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3.1 Symbol Tables

- *API*
- *elementary implementations*
- *ordered operations*

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3.1 SYMBOL TABLES

- *API*
- *elementary implementations*
- *ordered operations*

Symbol tables

Key-value pair abstraction.

- Insert a value with specified key.
- Given a key, search for the corresponding value.

Ex. DNS lookup.

- Insert domain name with specified IP address.
- Given domain name, find corresponding IP address.

domain name	IP address
www.cs.princeton.edu	128.112.136.11
www.princeton.edu	128.112.128.15
www.yale.edu	130.132.143.21
www.harvard.edu	128.103.060.55
www.simpsons.com	209.052.165.60

key value

Symbol table applications

application	purpose of search	key	value
dictionary	find definition	word	definition
book index	find relevant pages	term	list of page numbers
file share	find song to download	name of song	computer ID
financial account	process transactions	account number	transaction details
web search	find relevant web pages	keyword	list of page names
compiler	find properties of variables	variable name	type and value
routing table	route Internet packets	destination	best route
DNS	find IP address	domain name	IP address
reverse DNS	find domain name	IP address	domain name
genomics	find markers	DNA string	known positions
file system	find file on disk	filename	location on disk

Symbol tables: context

Also known as: maps, dictionaries, associative arrays.

Generalizes arrays. Keys need not be between 0 and $N - 1$.

Language support.

- External libraries: C, VisualBasic, Standard ML, bash, ...
- Built-in libraries: Java, C#, C++, Scala, ...
- Built-in to language: Awk, Perl, PHP, Tcl, JavaScript, Python, Ruby, Lua.

every array is an associative array

every object is an associative array

table is the only primitive data structure

```
hasNiceSyntaxForAssociativeArrays["Python"] = true
hasNiceSyntaxForAssociativeArrays["Java"]   = false
```

legal Python code

Basic symbol table API

Associative array abstraction. Associate one value with each key.

`public class ST<Key, Value>`		
	`ST()`	*create an empty symbol table*
`void`	`put(Key key, Value val)`	*put key-value pair into the table* ← `a[key] = val;`
`Value`	`get(Key key)`	*value paired with key* ← `a[key]`
`boolean`	`contains(Key key)`	*is there a value paired with key?*
`void`	`delete(Key key)`	*remove key (and its value) from table*
`boolean`	`isEmpty()`	*is the table empty?*
`int`	`size()`	*number of key-value pairs in the table*
`Iterable<Key>`	`keys()`	*all the keys in the table*

Conventions

- Values are not `null`. ← Java allows null value
- Method `get()` returns `null` if key not present.
- Method `put()` overwrites old value with new value.

Intended consequences.

- Easy to implement `contains()`.

```
public boolean contains(Key key)
{  return get(key) != null;  }
```

- Can implement lazy version of `delete()`.

```
public void delete(Key key)
{  put(key, null);  }
```

Keys and values

Value type. Any generic type.

specify Comparable in API.

Key type: several natural assumptions.

- Assume keys are `Comparable`, use `compareTo()`.
- Assume keys are any generic type, use `equals()` to test equality.
- Assume keys are any generic type, use `equals()` to test equality; use `hashCode()` to scramble key.

built-in to Java
(stay tuned)

Best practices. Use immutable types for symbol table keys.

- Immutable in Java: `Integer`, `Double`, `String`, `java.io.File`, …
- Mutable in Java: `StringBuilder`, `java.net.URL`, arrays, ...

Equality test

All Java classes inherit a method `equals()`.

Java requirements. For any references `x`, `y` and `z`:

- Reflexive: `x.equals(x)` is `true`.
- Symmetric: `x.equals(y)` iff `y.equals(x)`.
- Transitive: if `x.equals(y)` and `y.equals(z)`, then `x.equals(z)`.
- Non-null: `x.equals(null)` is `false`.

equivalence relation

do x and y refer to the same object?

Default implementation. `(x == y)`

Customized implementations. `Integer`, `Double`, `String`, `java.io.File`, ...

User-defined implementations. Some care needed.

Implementing equals for user-defined types

Seems easy.

```
public       class Date implements Comparable<Date>
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Date that)
   {

      if (this.day   != that.day  ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

check that all significant fields are the same

Implementing equals for user-defined types

Seems easy, but requires some care.

typically unsafe to use equals() with inheritance (would violate symmetry)

```
public final class Date implements Comparable<Date>
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Object y)
   {
      if (y == this) return true;

      if (y == null) return false;

      if (y.getClass() != this.getClass())
         return false;

      Date that = (Date) y;
      if (this.day   != that.day  ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

must be Object.
Why? Experts still debate.

optimize for true object equality

check for null

objects must be in the same class
(religion: getClass() vs. instanceof)

cast is guaranteed to succeed

check that all significant fields are the same

Equals design

"Standard" recipe for user-defined types.

- Optimization for reference equality.
- Check against `null`.
- Check that two objects are of the same type and cast.
- Compare each significant field:
  - if field is a primitive type, use == ← but use Double.compare() with double (or otherwise deal with -0.0 and NaN)
  - if field is an object, use `equals()` ← apply rule recursively
  - if field is an array, apply to each entry ← can use `Arrays.deepEquals(a, b)` but not `a.equals(b)`

Best practices.

e.g., cached Manhattan distance

- No need to use calculated fields that depend on other fields.
- Compare fields mostly likely to differ first.
- Make `compareTo()` consistent with `equals()`.

`x.equals(y)` if and only if `(x.compareTo(y) == 0)`

ST test client for traces

Build ST by associating value i with i^{th} string from standard input.

```
public static void main(String[] args)
{
   ST<String, Integer> st = new ST<String, Integer>();
   for (int i = 0; !StdIn.isEmpty(); i++)
   {
     String key = StdIn.readString();
     st.put(key, i);
   }
   for (String s : st.keys())
      StdOut.println(s + " " + st.get(s));
}
```

keys	S	E	A	R	C	H	E	X	A	M	P	L	E
values	0	1	2	3	4	5	6	7	8	9	10	11	12

output

```
A  8
C  4
E  12
H  5
L  11
M  9
P  10
R  3
S  0
X  7
```

ST test client for analysis

Frequency counter. Read a sequence of strings from standard input and print out one that occurs with highest frequency.

```
% more tinyTale.txt
it was the best of times
it was the worst of times
it was the age of wisdom
it was the age of foolishness
it was the epoch of belief
it was the epoch of incredulity
it was the season of light
it was the season of darkness
it was the spring of hope
it was the winter of despair

% java FrequencyCounter 1 < tinyTale.txt
it 10

% java FrequencyCounter 8 < tale.txt
business 122

% java FrequencyCounter 10 < leipzig1M.txt
government 24763
```

tiny example
(60 words, 20 distinct)

real example
(135,635 words, 10,769 distinct)

real example
(21,191,455 words, 534,580 distinct)

Frequency counter implementation

```
public class FrequencyCounter
{
   public static void main(String[] args)
   {
      int minlen = Integer.parseInt(args[0]);
      ST<String, Integer> st = new ST<String, Integer>();
      while (!StdIn.isEmpty())
      {
         String word = StdIn.readString();
         if (word.length() < minlen) continue;
         if (!st.contains(word)) st.put(word, 1);
         else                    st.put(word, st.get(word) + 1);
      }
      String max = "";
      st.put(max, 0);
      for (String word : st.keys())
         if (st.get(word) > st.get(max))
            max = word;
      StdOut.println(max + " " + st.get(max));
   }
}
```

create ST

ignore short strings

read string and update frequency

print a string with max freq

Algorithms

Robert Sedgewick | Kevin Wayne

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3.1 Symbol Tables

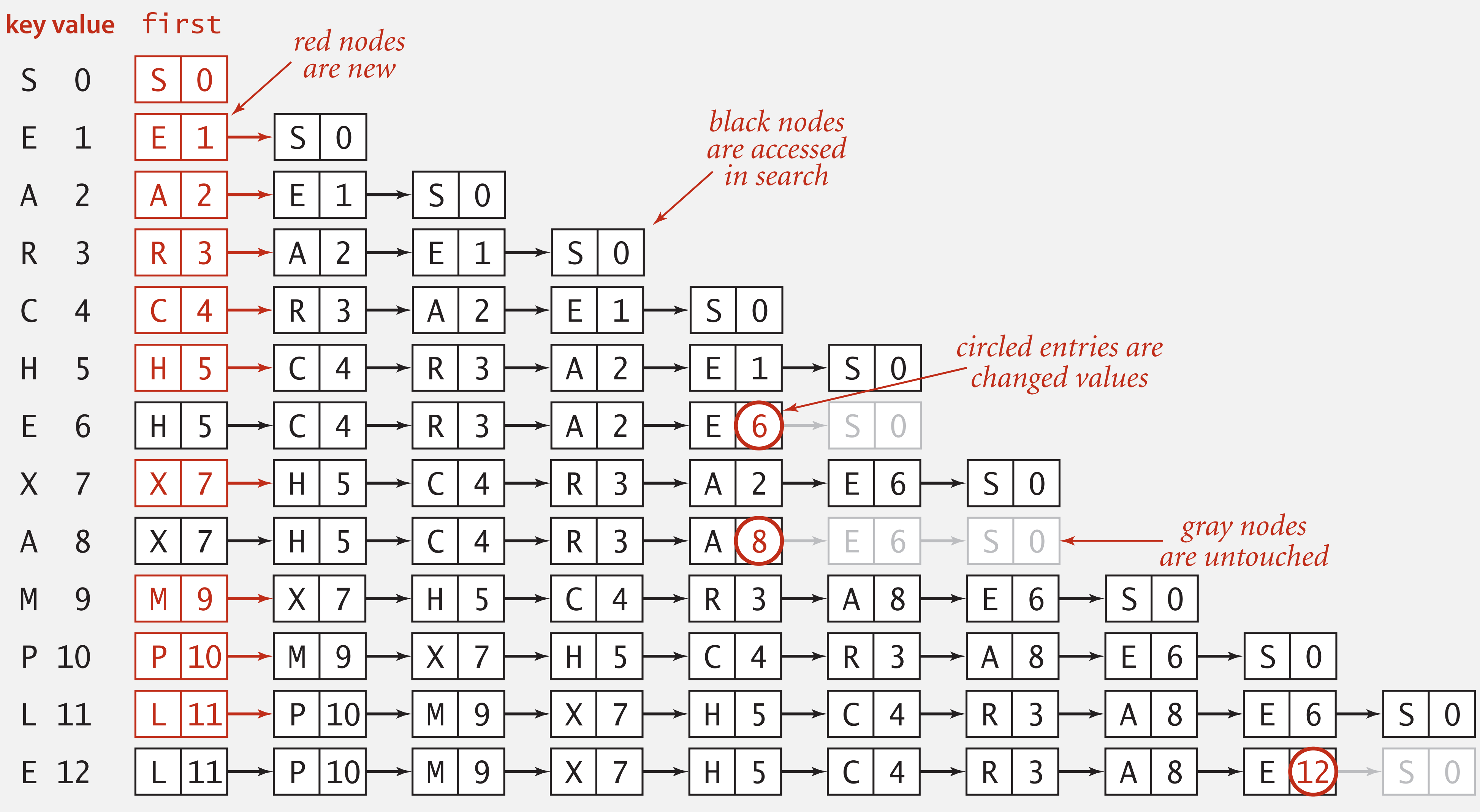
- *API*
- *elementary implementations*
- *ordered operations*

Sequential search in a linked list

Data structure. Maintain an (unordered) linked list of key-value pairs.

Search. Scan through all keys until find a match.

Insert. Scan through all keys until find a match; if no match add to front.



Trace of linked-list ST implementation for standard indexing client

Elementary ST implementations: summary

ST implementation	guarantee		average case		key interface
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	$N/2$	N	`equals()`

Challenge. Efficient implementations of both search and insert.

Binary search in an ordered array

Data structure. Maintain an ordered array of key-value pairs.

Rank helper function. How many keys $< k$?

`keys[]`

0	1	2	3	4	5	6	7	8	9
A	C	E	H	L	M	P	R	S	X

successful search for P

`lo`	`hi`	`m`										
0	9	4	A	C	E	H	L	M	P	R	S	X
5	9	7	A	C	E	H	L	M	P	R	S	X
5	6	5	A	C	E	H	L	M	P	R	S	X
6	6	6	A	C	E	H	L	M	P	R	S	X

entries in black are `a[lo..hi]`

entry in red is `a[m]`

loop exits with `keys[m] = P`: *return* 6

unsuccessful search for Q

`lo`	`hi`	`m`										
0	9	4	A	C	E	H	L	M	P	R	S	X
5	9	7	A	C	E	H	L	M	P	R	S	X
5	6	5	A	C	E	H	L	M	P	R	S	X
7	6	6	A	C	E	H	L	M	P	R	S	X

loop exits with `lo > hi`: *return* 7

Binary search: Java implementation

```
public Value get(Key key)
{
   if (isEmpty()) return null;
   int i = rank(key);
   if (i < N && keys[i].compareTo(key) == 0) return vals[i];
   else return null;
}
```

```
private int rank(Key key)                          number of keys < key
{
   int lo = 0, hi = N-1;
   while (lo <= hi)
   {
       int mid = lo + (hi - lo) / 2;
       int cmp = key.compareTo(keys[mid]);
       if      (cmp  < 0) hi = mid - 1;
       else if (cmp  > 0) lo = mid + 1;
       else if (cmp == 0) return mid;
  }
  return lo;
}
```

Binary search: trace of standard indexing client

Problem. To insert, need to shift all greater keys over.

		keys[]											vals[]									
key	value	0	1	2	3	4	5	6	7	8	9	N	0	1	2	3	4	5	6	7	8	9
S	0	S										1	0									
E	1	E	S									2	1	0								
A	2	A	E	S								3	2	1	0							
R	3	A	E	R	S							4	2	1	3	0						
C	4	A	C	E	R	S						5	2	4	1	3	0					
H	5	A	C	E	H	R	S					6	2	4	1	5	3	0				
E	6	A	C	E	H	R	S					6	2	4	6	5	3	0				
X	7	A	C	E	H	R	S	X				7	2	4	6	5	3	0	7			
A	8	A	C	E	H	R	S	X				7	8	4	6	5	3	0	7			
M	9	A	C	E	H	M	R	S	X			8	8	4	6	5	9	3	0	7		
P	10	A	C	E	H	M	P	R	S	X		9	8	4	6	5	9	10	3	0	7	
L	11	A	C	E	H	L	M	P	R	S	X	10	8	4	6	5	11	9	10	3	0	7
E	12	A	C	E	H	L	M	P	R	S	X	10	8	4	12	5	11	9	10	3	0	7
		A	C	E	H	L	M	P	R	S	X		8	4	12	5	11	9	10	3	0	7

entries in red were inserted

entries in gray did not move

entries in black moved to the right

circled entries are changed values

Elementary ST implementations: summary

ST implementation	guarantee		average case		key interface
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	$N / 2$	N	`equals()`
binary search (ordered array)	$\log N$	N	$\log N$	$N / 2$	`compareTo()`

Challenge. Efficient implementations of both search and insert.

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3.1 SYMBOL TABLES

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Examples of ordered symbol table API

	keys	*values*
min() →	09:00:00	Chicago
	09:00:03	Phoenix
get(09:00:13) →	09:00:13	Houston
	09:00:59	Chicago
	09:01:10	Houston
floor(09:05:00) →	09:03:13	Chicago
	09:10:11	Seattle
select(7) →	09:10:25	Seattle
	09:14:25	Phoenix
	09:19:32	Chicago
	09:19:46	Chicago
keys(09:15:00, 09:25:00) →	09:21:05	Chicago
	09:22:43	Seattle
	09:22:54	Seattle
	09:25:52	Chicago
ceiling(09:30:00) →	09:35:21	Chicago
	09:36:14	Seattle
max() →	09:37:44	Phoenix

size(09:15:00, 09:25:00) *is* 5

rank(09:10:25) *is* 7

Ordered symbol table API

```
public class ST<Key extends Comparable<Key>, Value>

                 ...

              Key  min()                   
              Key  max()                   
              Key  floor(Key key)          
              Key  ceiling(Key key)        
              int  rank(Key key)           
              Key  select(int k)           
             void  deleteMin()             
             void  deleteMax()             
              int  size(Key lo, Key hi)    
    Iterable<Key>  keys()                  
    Iterable<Key>  keys(Key lo, Key hi)    
```

Key	min()	*smallest key*
Key	max()	*largest key*
Key	floor(Key key)	*largest key less than or equal to key*
Key	ceiling(Key key)	*smallest key greater than or equal to key*
int	rank(Key key)	*number of keys less than key*
Key	select(int k)	*key of rank k*
void	deleteMin()	*delete smallest key*
void	deleteMax()	*delete largest key*
int	size(Key lo, Key hi)	*number of keys between lo and hi*
Iterable<Key>	keys()	*all keys, in sorted order*
Iterable<Key>	keys(Key lo, Key hi)	*keys between lo and hi, in sorted order*

Binary search: ordered symbol table operations summary

	sequential search	binary search
search	N	$\log N$
insert / delete	N	N
min / max	N	1
floor / ceiling	N	$\log N$
rank	N	$\log N$
select	N	1
ordered iteration	$N \log N$	N

order of growth of the running time for ordered symbol table operations

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3.2 Binary Search Trees

- BSTs
- ordered operations
- deletion

Algorithms

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3.2 Binary Search Trees

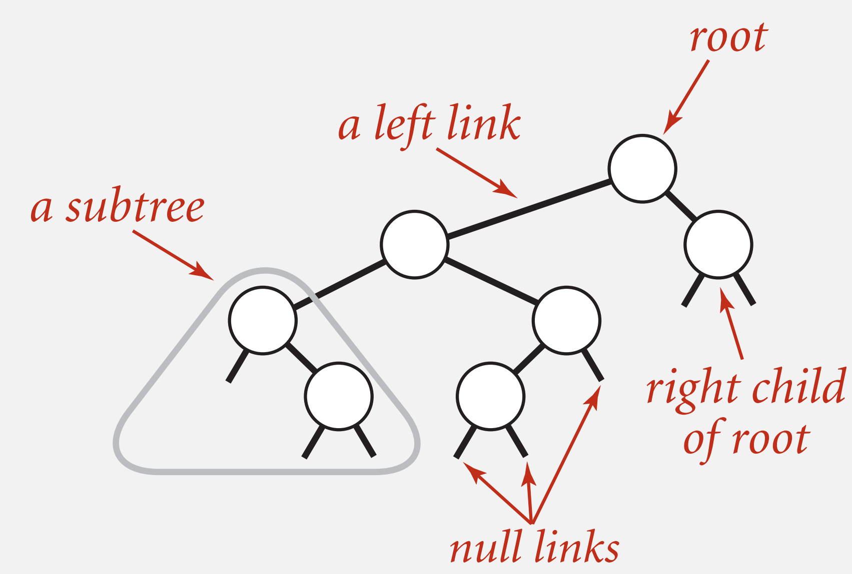
- *BSTs*
- *ordered operations*
- *deletion*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

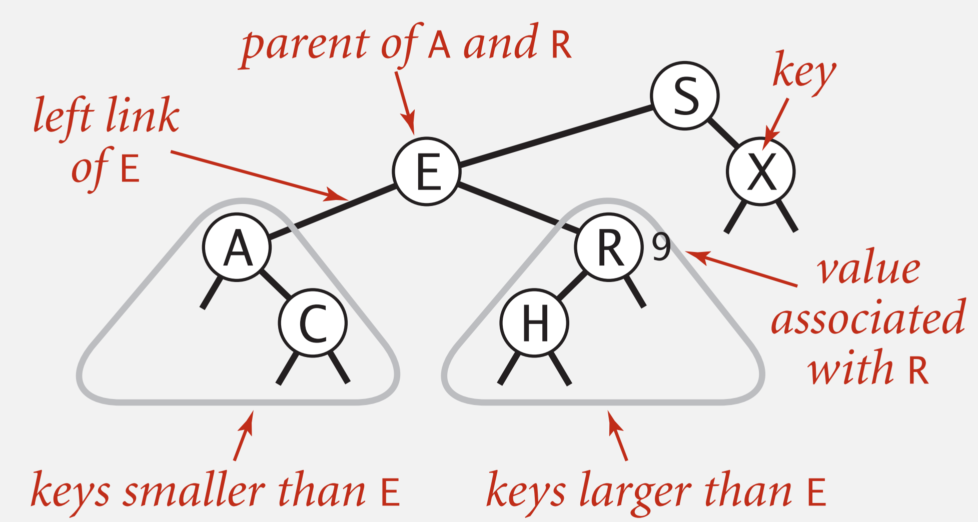
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



Symmetric order. Each node has a key, and every node's key is:

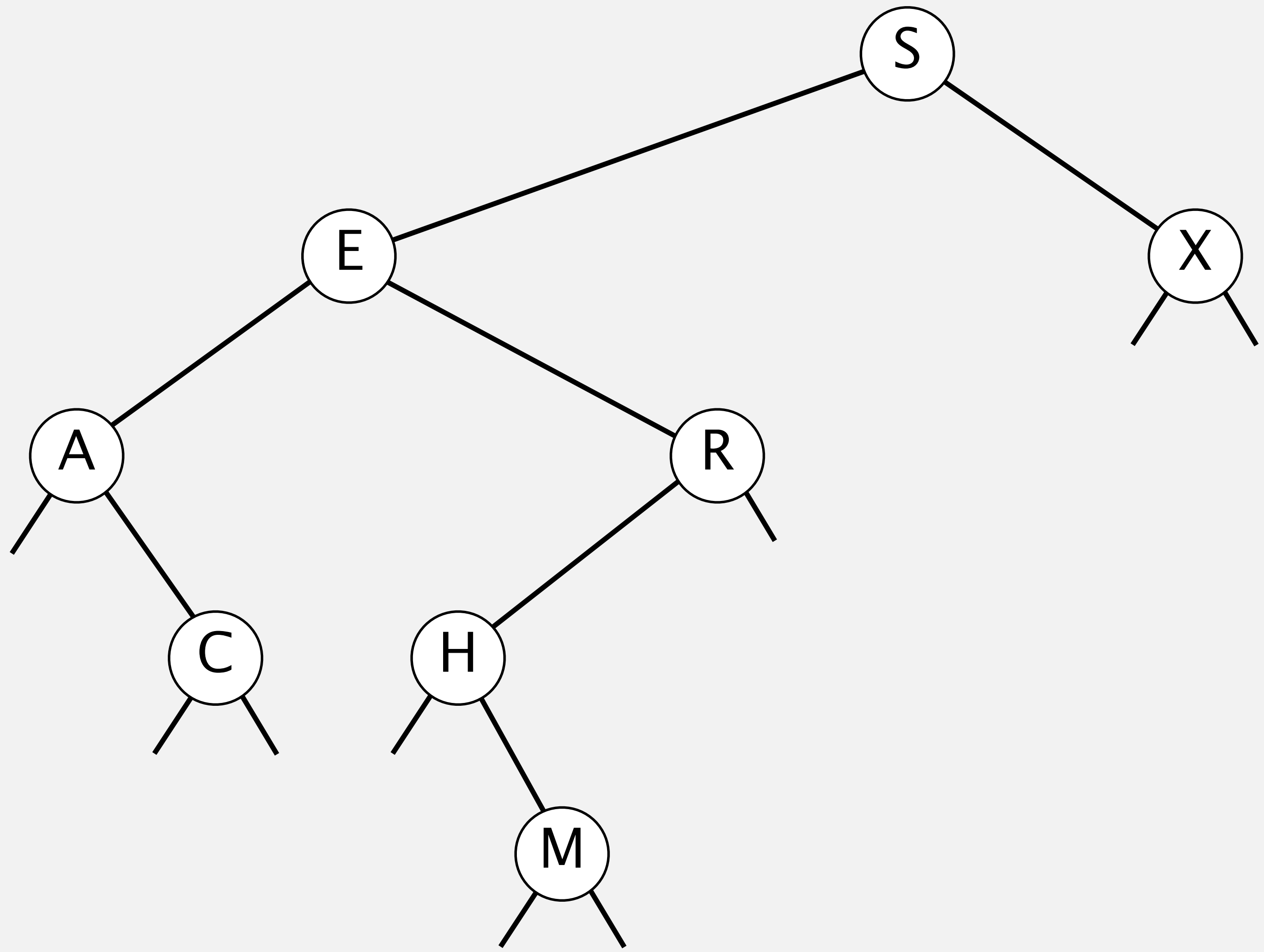
- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

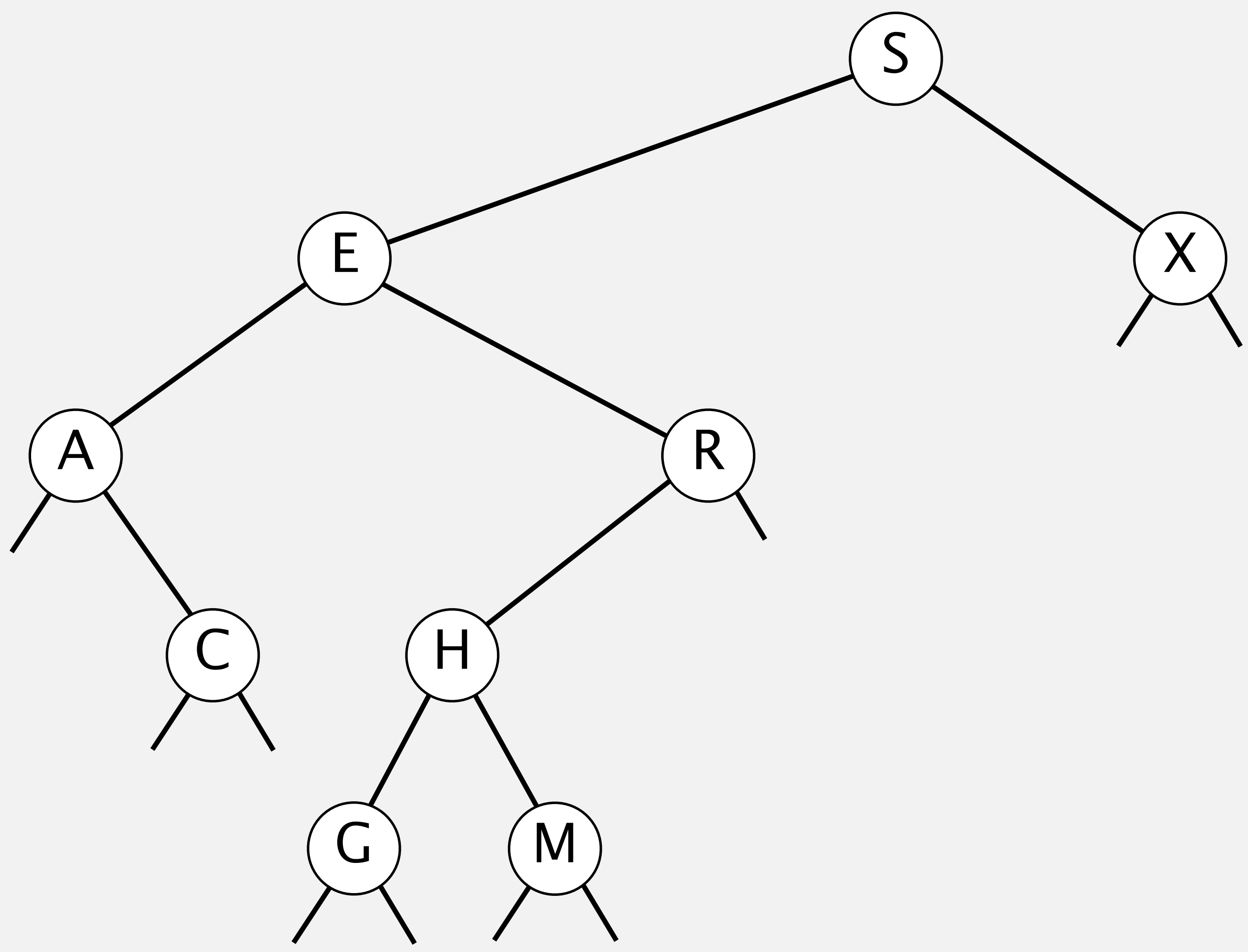
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G



BST representation in Java

Java definition. A BST is a reference to a root `Node`.

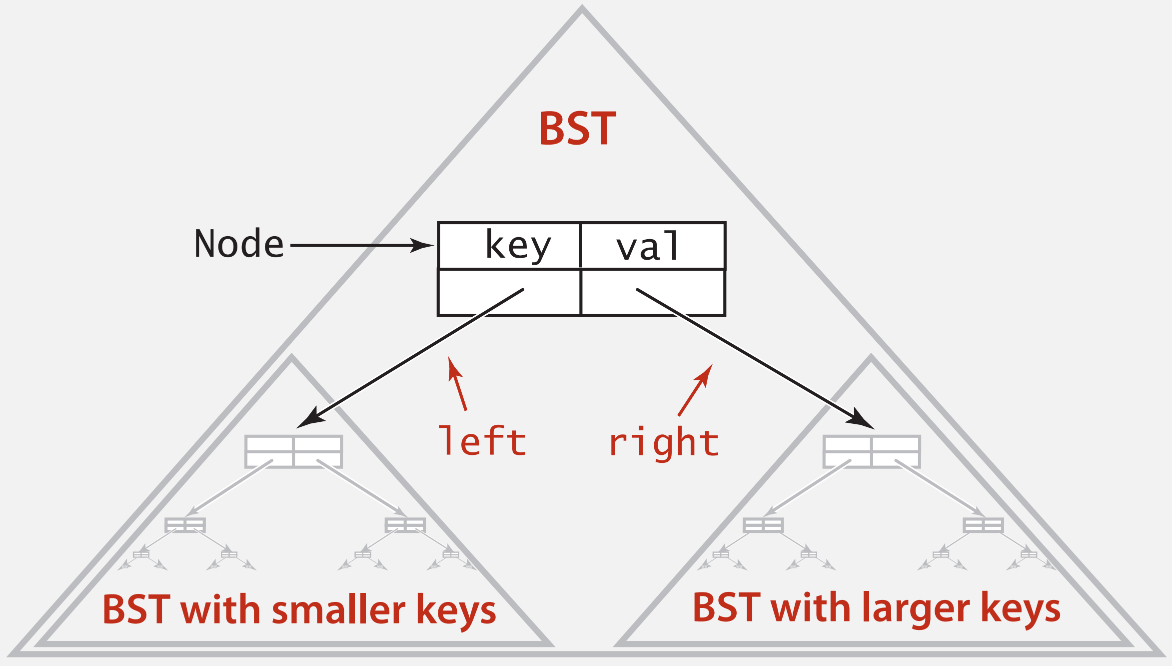
A `Node` is composed of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;

   private class Node
   {  /* see previous slide */  }

   public void put(Key key, Value val)
   {  /* see next slides */  }

   public Value get(Key key)
   {  /* see next slides */  }

   public void delete(Key key)
   {  /* see next slides */  }

   public Iterable<Key> iterator()
   {  /* see next slides */  }

}
```

root of BST

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

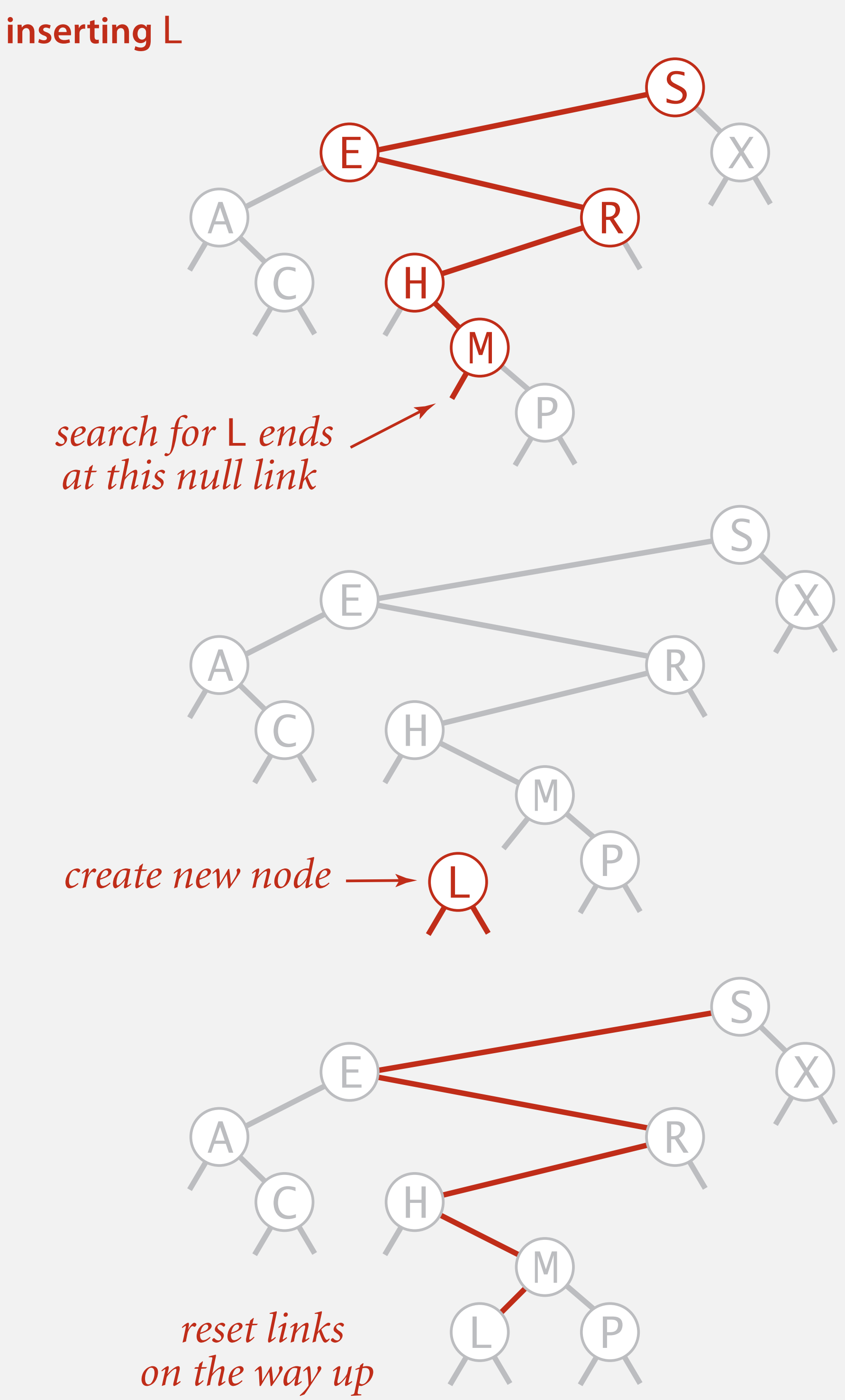
Cost. Number of compares is equal to 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree $\Rightarrow$ reset value.
- Key not in tree $\Rightarrow$ add new node.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

concise, but tricky, recursive code; read carefully!

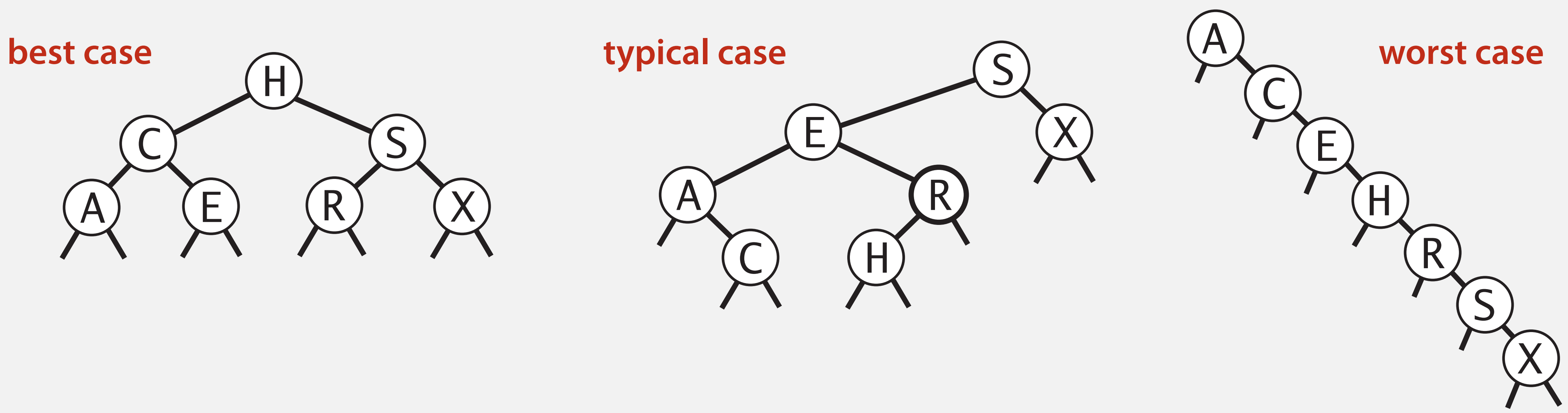
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0)
      x.left  = put(x.left,  key, val);
   else if (cmp  > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
}
```

Cost. Number of compares is equal to 1 + depth of node.

Tree shape

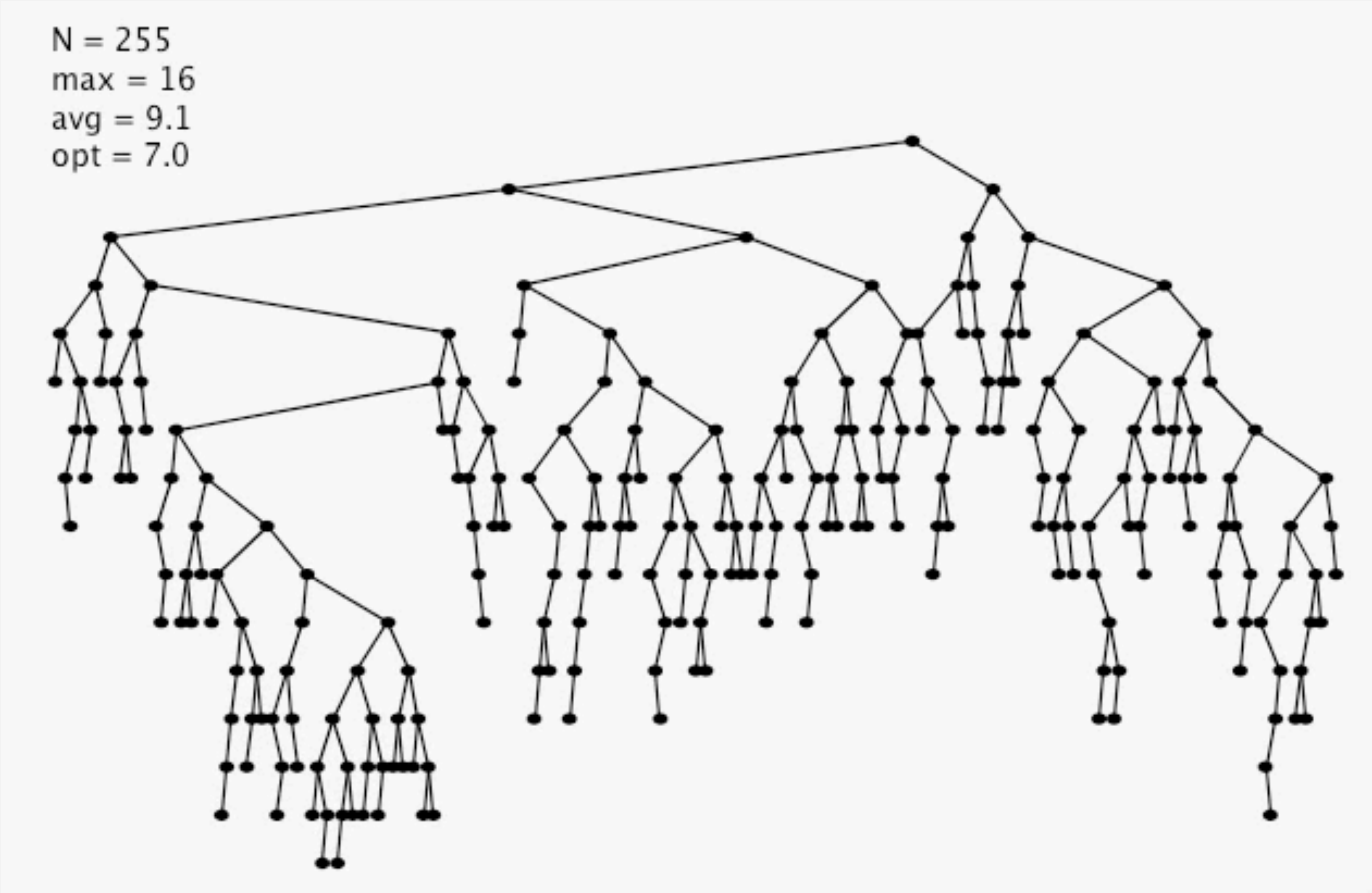
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.



Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.



Sorting with a binary heap

Q. What is this sorting algorithm?

```
0. Shuffle the array of keys.
1. Insert all keys into a BST.
2. Do an inorder traversal of BST.
```

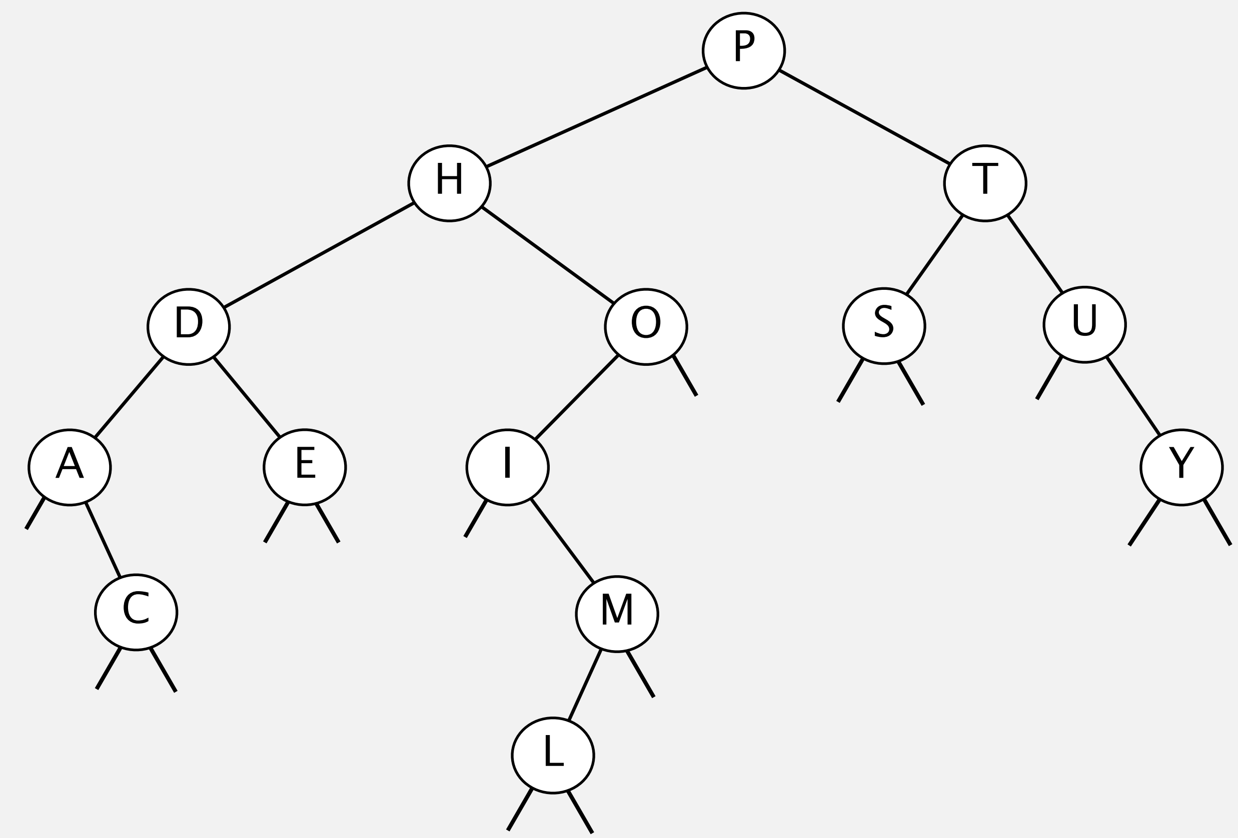
A. It's not a sorting algorithm (if there are duplicate keys)!

Q. OK, so what if there are no duplicate keys?

Q. What are its properties?

Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is $\sim 2 \ln N$.

Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order, expected height of tree is $\sim 4.311 \ln N$.

How Tall is a Tree?

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ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathrm{E}(H_n) = \alpha \log n - \beta \log\log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But... Worst-case height is N.

[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	½ N	N	`equals()`
binary search (ordered array)	$\lg N$	N	$\lg N$	½ N	`compareTo()`
BST	N	N	$1.39 \lg N$	$1.39 \lg N$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of 4.311 ln N?

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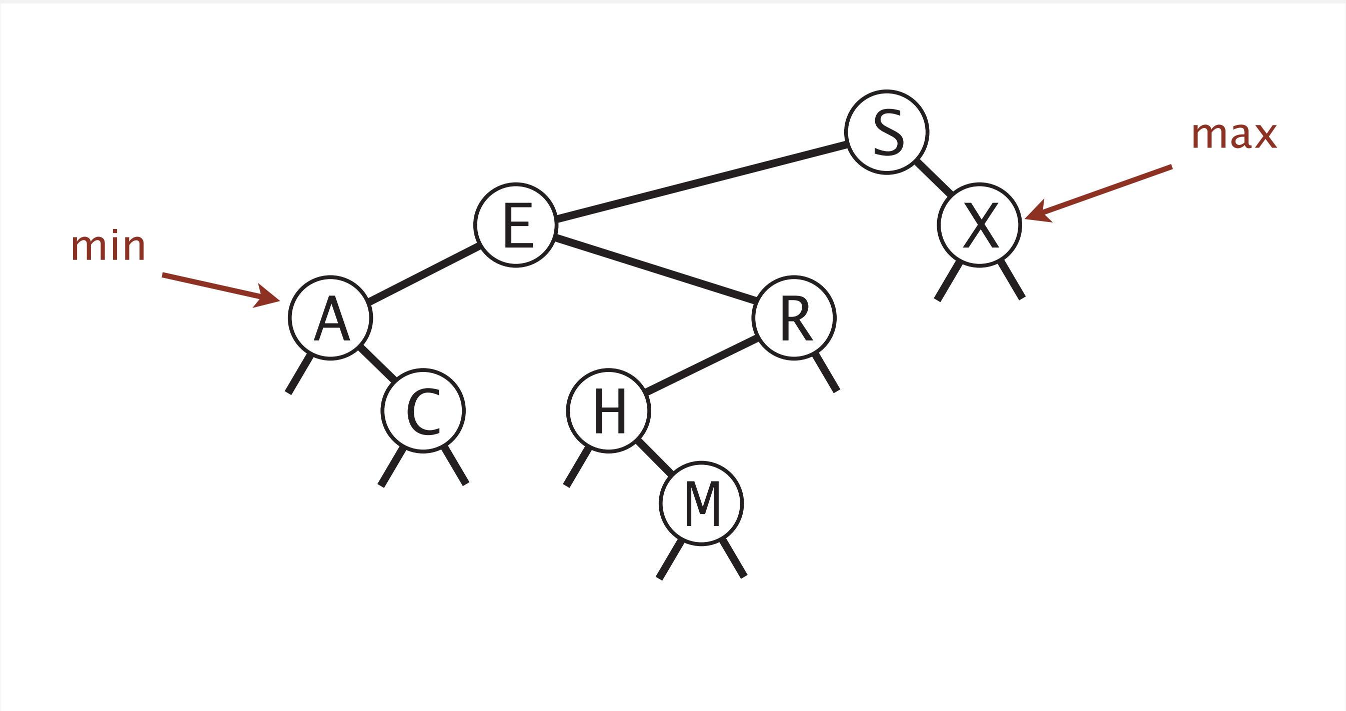
3.2 Binary Search Trees

- BSTs
- ordered operations
- deletion

Minimum and maximum

Minimum. Smallest key in table.

Maximum. Largest key in table.

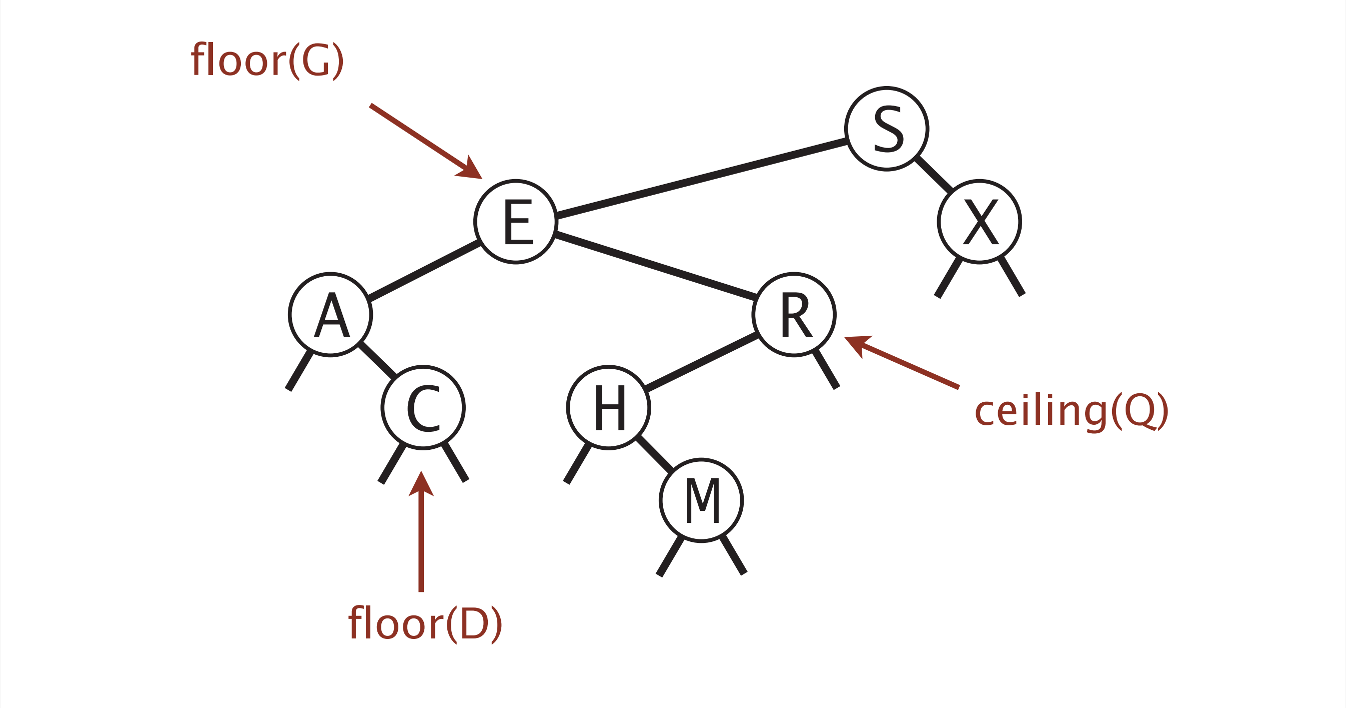


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ a given key.

Ceiling. Smallest key ≥ a given key.



Q. How to find the floor / ceiling?

Computing the floor

Case 1. [k equals the key in the node]

The floor of k is k.

Case 2. [k is less than the key in the node]

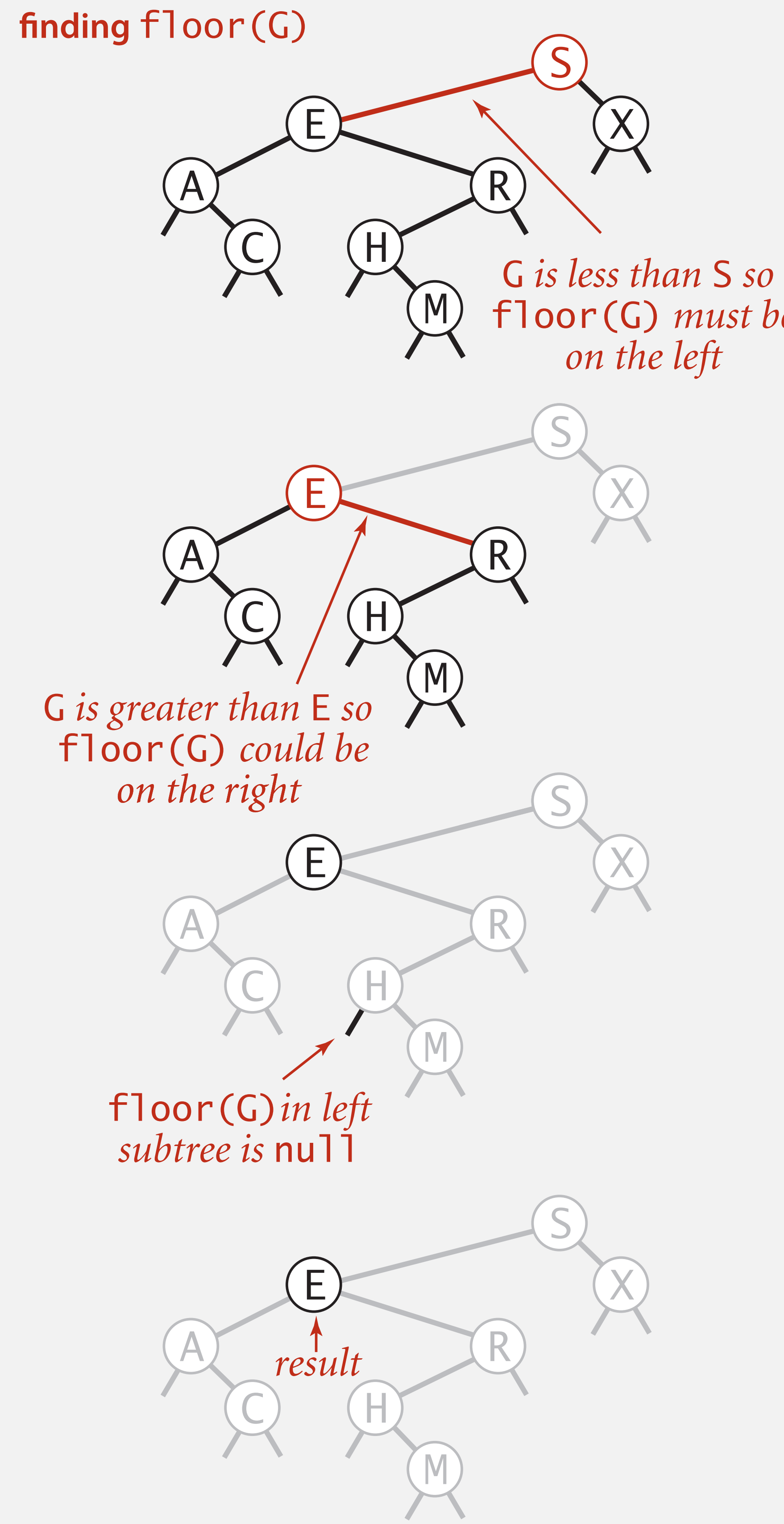
The floor of k is in the left subtree.

Case 3. [k is greater than the key in the node]

The floor of k is in the right subtree

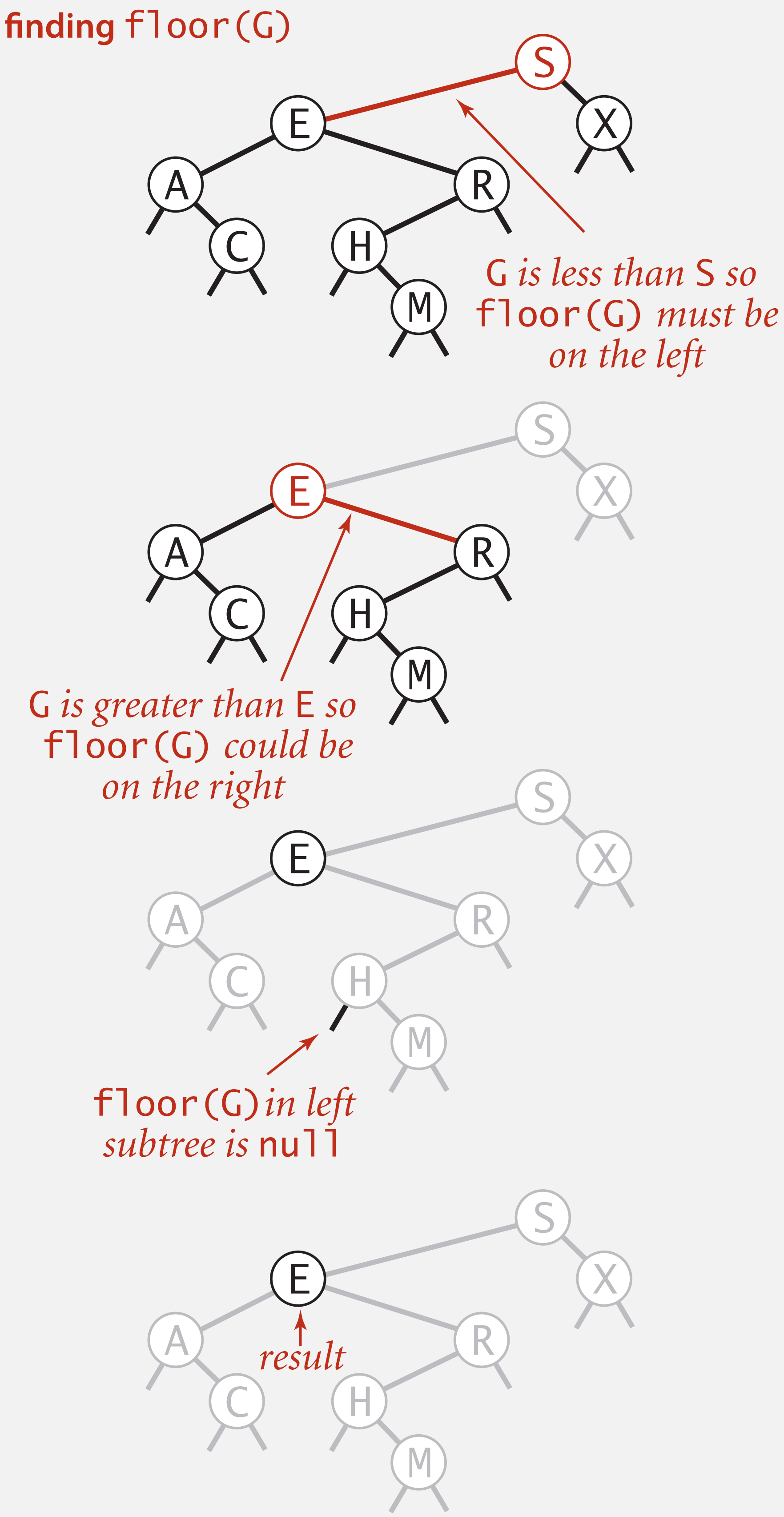
(if there is any key ≤ k in right subtree);

otherwise it is the key in the node.



Computing the floor

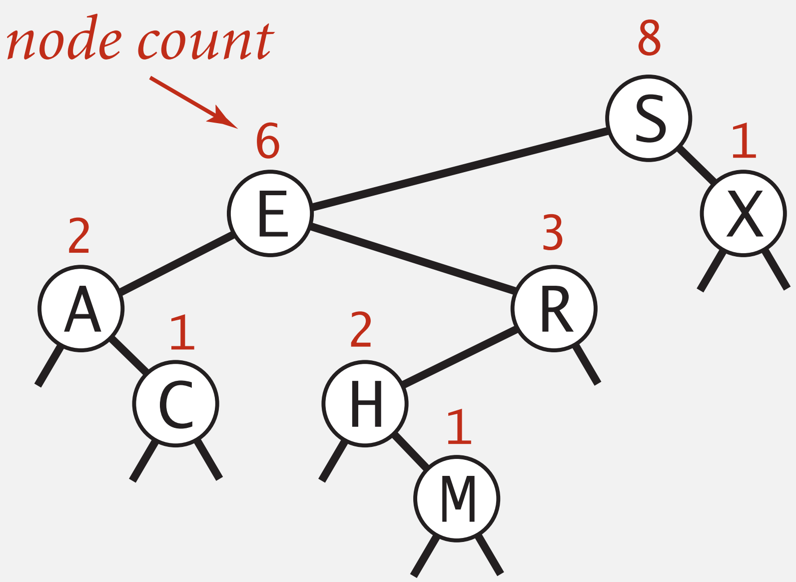
```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Rank and select

Q. How to implement `rank()` and `select()` efficiently?

A. In each node, we store the number of nodes in the subtree rooted at that node; to implement `size()`, return the count at the root.



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

ok to call when x is null

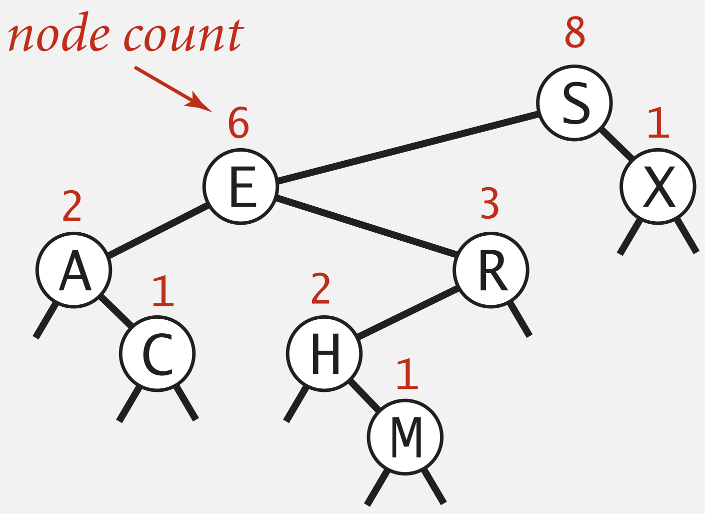
```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

initialize subtree count to 1

Rank

Rank. How many keys < k ?

Easy recursive algorithm (3 cases!)

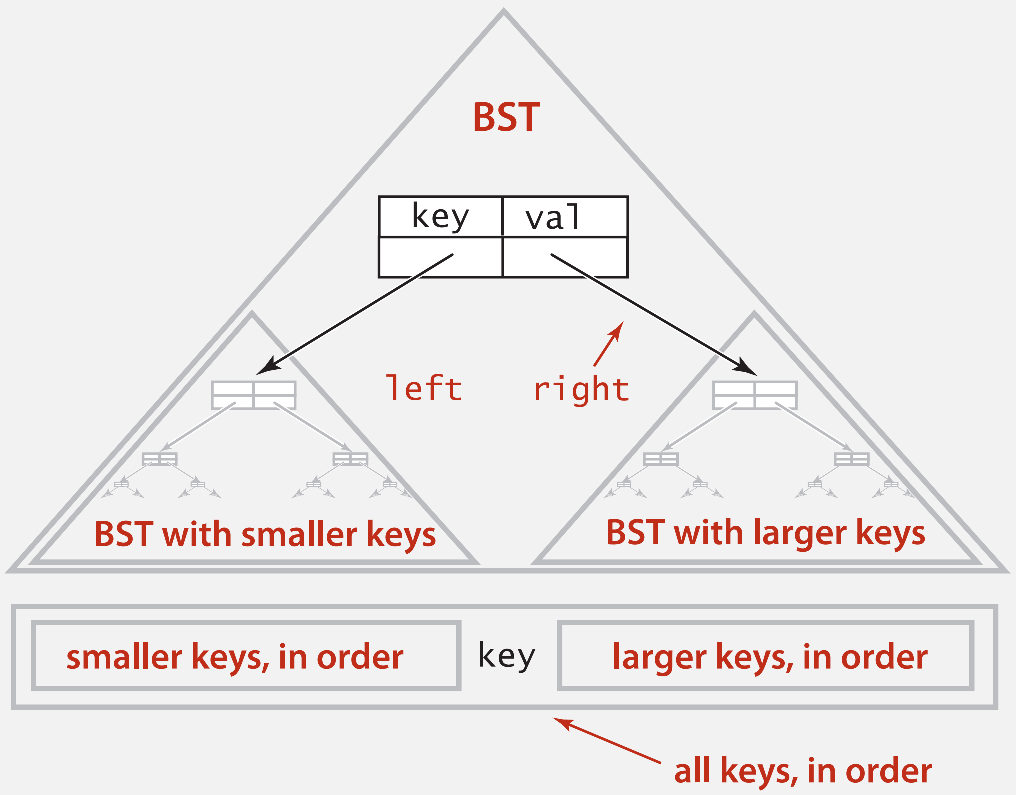



```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\lg N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\lg N$	h
rank	N	$\lg N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

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3.2 Binary Search Trees

- BSTs
- ordered operations
- deletion

ST implementations: summary

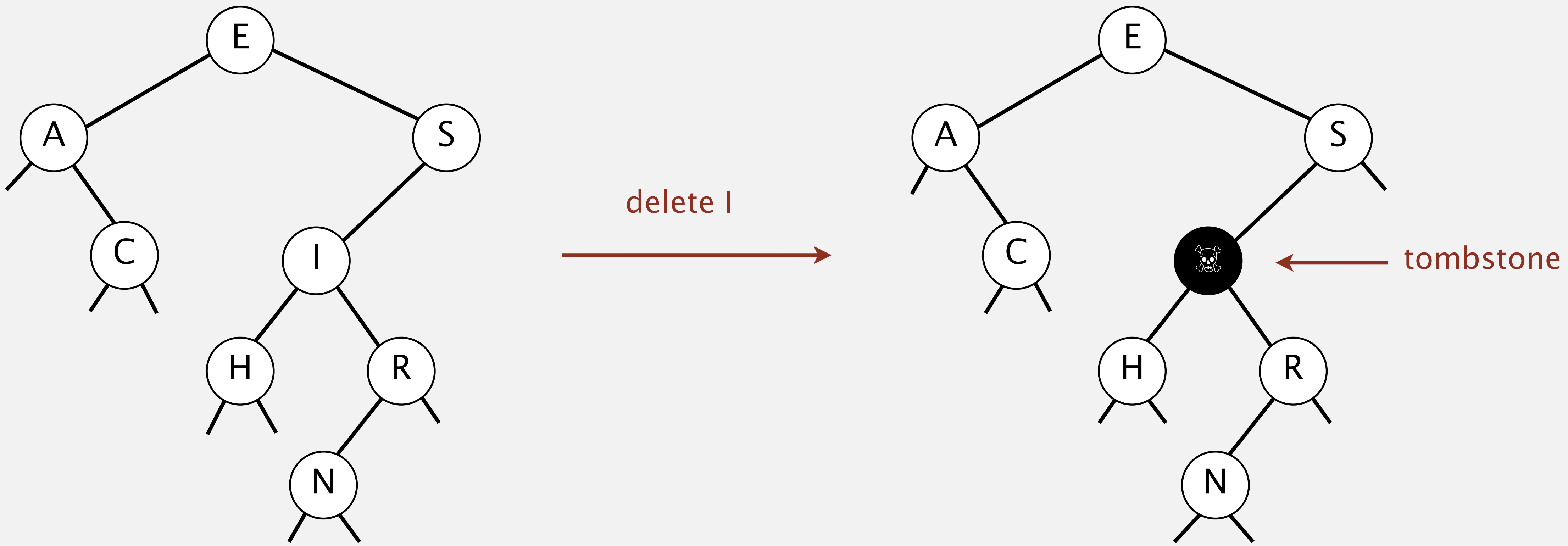
implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	lg N	N	N	lg N	½ N	½ N	✔	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	? ? ?	✔	`compareTo()`

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to `null`.
- Leave key in tree to guide search (but don't consider it equal in search).



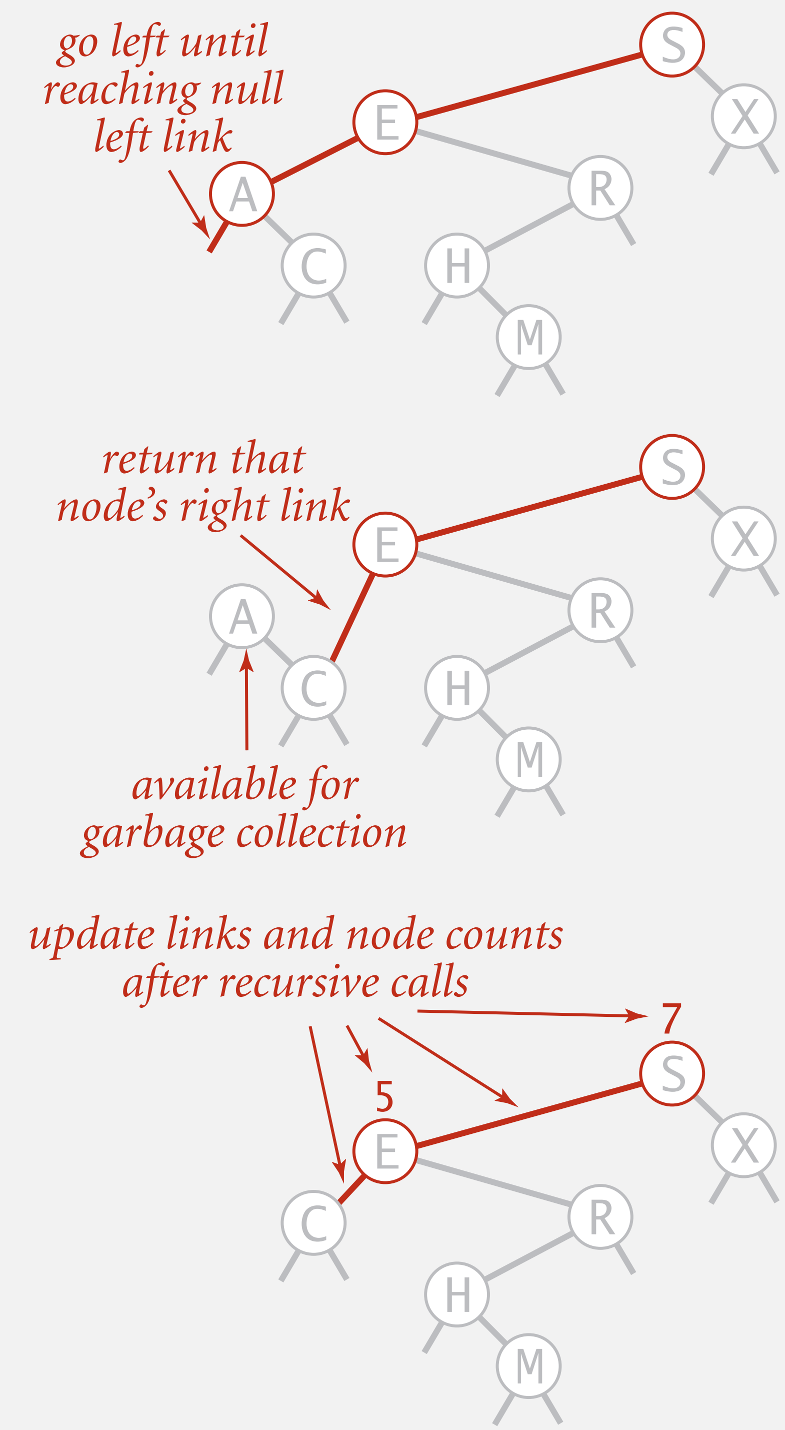
Cost. $\sim 2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

Unsatisfactory solution. Tombstone (memory) overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.



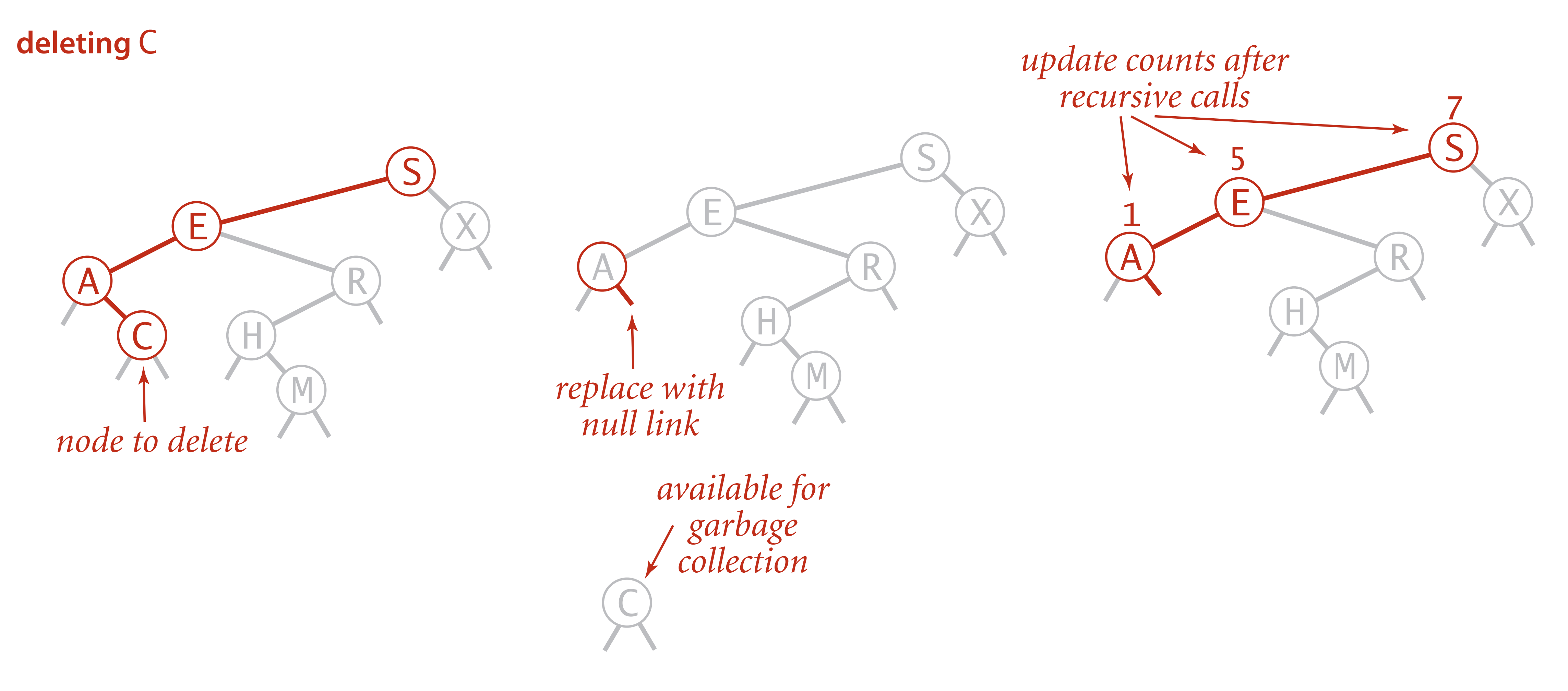

```
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
   if (x.left == null) return x.right;
   x.left = deleteMin(x.left);
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

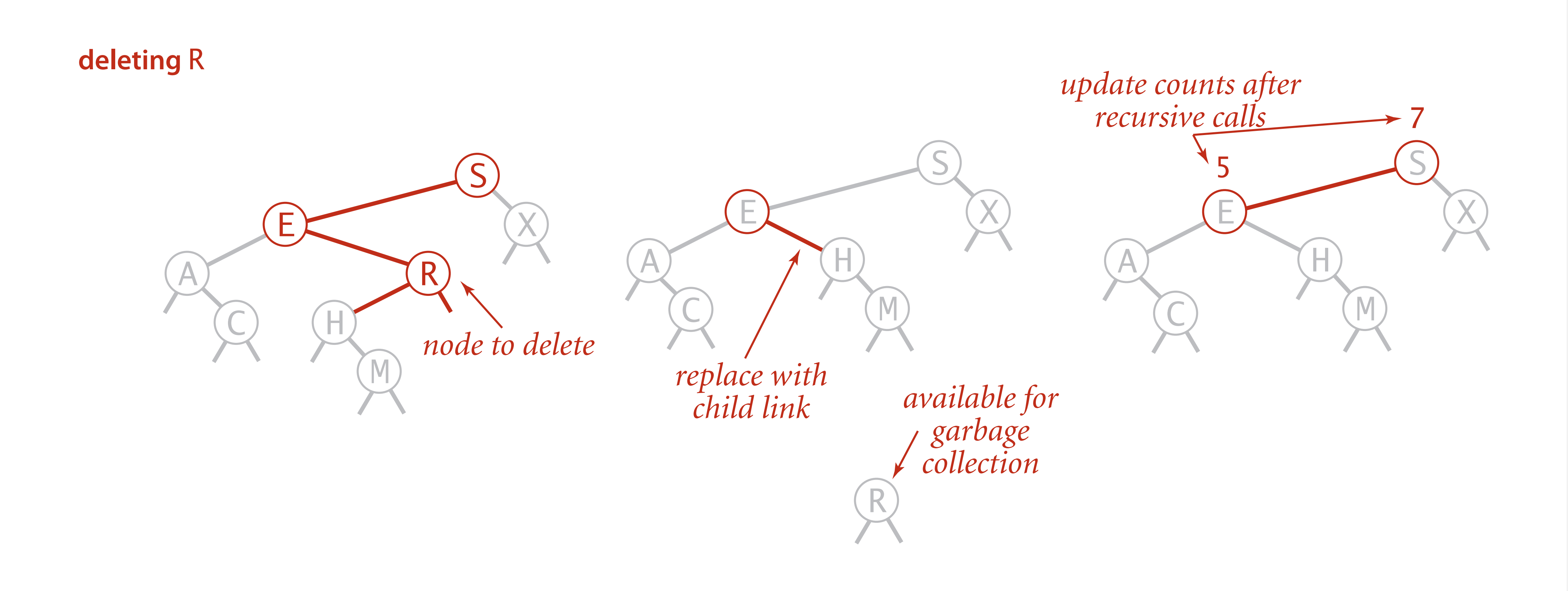
Case 0. [0 children] Delete `t` by setting parent link to null.



Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 1. [1 child] Delete `t` by replacing parent link.

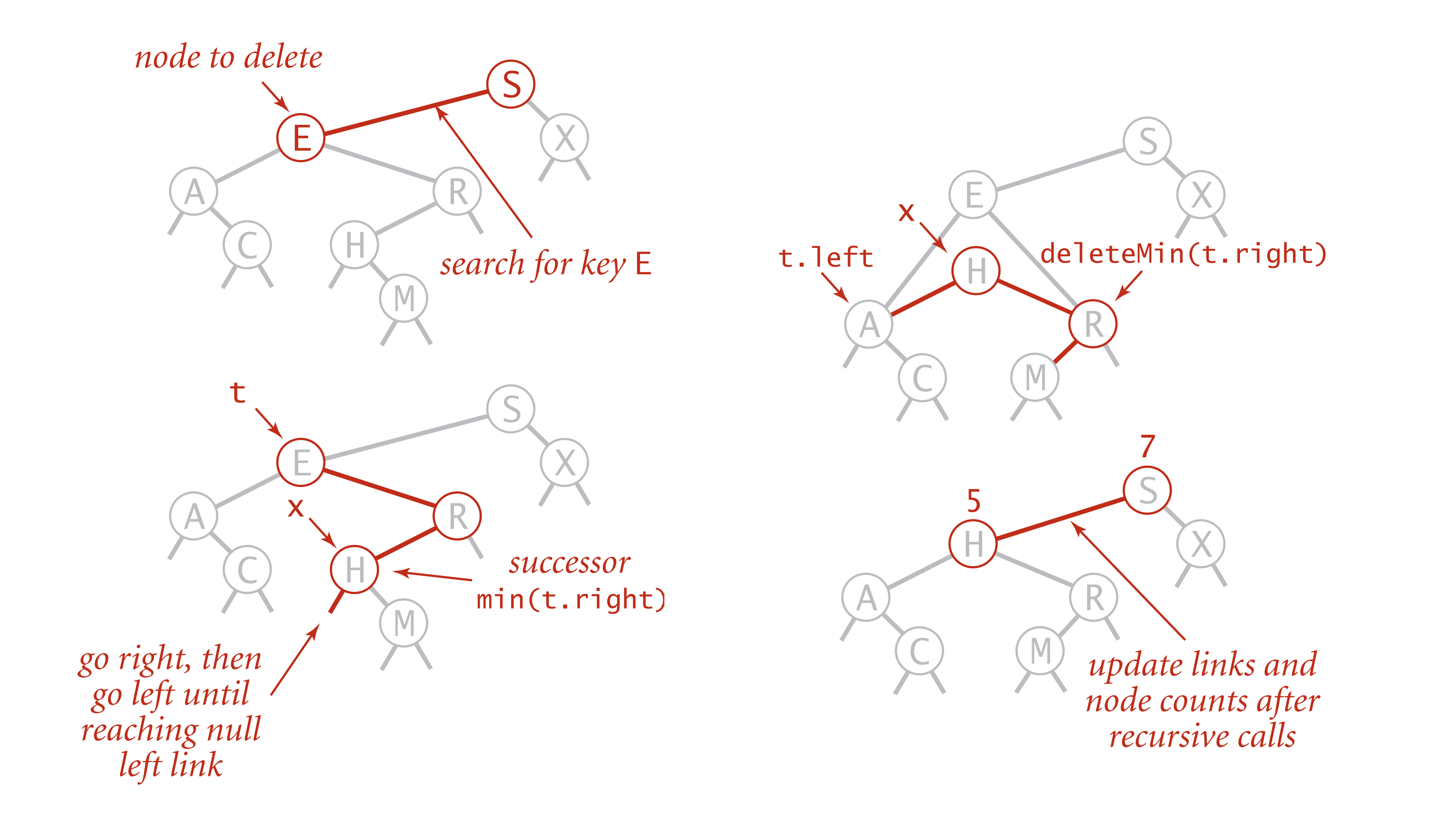


Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 2. [2 children]

- Find successor `x` of `t`. ← x has no left child
- Delete the minimum in `t`'s right subtree. ← but don't garbage collect x
- Put `x` in `t`'s spot. ← still a BST



Hibbard deletion: Java implementation

```
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = delete(x.left,  key);
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;
      if (x.left  == null) return x.right;

      Node t = x;
      x = min(t.right);
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
   x.count = size(x.left) + size(x.right) + 1;
   return x;
}
```

search for key

no right child

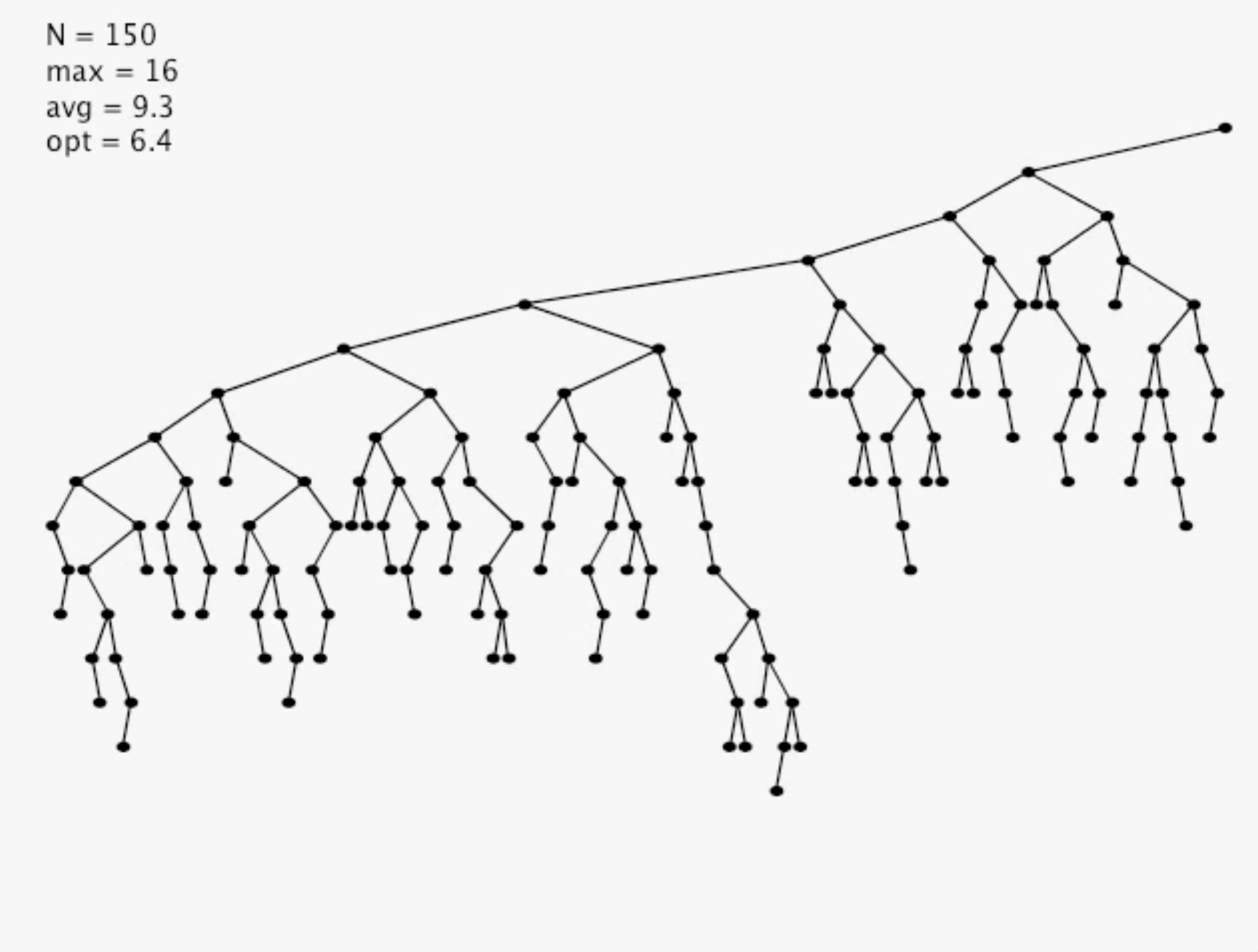
no left child

replace with successor

update subtree counts

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



Surprising consequence. Trees not random (!) $\Rightarrow \sqrt{N}$ per op.

Longstanding open problem. Simple and efficient delete for BSTs.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	lg N	N	N	lg N	½ N	½ N	✔	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	$\sqrt{N}$	✔	`compareTo()`

other operations also become √N
if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.