COS302/SML305-Princeton University-Spring 2022
Assignment #8

Due: April 4, 2022 at 11:59pm

Upload at: `https://www.gradescope.com/courses/355863/assignments`

Remember to append your Colab PDF as explained in the first homework, with all outputs visible.
When you print to PDF it may be helpful to scale at 95% or so to get everything on the page.

Problem 1 (18pts)

A coin is weighted so that its probability of landing on heads is 0.2. Suppose you flip the coin 20 times.

(A) Compute the probability that the coin lands on heads at least 16 times.

(B) Use Markov's inequality to bound the event that the coin lands on heads at least 16 times.

(C) Use Chebyshev's inequality to bound the event that the coin lands on heads at least 16 times.

Problem 2 (20pts)

The moment generating function (MGF) for a random variable X is:

$$M_X(t) = \mathbb{E}[e^{tX}] .$$

One useful property of moment generating functions is that they make it relatively easy to compute weighted sums of independent random variables. Specifically, if $Z = \alpha X + \beta Y$ is the weighted sum of random variables X and Y for weights $\alpha, \beta \in \mathbb{R}$, then

$$M_X(\alpha t) M_Y(\beta t).$$

(A) Derive the MGF for a Poisson random variable X with parameter λ.

(B) Let X be a Poisson random variable with parameter λ, as above, and let Y be a Poisson random variable with parameter γ. Suppose X and Y are independent. Use the MGF to show that the sum $X + Y$ is also Poisson. What is the parameter of the resulting distribution?

(C) Derive the MGF for an exponential random variable X with parameter λ. For what values of t does the MGF $M_X(t)$ exist?

(D) Let X be an exponential random variable with parameter λ, and let $\alpha > 0$ be a constant. Show that the random variable αX is also exponentially distributed. What is its parameter?

(E) Imagine that you have k i.i.d. random variables $X_1, X_2, \ldots, X_k$, each of which is exponentially distributed with parameter λ. Show that this sum is gamma distributed and compute its parameters. This will require finding the MGF of the gamma distribution. Use the following parameterization of the gamma density:

$$f(z\,;\,\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}.$$

Problem 3 (20pts)

One remarkable Monte Carlo trick is called *importance sampling*. Imagine that there is a random variable $X : \Omega \to \mathbb{R}$ with probability density function $\pi(x)$. You would like to compute the expectation under X of some function $f(x)$ using Monte Carlo, i.e.,

$$\mathbb{E}_\pi[f(x)] = \int_{-\infty}^{\infty} \pi(x)\, f(x)\, dx \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i) \qquad \text{where } x_i \sim \pi(x)\,.$$

The above description has some typical "abuses of notation": 1) $\mathbb{E}_\pi[f(x)]$ denotes *"expectation of $f(x)$ under distribution with density $\pi(x)$"*, and 2) $x_i \sim \pi(x)$ means *"x_i are drawn according to distribution with density $\pi(x)$"*.

However, suppose that you cannot easily sample from the distribution given by $\pi(x)$? The trick is to find some other distribution that has the same support as $\pi(x)$ with density, say $q(x)$, and then multiply and divide by $q(x)$:

$$\mathbb{E}_\pi[f(x)] = \int_{-\infty}^{\infty} \pi(x)\, f(x)\, dx = \int_{-\infty}^{\infty} q(x)\,\frac{\pi(x)}{q(x)}\, f(x)\, dx = \mathbb{E}_q\left[\frac{\pi(x)}{q(x)}\, f(x)\right] \approx \frac{1}{n}\sum_{i=1}^{n} \frac{\pi(x_i)}{q(x_i)}\, f(x_i) \quad \text{where } x_i \sim q(x)$$

This turns an expectation under $\pi(x)$ into an expectation under the more convenient $q(x)$. This procedure is called *importance sampling* because the evaluations of $f(x)$ are weighted according to the ratio between the *target density* $\pi(x)$ and the *proposal density* $q(x)$. In this problem, we will use importance sampling to explore properties of the distribution with two-parameter density

$$\pi(x\,;\,\beta) = \frac{\beta}{2\Gamma(1/\beta)} e^{-|x|^\beta},$$

where $\beta > 0$. The mean of this distribution is zero. Do the following problems in a `Colab` notebook and turn in your answer via a PDF and shared link as usual.

(A) Estimate the variance of this distribution for $\beta = 2$ using importance sampling. Use a standard normal distribution (Gaussian with mean zero and variance one) for the proposal distribution $q(x)$. Write down the formula for the function whose expectation you'll be taking under $q(x)$.

(B) Use `numpy` to estimate the expectation numerically with 10,000 samples from $q(x)$.

(C) One way to investigate how well your importance sampler is working is to look at the distribution of the weights $\pi(x_i)/q(x_i)$. If a few of these weights are much larger than the others, then your Monte Carlo estimate will really only be incorporating information about those samples with big weights. That is bad. Print the variance of the weights from your samples in part (B), and also create a histogram of the log weights.

(D) Now perform the procedure from (B) and (C) above again, but where $q(x)$ is Gaussian with a mean of 5 and variance one. What happens to the estimate and the weights? Why do you think that happened?

(E) Do (B) and (C) a third time, but put $q(x)$ back to a standard normal and make $\beta = 0.5$ in the target distribution. The means of the proposal and target are the same, but it is still behaving badly. Why do you think that is?

Problem 4 (40pts)

In 1777, Georges-Louis Leclerc, Comte de Buffon introduced a simple Monte Carlo method for estimation, now called the Buffon's needle problem. Buffon's needle problem is a fun way to estimate π via physical simulation. In this problem, we'll look at the following variant: you have a piece of paper on a tabletop and drawn on it is a square with sides of length w. Centered and entirely contained in the square is a circle with diameter d, with $d \leq w$. You drop points (say, grains of sand) onto the piece of paper in a spatially uniform way. You count the grains and find that the ratio of grains inside the circle to the total number grains inside the square is r:

$$r = \frac{\text{\# of grains inside circle}}{\text{\# of grains inside square}}.$$

(A) Write an estimate for π in terms of w, d, and r.

(B) Assume that the diameter of the circle is 2. What box width w minimizes the variance of your estimate of π when you form this Monte Carlo estimate?

(C) Perform a real-life version of this experiment and estimate π with physical simulation. You could do your experiment with paper and sand or something, as described above. Or you can be more creative! Some ideas:

- Grate a known amount of cheese onto a square pan that has a pizza in the middle. Weigh the cheese that doesn't fall onto the pizza.
- Put a circular cup into a box and drop coins or peas or something into the box and count the fraction that fall into the cup.
- Make a batch of cookies that are all perfect circles. Spread sprinkles over the entire baking pan and count (or weigh) how many don't make it onto cookies. You'll need to work out the compensation for having more than one cookie.
- Put a round bucket inside a square bucket and stick it out in the rain. Measure the difference in water collected.

You will need to measure the dimensions of the rectangular region and the diameter of the circle. Draw a couple of dozen samples and report your estimate. Explain what you did and take a photograph of your setup. The course staff will vote on their favorites.

Problem 5 (2pts)

Approximately how many hours did this assignment take you to complete?

My notebook URL: `https://colab.research.google.com/XXXXXXXXXXXXXXXXXXXXXXX`