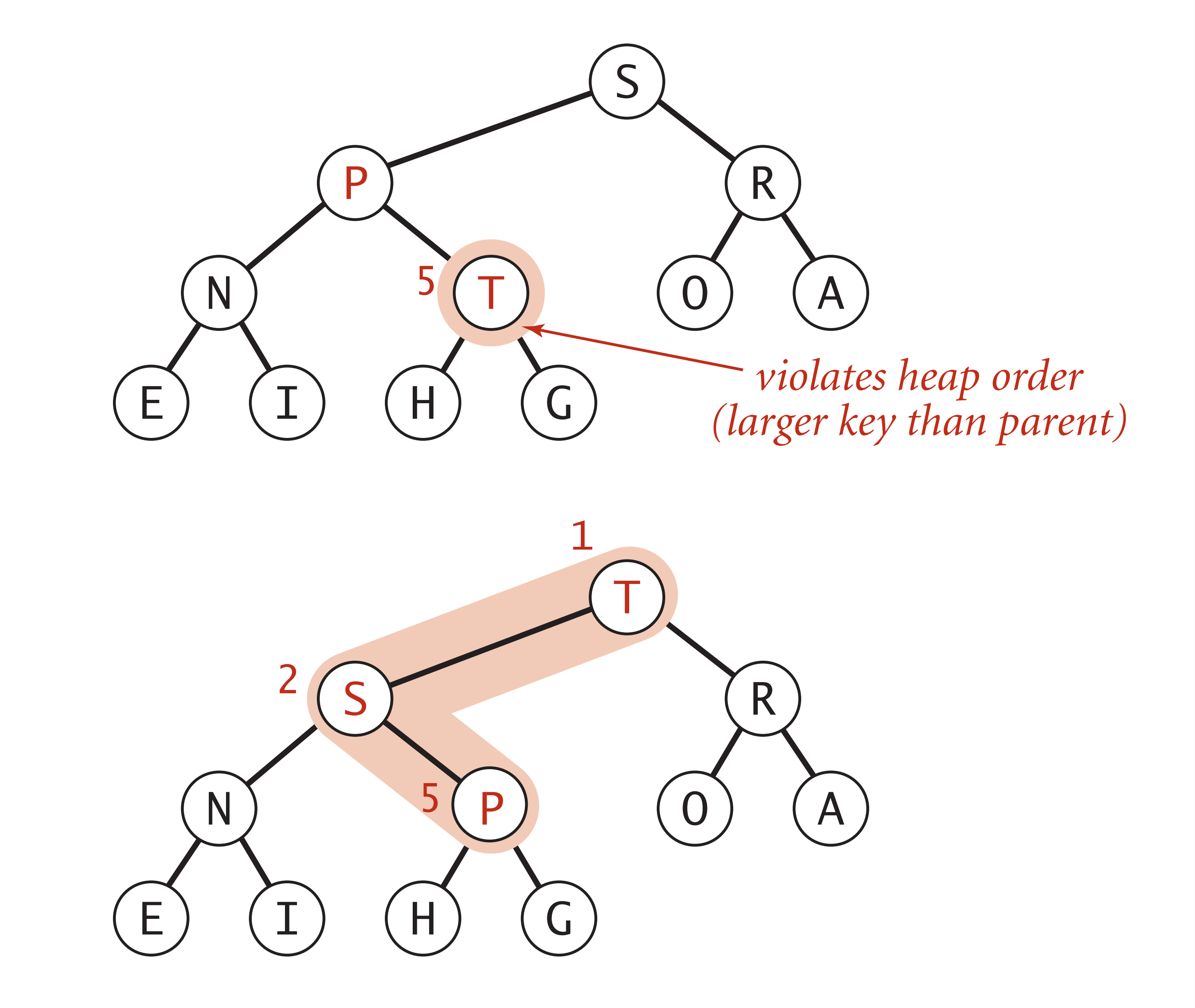
Scenario. Key in node becomes larger than key in parent's node.

To eliminate the violation:

- Exchange key in child node with key in parent node.
- Repeat until heap order restored.

```
private void swim(int k) {
   while (k > 1 && less(k/2, k)) {
      exch(k, k/2);
      k = k/2;
   }
}
```

*parent of node at* k *is at* k/2



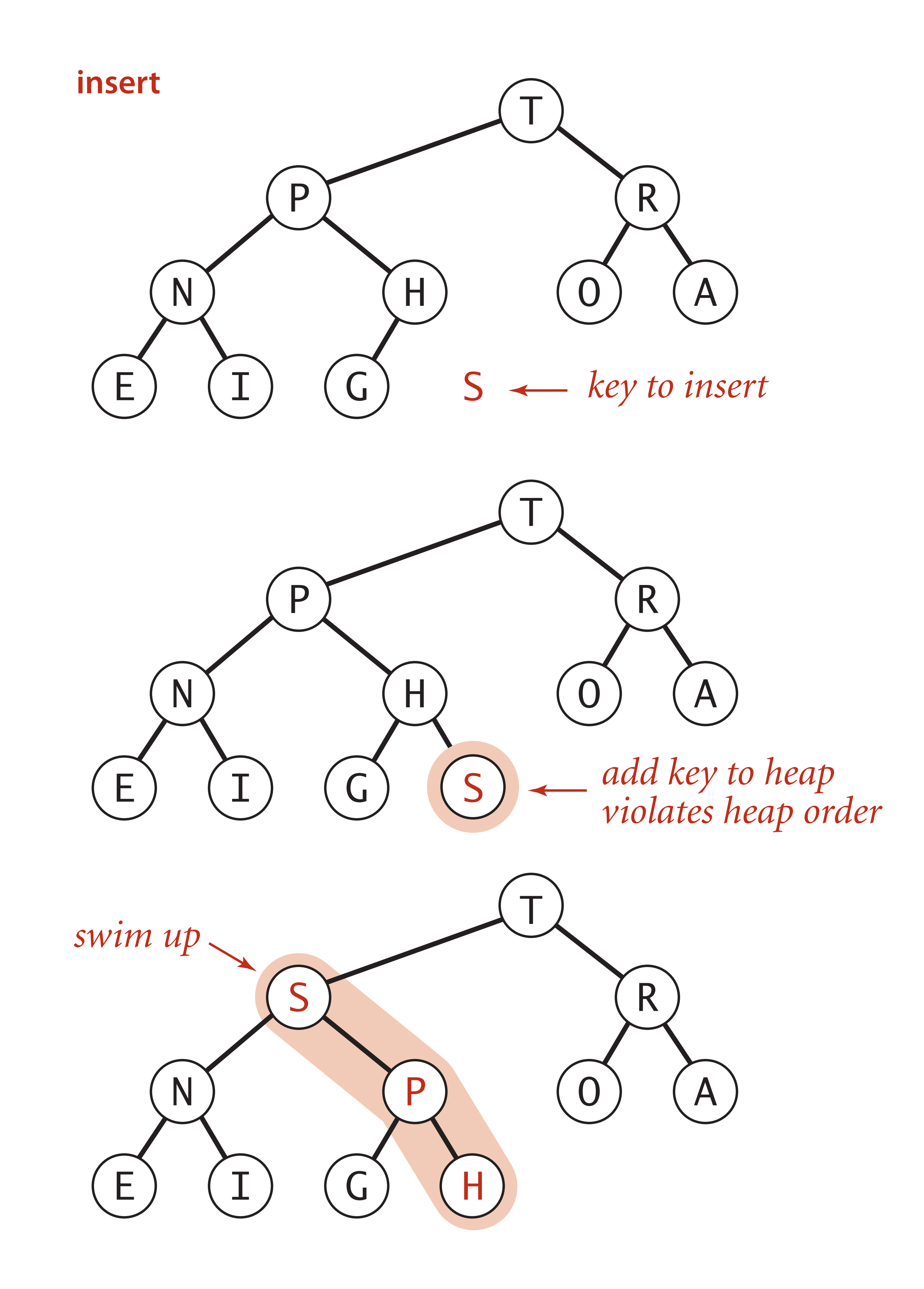

Peter principle. Node promoted to level of incompetence.

# Binary heap: insertion

Insert. Add node at end in bottom level; then, swim it up.

Cost. At most $1 + \log_2 n$ compares.

```
public void insert(Key x) {
   pq[++n] = x;
   swim(n);
}
```



# Binary heap: demotion

Scenario. Key in node becomes smaller than one (or both) of keys in childrens' nodes.

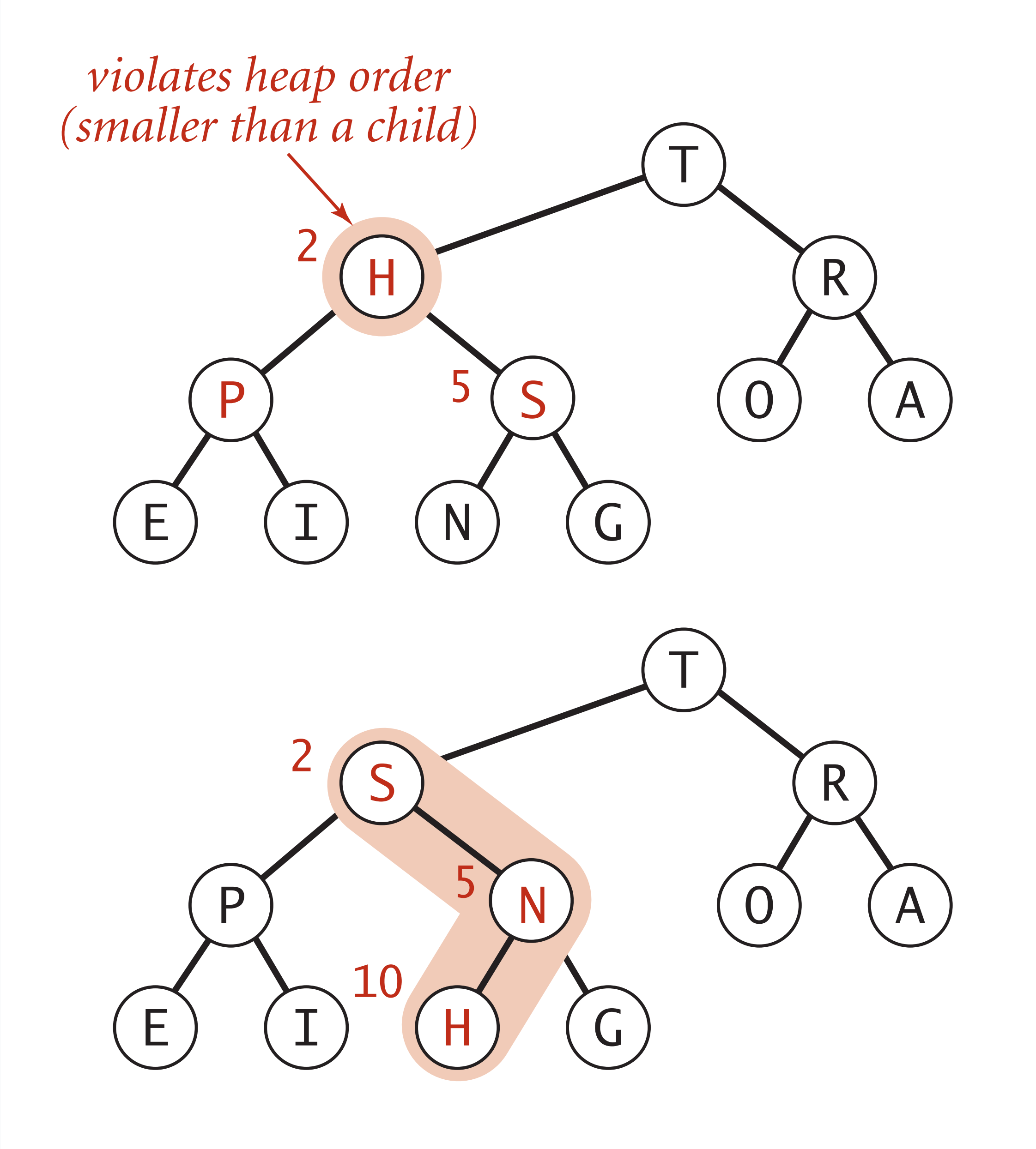
To eliminate the violation:

*why not smaller child ?*

- Exchange key in parent node with key in larger child's node.
- Repeat until heap order restored.

```
private void sink(int k) {
   while (2*k <= n) {
      int j = 2*k;
      if (j < n && less(j, j+1))
         j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

*children of node at k are at 2*k and 2*k+1*




Power struggle. Better subordinate promoted.

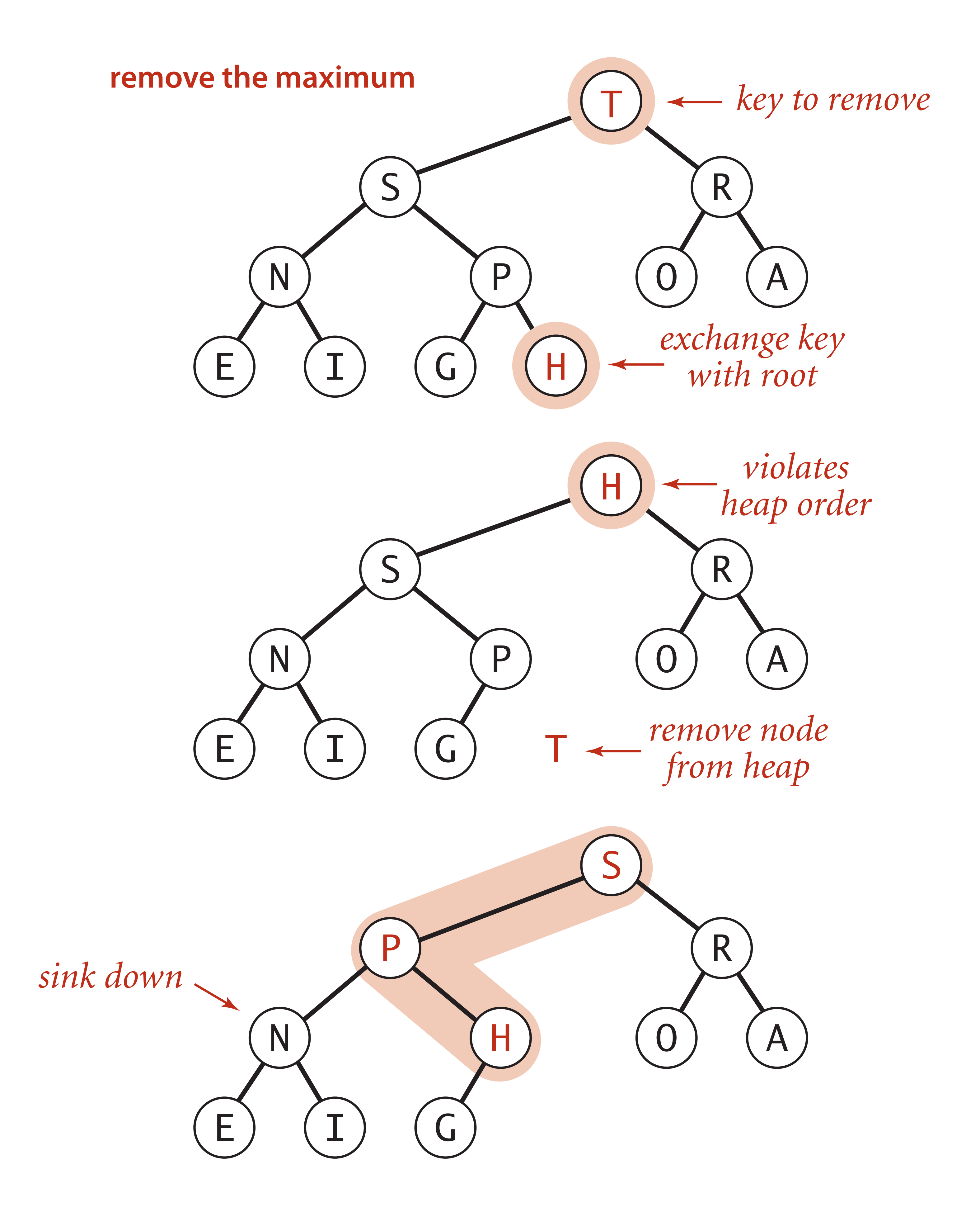
# Binary heap: delete the maximum

Delete max. Exchange root with node at end; then, sink it down.

Cost. At most $2 \log_2 n$ compares.

```
public Key delMax() {
   Key max = pq[1];
   exch(1, n--);
   sink(1);
   pq[n+1] = null;
   return max;
}
```

← *prevent loitering*



# Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>> {
   private Key[] a;
   private int n;

   public MaxPQ(int capacity) {
      a = (Key[]) new Comparable[capacity+1];
   }

   public void insert(Key key)   // see previous code
   public Key delMax()           // see previous code

   private void swim(int k)      // see previous code
   private void sink(int k)      // see previous code

   private boolean less(int i, int j) {
      return a[i].compareTo(a[j]) < 0;
   }

   private void exch(int i, int j)
   {  Key temp = a[i]; a[i] = a[j]; a[j] = temp;  }
}
```

*fixed capacity (for simplicity)*

*PQ ops*

*heap helper functions*

*array helper functions*

https://algs4.cs.princeton.edu/24pq/MaxPQ.java.html

# Priority queue: implementations cost summary

Goal. Implement both INSERT and DELETE-MAX in $\Theta(\log n)$ time.

| implementation | INSERT | DELETE-MAX | MAX |
|---|---|---|---|
| **unordered list** | $1$ | $n$ | $n$ |
| **ordered array** | $n$ | $1$ | $1$ |
| **goal** | $\log n$ | $\log n$ | $1$ |

**order of growth of running time for priority queue with n items**

# Binary heap: considerations

## Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

*leads to $O(\log n)$ amortized time per op (how to make worst case?)*

## Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

## Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

*can implement efficiently with* `sink()` *and* `swim()`
*[ stay tuned for Prim/Dijkstra ]*

## Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

*immutable in Java:* `String`, `Integer`, `Double`, …

# Priority queue with Delete-Random

Goal. Design an efficient data structure to support the following API:

- Insert: insert a key.
- Delete–Max: return and remove a largest key.
- Sample: return a random key.
- Delete–Random: return and remove a random key.

Midterm exam
Spring 2012

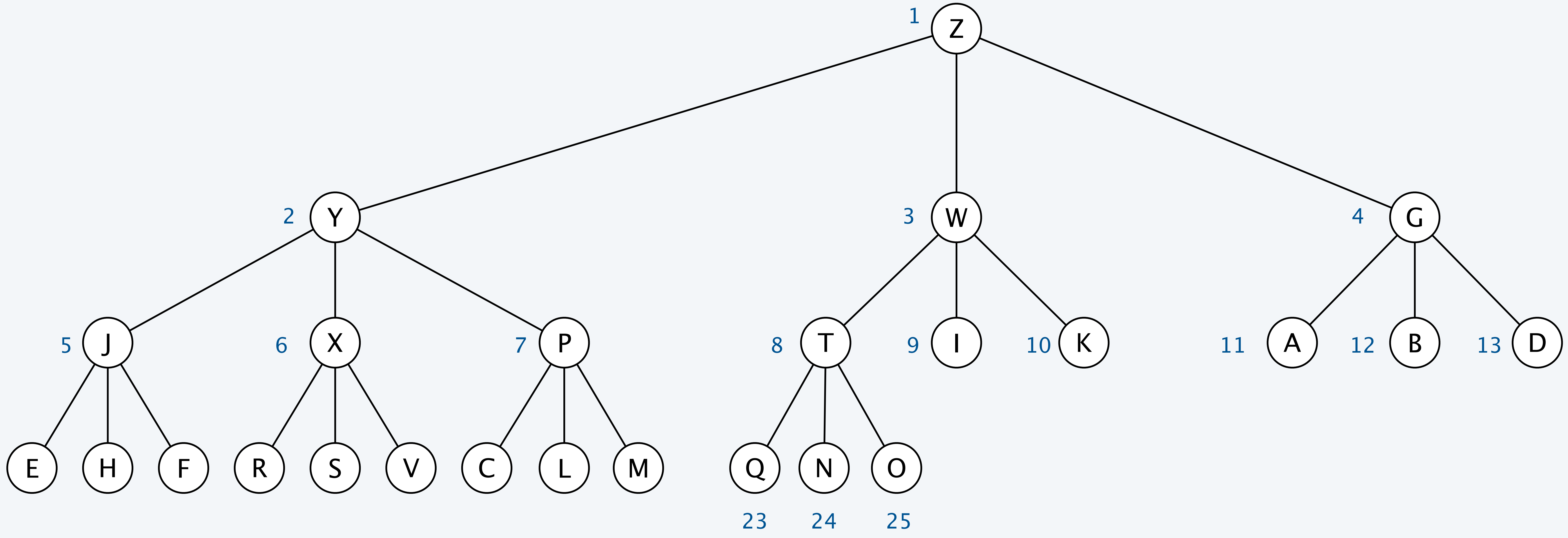
# Multiway heaps

Multiway heaps.

- Complete $d$-way tree.
- Child's key no larger than parent's key.

Property. Height of complete $d$-way tree on $n$ nodes is $\sim \log_d n$.

Property. Children of key at index $k$ at indices $3k - 1$, $3k$, and $3k + 1$; parent at $\lfloor (k+1) / 3 \rfloor$.




**3-way heap**

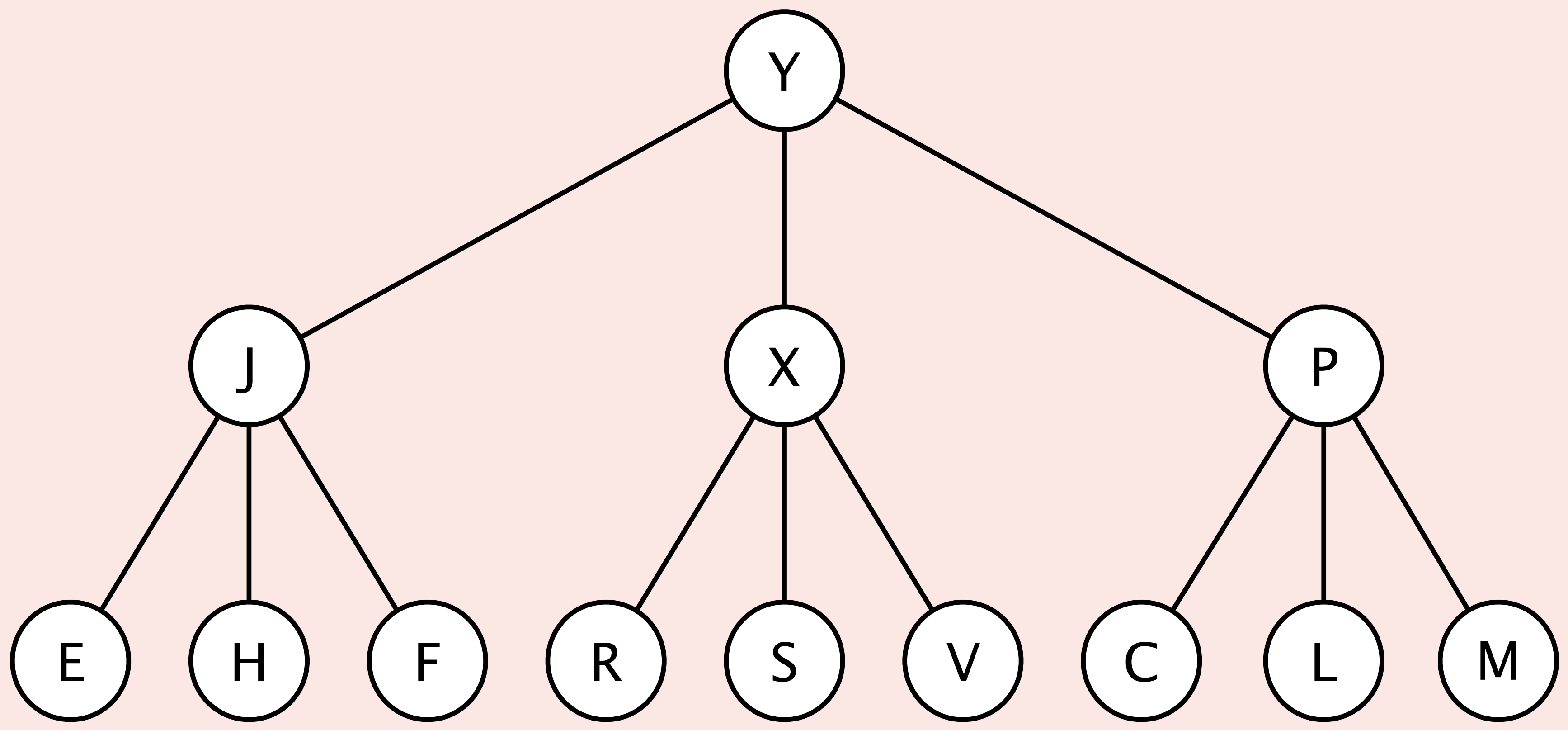
**In the worst case, how many compares to INSERT and DELETE-MAX in a $d$-way heap as function of both $n$ and $d$?**

**A.** $\sim \log_d n$ and $\sim \log_d n$

**B.** $\sim \log_d n$ and $\sim d \log_d n$

**C.** $\sim d \log_d n$ and $\sim \log_d n$

**D.** $\sim d \log_d n$ and $\sim d \log_d n$



# Priority queue: implementation cost summary

| implementation | INSERT | DELETE-MAX | MAX | |
|---|---|---|---|---|
| unordered list | $1$ | $n$ | $n$ | |
| ordered array | $n$ | $1$ | $1$ | |
| binary heap | $\log n$ | $\log n$ | $1$ | |
| d-ary heap | $\log_d n$ | $d \log_d n$ | $1$ | ← *sweet spot: $d = 4$* |
| Fibonacci | $1$ | $\log n$ | $1$ | ← *see* COS 423 |
| impossible | $1$ | $1$ | $1$ | ← *why impossible ?* |

order-of-growth of running time for priority queue with n items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# 2.4 Priority Queues

- ▸ *APIs*
- ▸ *elementary implementations*
- ▸ *binary heaps*
- ▸ ***heapsort***

**What are the properties of this sorting algorithm?**

```
public void sort(String[] a) {
    int n = a.length;
    MinPQ<String> pq = new MinPQ<String>();

    for (int i = 0; i < n; i++)
        pq.insert(a[i]);

    for (int i = 0; i < n; i++)
        a[i] = pq.delMin();
}
```

**A.** $\Theta(n \log n)$ compares in the worst case.

**B.** In-place.

**C.** Stable.

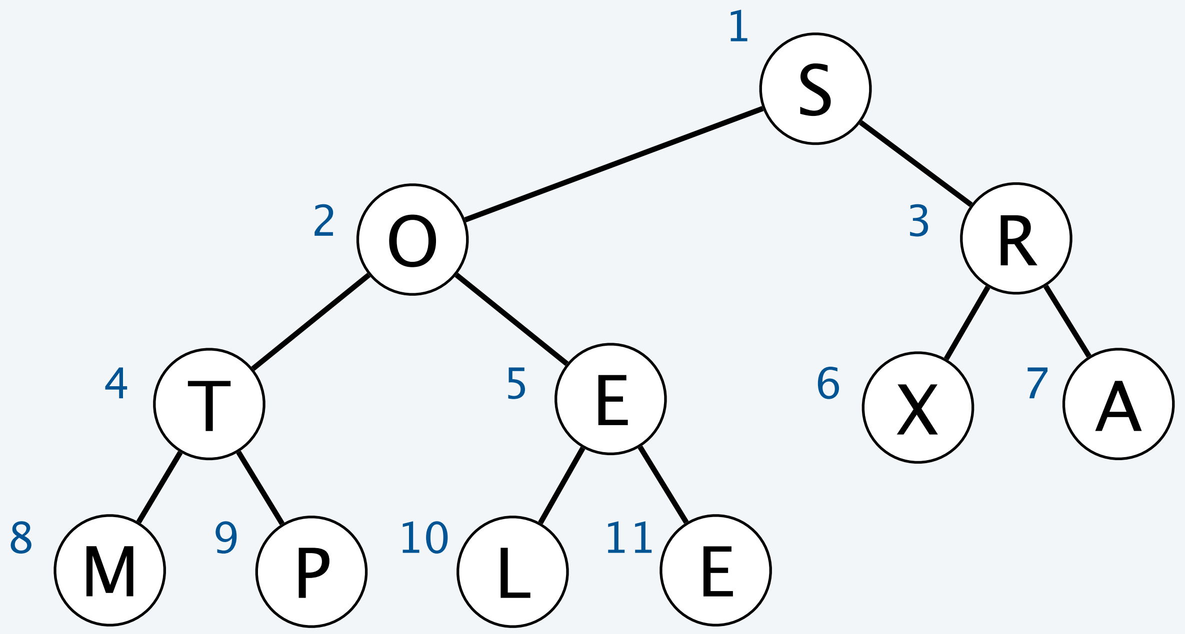
**D.** *All of the above.*

# Heapsort

## Basic plan for in-place sort.

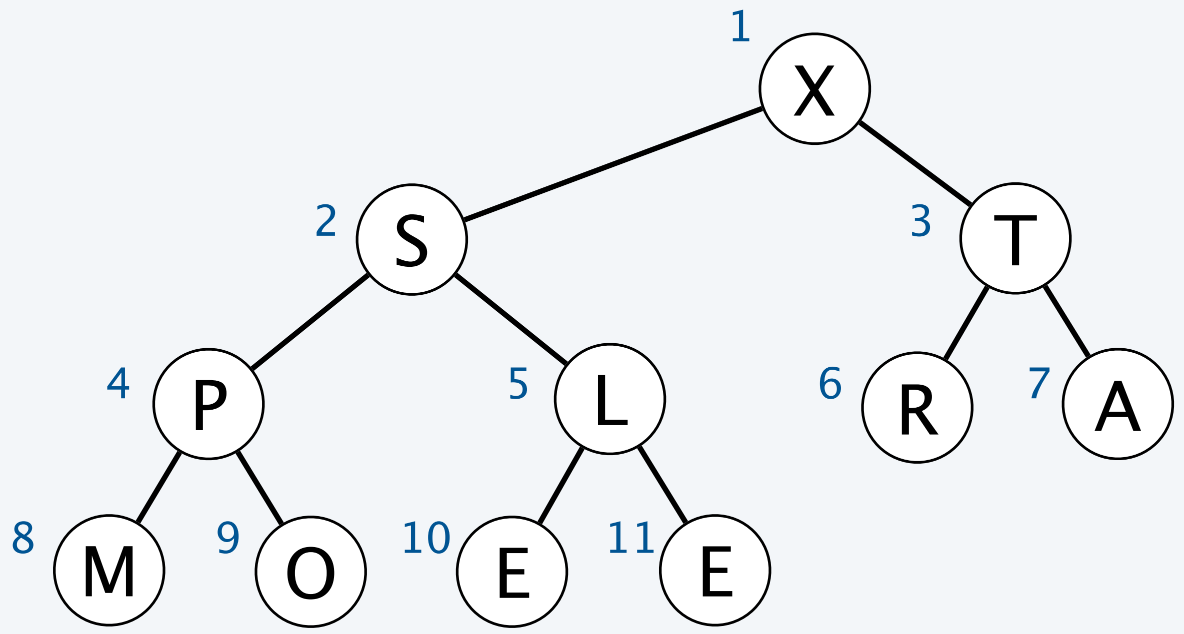
- View input array as a complete binary tree. ← *we'll assume 1-indexed for now*
- Phase 1 (heap construction): build a max-oriented heap.
- Phase 2 (sortdown): repeatedly remove the maximum key. ← *a version of selection sort*

**keys in arbitrary order**



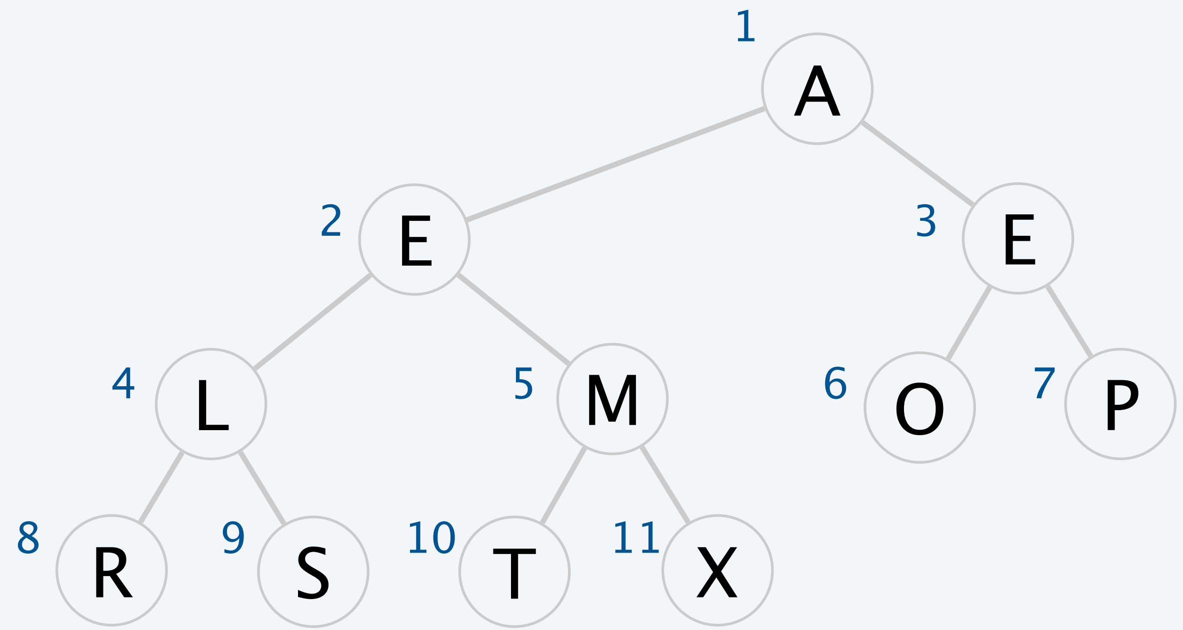

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| S | O | R | T | E | X | A | M | P | L | E |

**build max heap (in place)**




| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| X | S | T | P | L | R | A | M | O | E | E |

**sorted result (in place)**

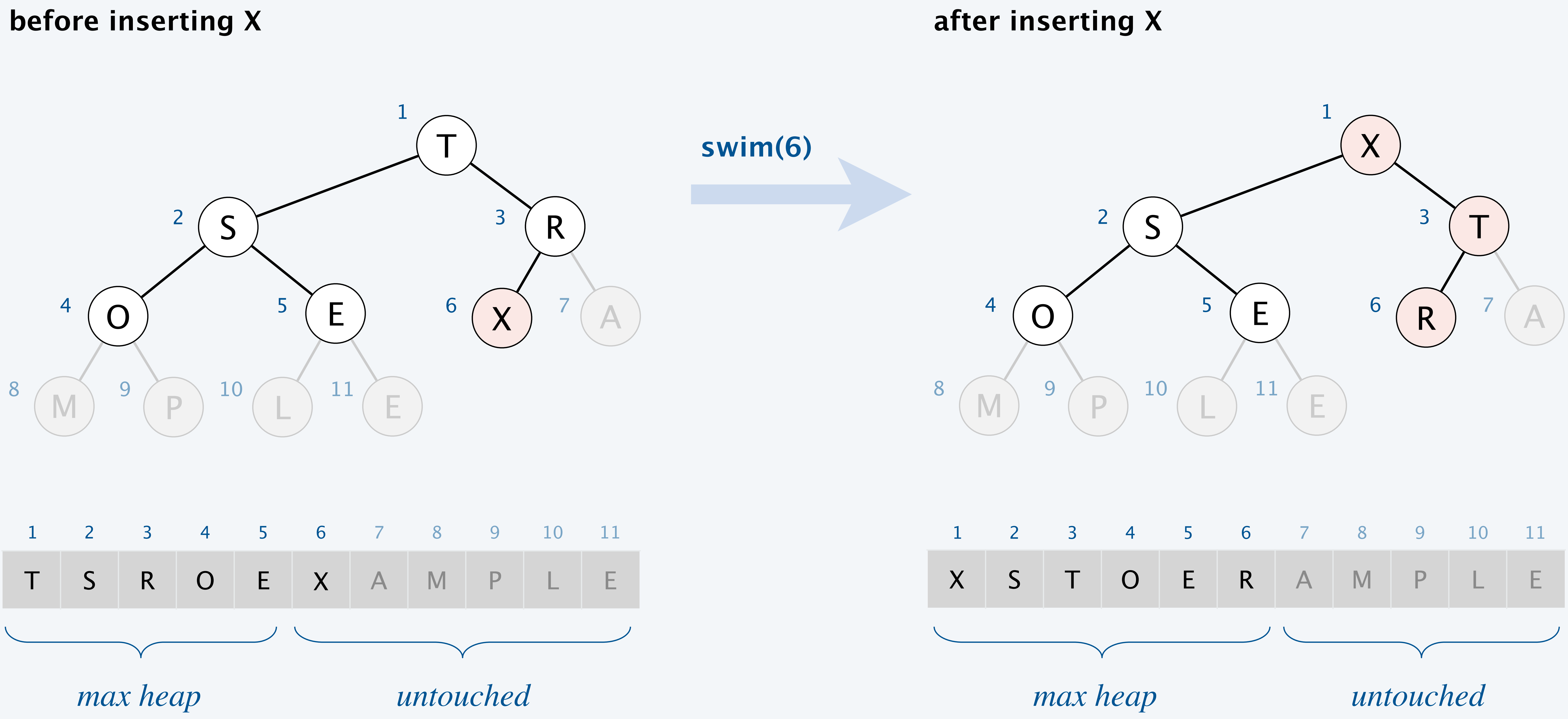



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | E | E | L | M | O | P | R | S | T | X |

# Heapsort: top-down heap construction

## Phase 1 (top-down heap construction).

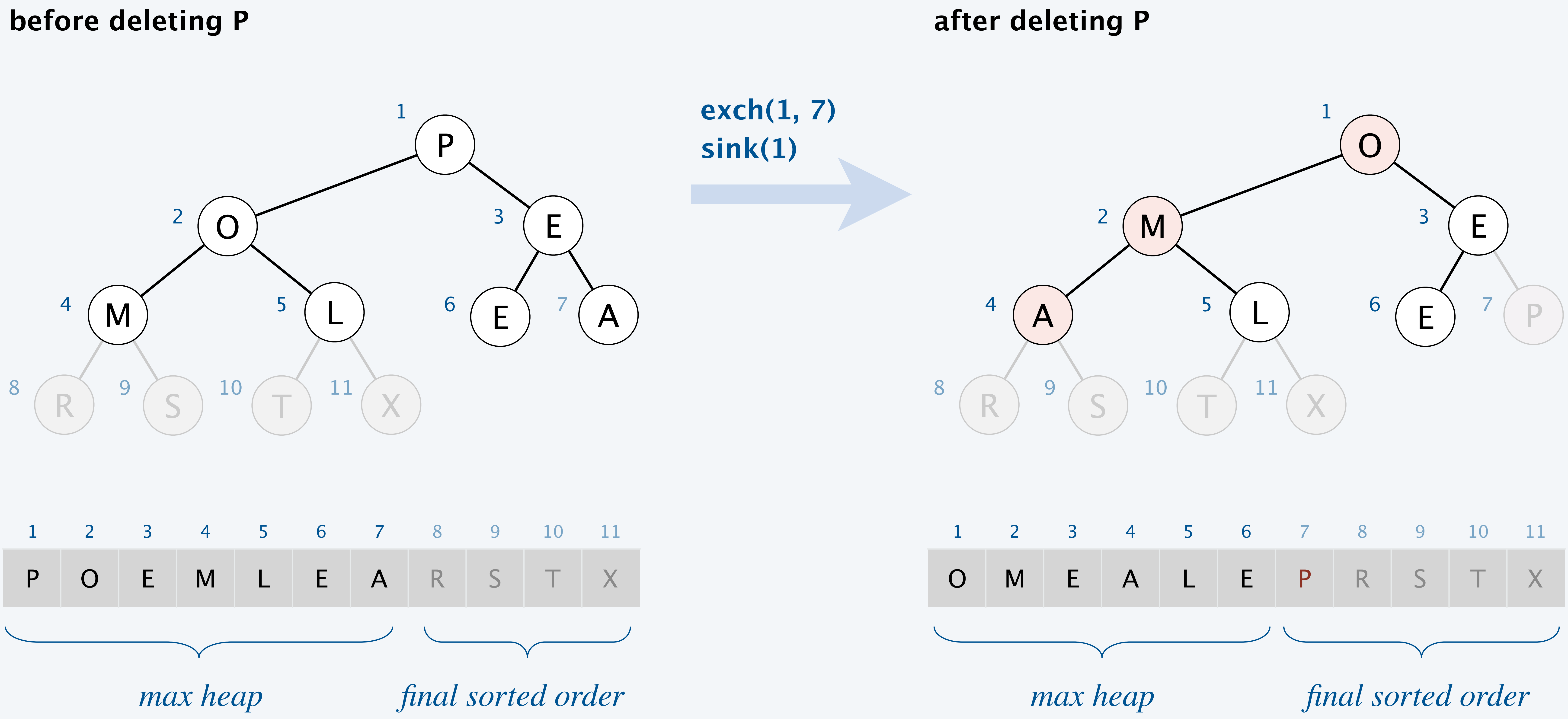
- View input array as complete binary tree.
- Insert keys into a max heap, one at a time.



# Heapsort: sortdown

## Phase 2 (sortdown).

- Remove the maximum, one at a time.
- Leave in array (instead of nulling out).



# Heapsort: Java implementation

```
public class HeapTopDown {

   public static void sort(Comparable[] a) {
      // top-down heap construction
      int n = a.length;
      for (int k = 1; k <= n; k++)
         swim(a, k);

      // sortdown
      int k = n;
      while (k > 1) {
         exch(a, 1, k--);
         sink(a, 1, k);
      }
   }

   ...
}
```

https://algs4.cs.princeton.edu/24pq/HeapTopDown.java.html

```
private static void sink(Comparable[] a, int k, int n)
{  /* as before */  }

private static void swim(Comparable[] a, int k)
{  /* as before */  }

private static boolean less(Comparable[] a, int i, int j)
{  /* as before */  }

private static void exch(Object[] a, int i, int j)
{  /* as before */  }
```

*but make static (and pass arguments* `a[]` *and* `n`*)* (applies to `sink` and `swim`)

*but convert from 1-based indexing to 0-base indexing* (applies to `less` and `exch`)

# Heapsort: mathematical analysis

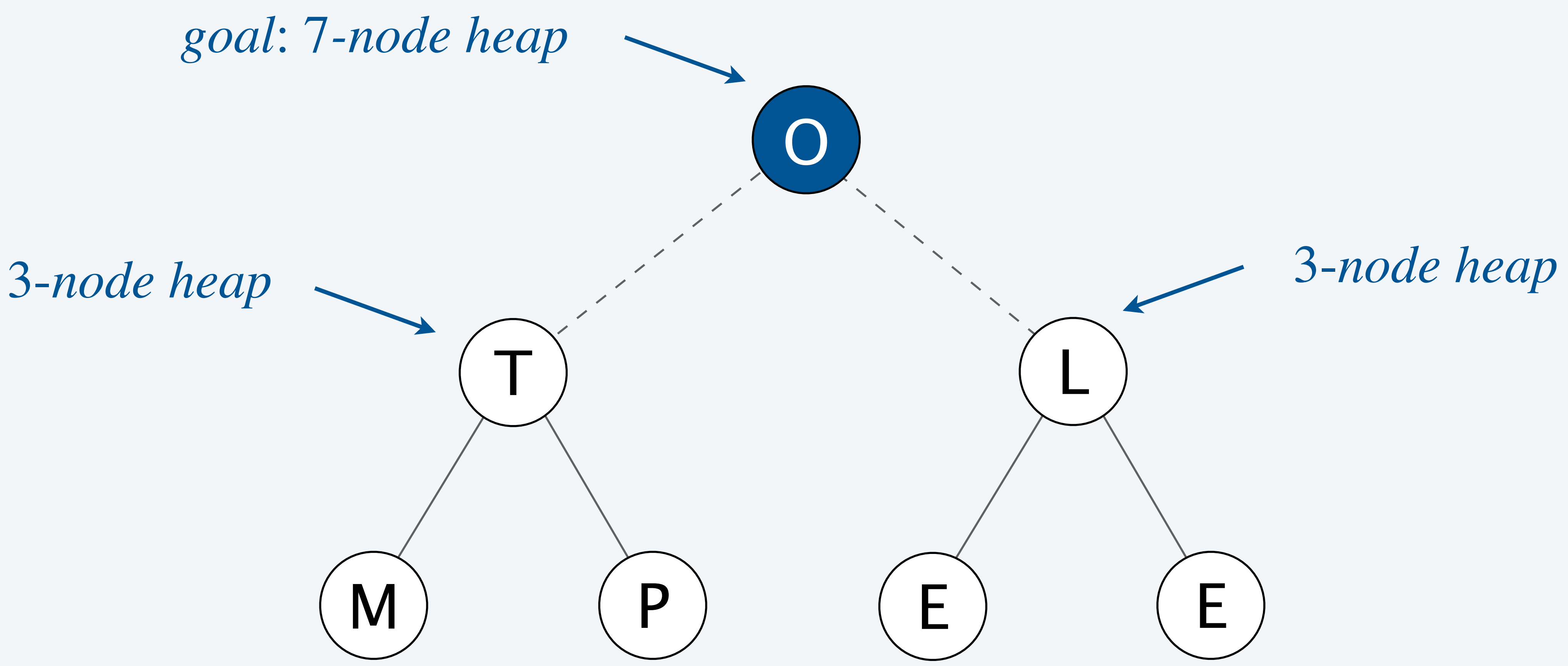
Proposition. Heapsort uses only $\Theta(1)$ extra space.

Proposition. Heapsort makes $\le 3\, n \log_2 n$ compares (and $\le 2\, n \log_2 n$ exchanges).

- Top-down heap construction: $\le n \log_2 n$ compares (and exchanges).
- Sortdown: $\le 2n \log_2 n$ compares (and $\le n \log_2 n$ exchanges).

Bottom-up heap construction. [see book] Successively building larger heap from smaller ones.

Proposition. Makes $\le 2\, n$ compares (and $\le n$ exchanges).



# Heapsort: context

Significance. In-place sorting algorithm with $\Theta(n \log n)$ worst-case.

- Mergesort: no, $\Theta(n)$ extra space. ⟵ *in-place merge possible; not practical*
- Quicksort: no, $\Theta(n^2)$ time in worst case. ⟵ *$\Theta(n \log n)$ worst-case quicksort possible; not practical*
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

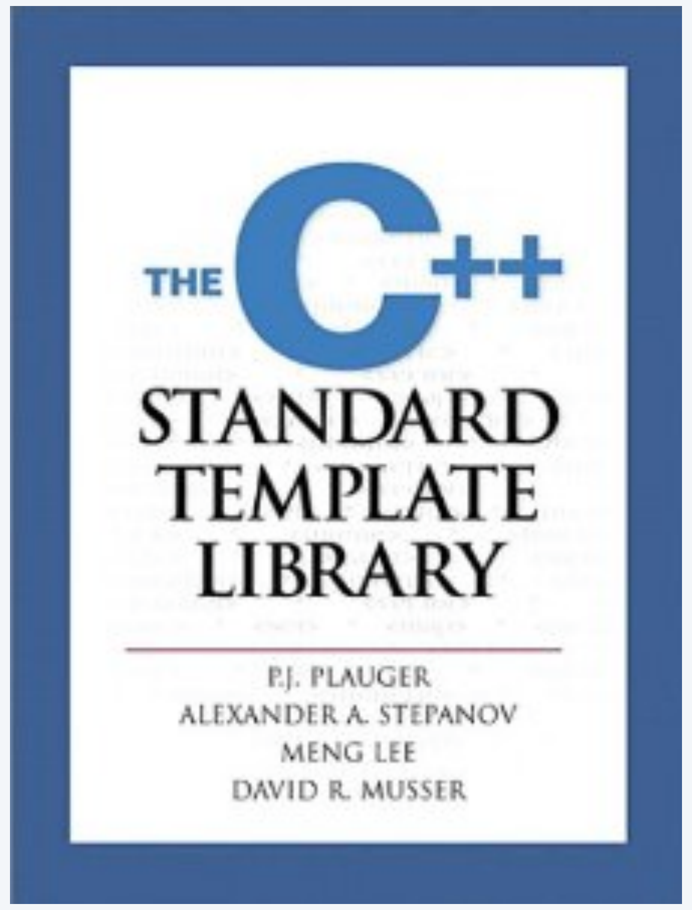
- Inner loop longer than quicksort's.
- Not stable.

# Introsort

Goal. As fast as quicksort in practice; in place; $\Theta(n \log n)$ worst case.

Introsort.

- Run quicksort.
- Cutoff to heapsort if function-call stack depth exceeds $2 \log_2 n$.
- Cutoff to insertion sort for $n \leq 16$.




In the wild. C++ STL, Microsoft .NET Framework, Go.

# Sorting algorithms: summary

| | inplace? | stable? | best | average | worst | remarks |
|---|---|---|---|---|---|---|
| **selection** | ✔ | | $\frac{1}{2} n^2$ | $\frac{1}{2} n^2$ | $\frac{1}{2} n^2$ | *n exchanges* |
| **insertion** | ✔ | ✔ | $n$ | $\frac{1}{4} n^2$ | $\frac{1}{2} n^2$ | *use for small n or partially ordered* |
| **merge** | | ✔ | $\frac{1}{2} n \log_2 n$ | $n \log_2 n$ | $n \log_2 n$ | $\Theta(n \log n)$ *guarantee; stable* |
| **timsort** | | ✔ | $n$ | $n \log_2 n$ | $n \log_2 n$ | *improves mergesort when pre-existing order* |
| **quick** | ✔ | | $n \log_2 n$ | $2 n \ln n$ | $\frac{1}{2} n^2$ | $\Theta(n \log n)$ *probabilistic guarantee; fastest in practice* |
| **3-way quick** | ✔ | | $n$ | $2 n \ln n$ | $\frac{1}{2} n^2$ | *improves quicksort when duplicate keys* |
| **heap** | ✔ | | $3 n$ | $2 n \log_2 n$ | $2 n \log_2 n$ | $\Theta(n \log n)$ *guarantee; in-place* |
| **?** | ✔ | ✔ | $n$ | $n \log_2 n$ | $n \log_2 n$ | *holy sorting grail* |

**number of compares to sort an array of n elements**

# Credits

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| *Emergency Room Triage* | unknown | |
| *Car GPS* | Adobe Stock | Education License |
| *Complete Binary Tree* | Shlomit Pinter | by author |
| *Computer and Supercomputer* | New York Times | |

# A final thought