COS 226, SPRING 2014

ALGORITHMS AND DATA STRUCTURES

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PRINCETON UNIVERSITY

http://www.princeton.edu/~cos226

COS 226 course overview

What is COS 226?

- Intermediate-level survey course.
- Programming and problem solving, with applications.
- Algorithm: method for solving a problem.
- Data structure: method to store information.

topic	data structures and algorithms
data types	stack, queue, bag, union-find, priority queue
sorting	quicksort, mergesort, heapsort, radix sorts
searching	BST, red-black BST, hash table
graphs	BFS, DFS, Prim, Kruskal, Dijkstra
strings	KMP, regular expressions, tries, data compression
advanced	B-tree, k-d tree, suffix array, maxflow

Why study algorithms?

Their impact is broad and far-reaching.

Internet. Web search, packet routing, distributed file sharing, ...

Biology. Human genome project, protein folding, ...

Computers. Circuit layout, file system, compilers, ...

Computer graphics. Movies, video games, virtual reality, ...

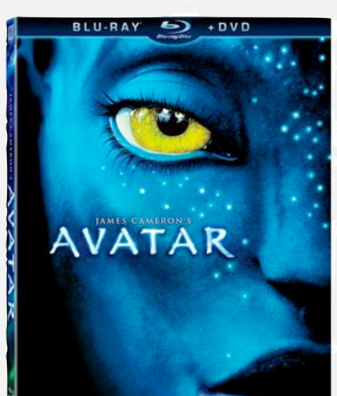
Security. Cell phones, e-commerce, voting machines, ...

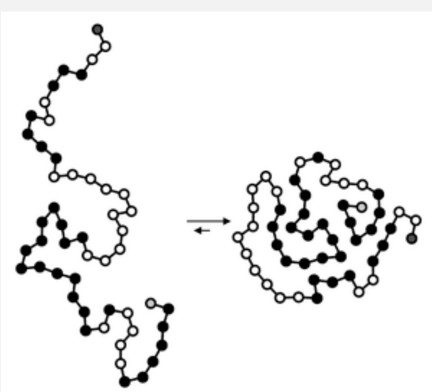
Multimedia. MP3, JPG, DivX, HDTV, face recognition, ...

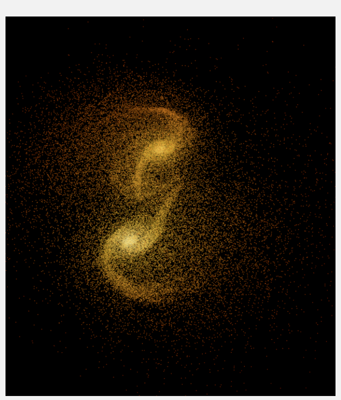
Social networks. Recommendations, news feeds, advertisements, ...

Physics. N-body simulation, particle collision simulation, ...

⋮







Why study algorithms?

Their impact is broad and far-reaching.

Mysterious algorithm was 4% of trading activity last week

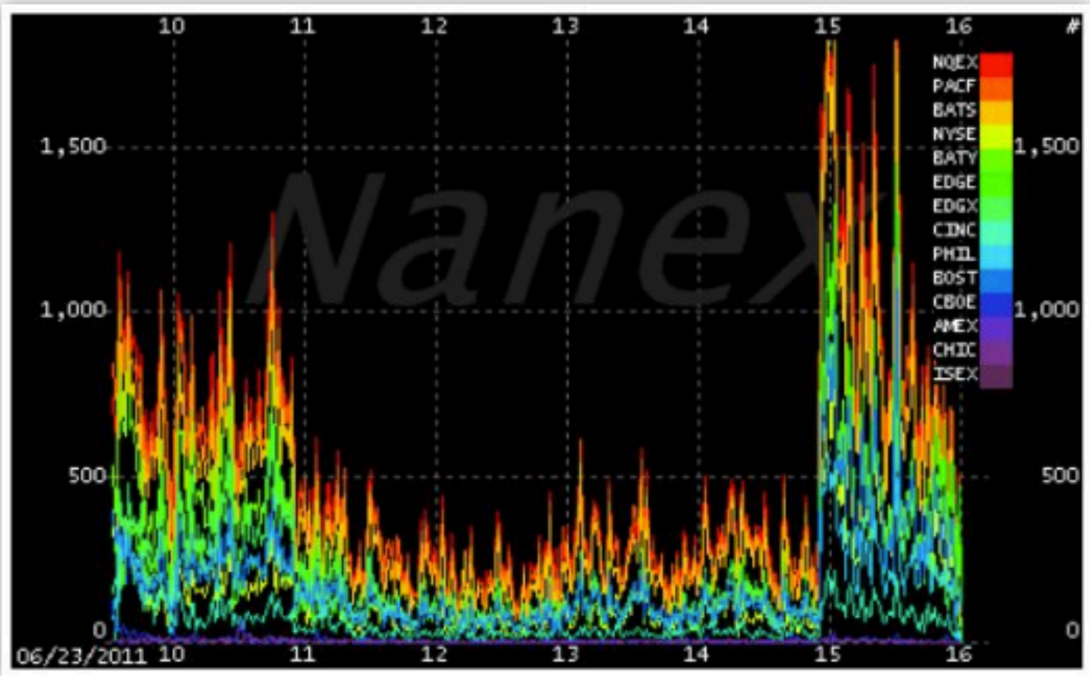
October 11, 2012

A single mysterious computer program that placed orders — and then subsequently canceled them — made up 4 percent of all quote traffic in the U.S. stock market last week, according to the top tracker of high-frequency trading activity.

The motive of the algorithm is still unclear, *CNBC* reports.

The program placed orders in 25-millisecond bursts involving about 500 stocks, according to Nanex, a market data firm. The algorithm never executed a single trade, and it abruptly ended at about 10:30 a.m. ET Friday.

[+]



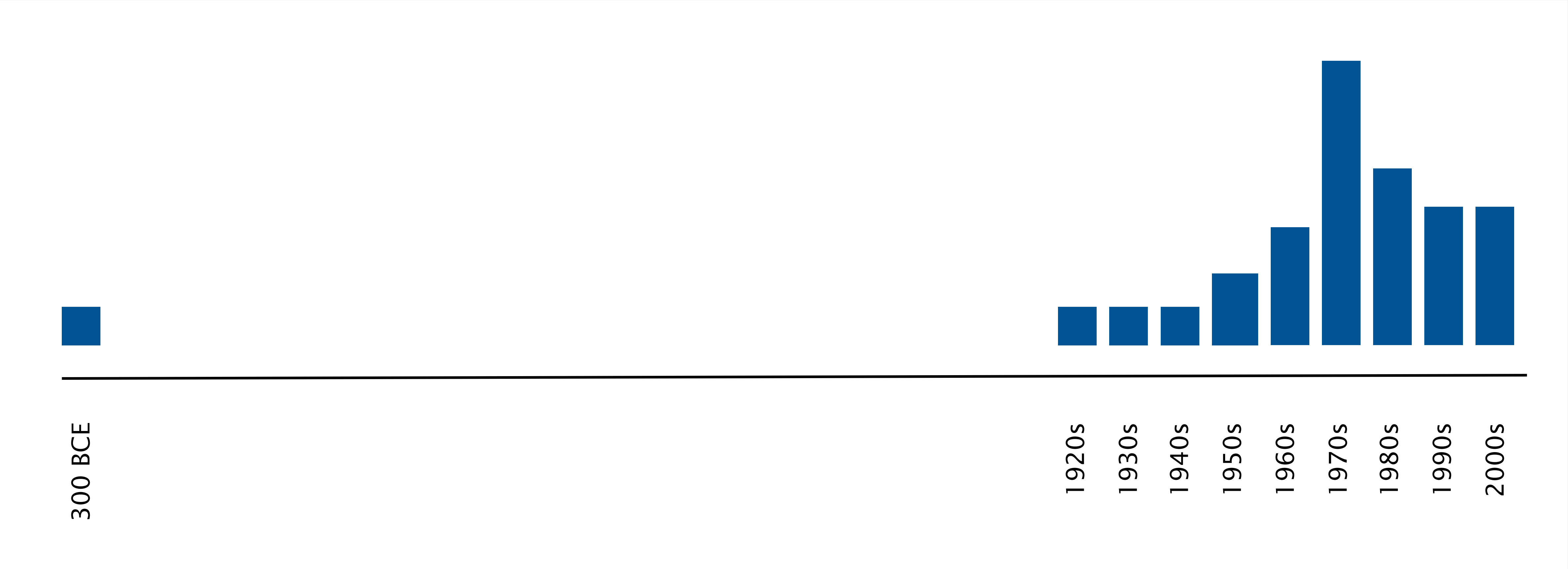
Generic high frequency rrading chart (credit: Nanex)

"My guess is that the algo was testing the market, as high-frequency frequently does," says Jon Najarian, co-founder of TradeMonster.com. "As soon as they add bandwidth, the HFT crowd sees how quickly they can top out to create latency." (*Read More*: Unclear What Caused Kraft Spike: Nanex Founder.)

Why study algorithms?

Old roots, new opportunities.

- Study of algorithms dates at least to Euclid.
- Formalized by Church and Turing in 1930s.
- Some important algorithms were discovered by undergraduates in a course like this!



Why study algorithms?

For intellectual stimulation.

FROM THE EDITORS

The Joy of Algorithms

Francis Sullivan, Associate Editor-in-Chief

THE THEME OF THIS FIRST-OF-THE-CENTURY ISSUE OF *COMPUTING IN SCIENCE & ENGINEERING* IS ALGORITHMS. IN FACT, WE WERE BOLD ENOUGH—AND PERHAPS FOOLISH ENOUGH—TO CALL THE 10 EXAMPLES WE'VE SELECTED "THE TOP 10 ALGORITHMS OF THE CENTURY."

Computational algorithms are probably as old as civilization. Sumerian cuneiform, one of the most ancient written records, consists partly of algorithm descriptions for reckoning in base 60. And I suppose we could claim that the Druid algorithm for estimating the start of summer is embodied in Stonehenge. (That's really hard hardware!)

Like so many other things that technology affects, algorithms have advanced in startling and unexpected ways in the 20th century—at least it looks that way to us now. The algorithms we chose for this issue have been essential for progress in communications, health care, manufacturing, economics, weather prediction, defense, and fundamental science. Conversely, progress in these areas has stimulated the search for ever-better algorithms. I recall one late-night bull session on the Maryland Shore when someone asked, "Who first ate a crab? After all, they don't look very appetizing." After the usual speculations about the observed behavior of sea gulls, someone gave what must be the right answer—namely, "A very hungry person first ate a crab."

The flip side to "necessity is the mother of invention" is "invention creates its own necessity." Our need for powerful machines always exceeds their availability. Each significant computation brings insights that suggest the next, usually much larger, computation to be done. New algorithms are an attempt to bridge the gap between the demand for cycles and the available supply of them. We've become accustomed to gaining the Moore's Law factor of two every 18 months. In effect, Moore's Law changes the constant in front of the estimate of running time as a function of problem size. Important new algorithms do not come along every 1.5 years, but when they do, they can change the exponent of the complexity!

For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. A colleague recently claimed that he'd done only 15 minutes of productive work in his whole life. He wasn't joking, because he was referring to the 15 minutes during which he'd sketched out a fundamental optimization algorithm. He regarded the previous years of thought and investigation as a sunk cost that might or might not have paid off.

Researchers have cracked many hard problems since 1 January 1900, but we are passing some even harder ones on to the next century. In spite of a lot of good work, the question of how to extract information from extremely large masses of data is still almost untouched. There are still very big challenges coming from more "traditional" tasks, too. For example, we need efficient methods to tell when the result of a large floating-point calculation is likely to be correct. Think of the way that check sums function. The added computational cost is very small, but the added confidence in the answer is large. Is there an analog for things such as huge, multidisciplinary optimizations? At an even deeper level is the issue of reasonable methods for solving specific cases of "impossible" problems. Instances of NP-complete problems crop up in attempting to answer many practical questions. Are there efficient ways to attack them?

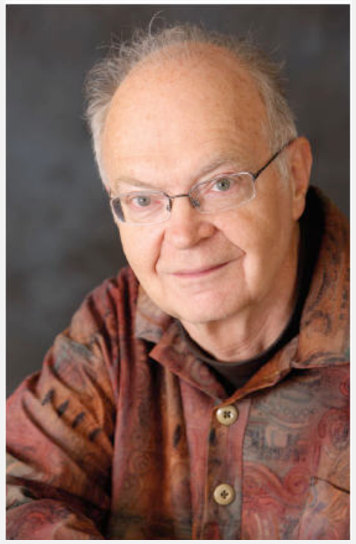
I suspect that in the 21st century, things will be ripe for another revolution in our understanding of the foundations of computational theory. Questions already arising from quantum computing and problems associated with the generation of random numbers seem to require that we somehow tie together theories of computing, logic, and the nature of the physical world.

The new century is not going to be very restful for us, but it is not going to be dull either!

2 COMPUTING IN SCIENCE & ENGINEERING

> "*For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing.*" — *Francis Sullivan*

> "*An algorithm must be seen to be believed.*" — *Donald Knuth*



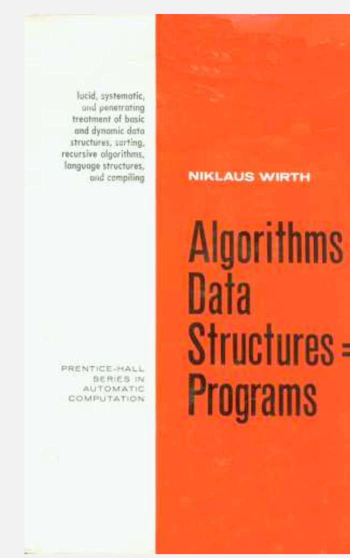
Why study algorithms?

To become a proficient programmer.

“ I will, in fact, claim that the difference between a bad programmer and a good one is whether he considers his code or his data structures more important. Bad programmers worry about the code. Good programmers worry about data structures and their relationships. ”

— Linus Torvalds (creator of Linux)

“ Algorithms + Data Structures = Programs. ” *— Niklaus Wirth*

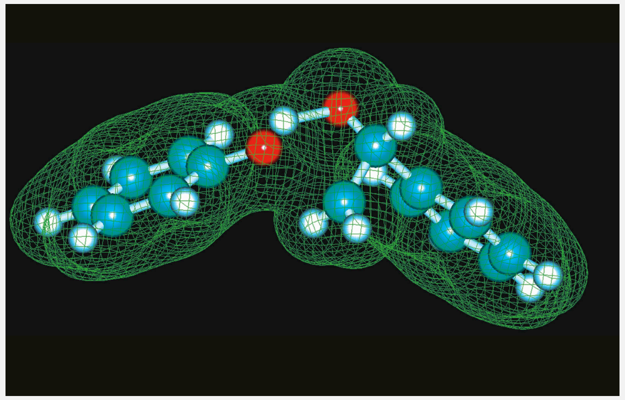


Why study algorithms?

They may unlock the secrets of life and of the universe.

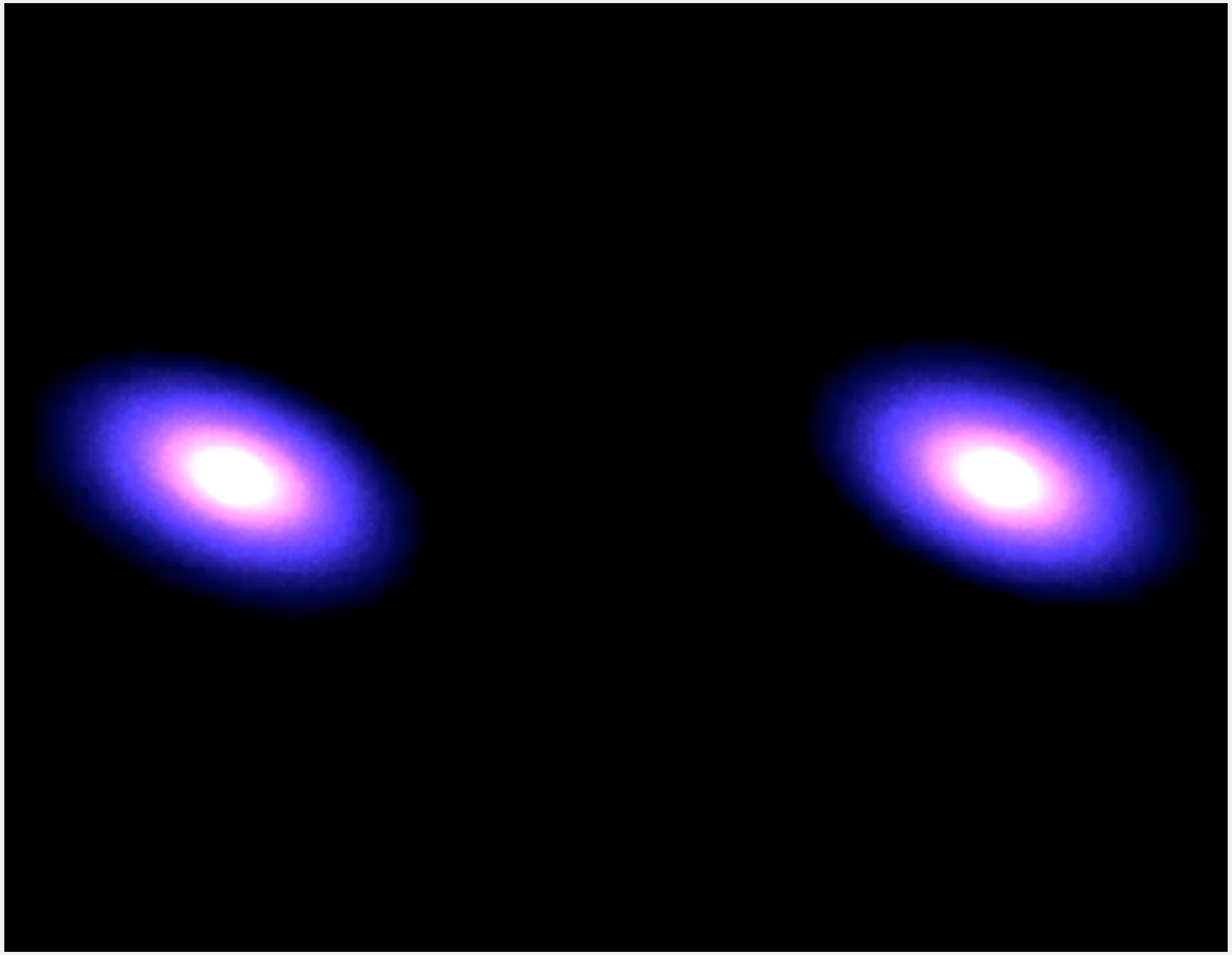
> *" Computer models mirroring real life have become crucial for most advances made in chemistry today…. Today the computer is just as important a tool for chemists as the test tube. "*
>
> — *Royal Swedish Academy of Sciences (Nobel Prize in Chemistry 2013)*

Martin Karplus, Michael Levitt, and Arieh Warshel



Why study algorithms?

To solve problems that could not otherwise be addressed.



http://www.youtube.com/watch?v=ua7YlN4eL_w

Why study algorithms?

Everybody else is doing it.

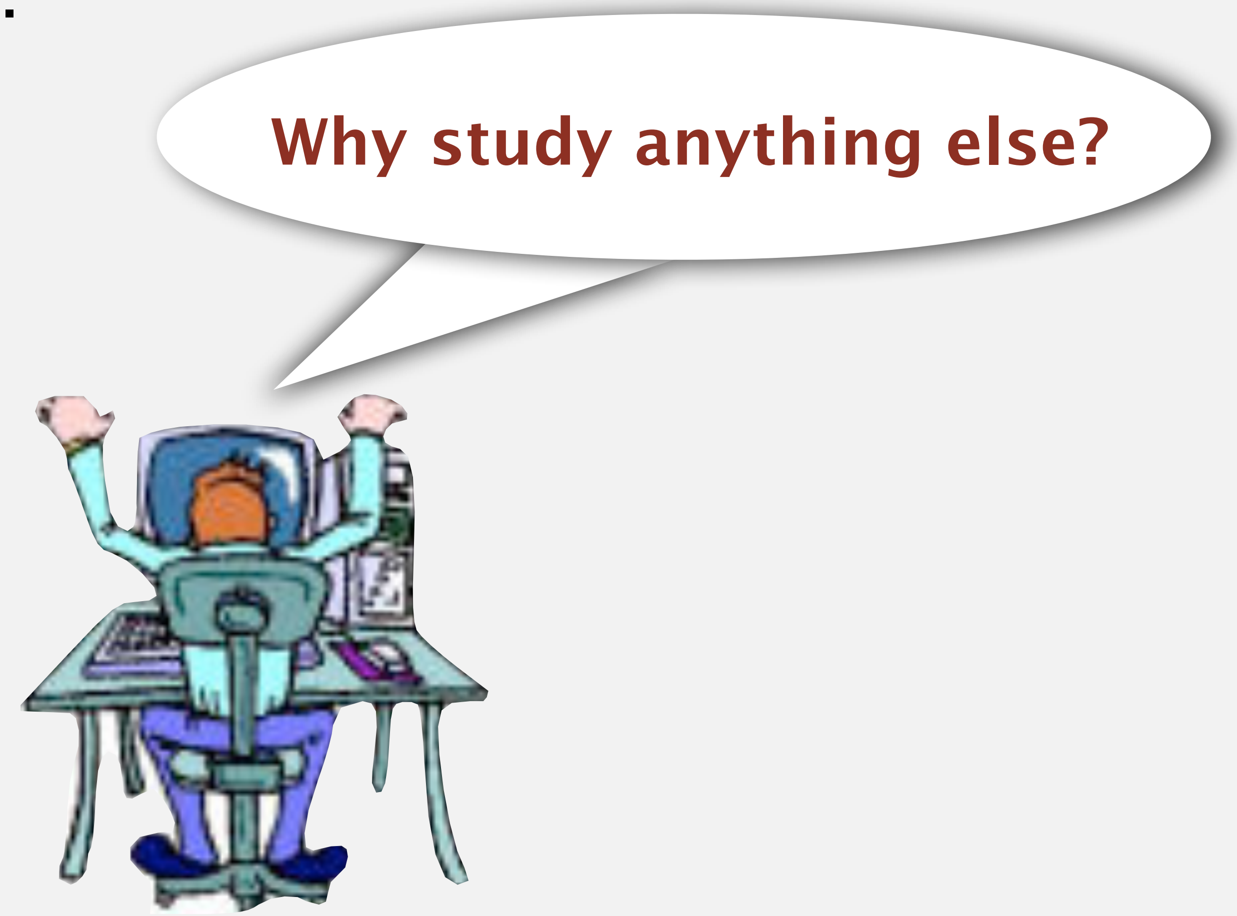
```
% sort -rn PU2013-14.txt
   774   COS 126   General Computer Science
   615   ECO 100   Introduction to Microeconomics
   471   ECO 101   Introduction to Macroeconomics
   444   ENG 385   Children's Literature
   440   MAT 202   Linear Algebra with Applications
   414   COS 226   Algorithms and Data Structures
   405   MAT 201   Multivariable Calculus
   384   CHV 310   Practical Ethics
   344   REL 261   Christian Ethics and Modern Society
   320   PSY 101   Introduction to Psychology
   300   COS 217   Introduction to Programming Systems
   ...
```

Why study algorithms?

For fun and profit.

Why study algorithms?

- Their impact is broad and far-reaching.
- Old roots, new opportunities.
- For intellectual stimulation.
- To become a proficient programmer.
- They may unlock the secrets of life and of the universe.
- To solve problems that could not otherwise be addressed.
- Everybody else is doing it.
- For fun and profit.



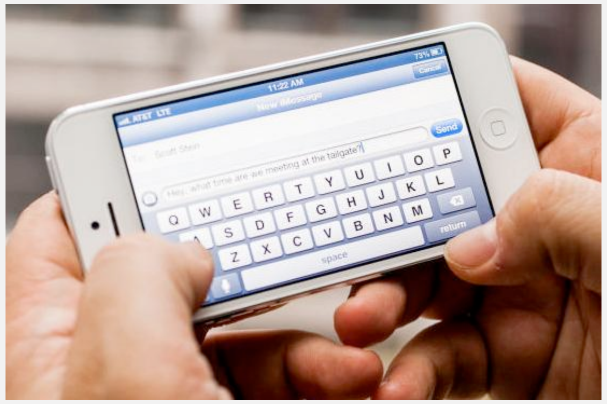
Lectures

Traditional lectures. Introduce new material.

Electronic devices. Permitted, but only to enhance lecture.

no

no



no

What	When	Where	Who	Office Hours
L01	**MW 11-12:20**	**McCosh 10**	**Kevin Wayne**	see web

Lectures

Traditional lectures. Introduce new material.

Flipped lectures.

- Watch videos online before lecture.
- Complete pre-lecture activities.
- Attend only one "flipped" lecture per week (interactive, collaborative, experimental).
- Apply via web ASAP: results by 5pm today.

What	When	Where	Who	Office Hours
L01	**MW 11-12:20**	**McCosh 10**	**Kevin Wayne**	see web
L02	**W 11-12:20**	**Frist 307**	**Andy Guna** **Josh Hug**	see web

Precepts

Discussion, problem-solving, background for assignments.

What	When	Where	Who	Office Hours
P01	**Th 11–11:50**	**CS 102**	**Andy Guna †**	see web
P02	**Th 12:30–1:20**	**Bobst 105**	**Andy Guna †**	see web
P03	**Th 1:30–2:20**	**Bobst 105**	**Nevin Li**	see web
P04	**F 10–10:50**	**Bobst 105**	**Jennifer Guo**	see web
P05	**F 11–11:50**	**Bobst 105**	**Madhu Jayakumar**	see web
P05A	**F 11–11:50**	**Sherrerd 001**	**Ruth Dannenfelser**	see web
P06	**F 2:30–3:20**	**Friend 108**	**Chris Eubank**	see web
P06A	**F 2:30–3:20**	**Friend 111**	**TBA**	see web
P06B	**F 2:30–3:20**	**Friend 109**	**Josh Hug †**	see web
P07	**F 3:30–4:20**	**Friend 108**	**Josh Hug †**	see web

likely to change

† lead preceptor

Coursework and grading

Programming assignments. 45%

- Due on Tuesdays at 11pm via electronic submission.
- Collaboration/lateness policies: see web.

Exercises. 10%

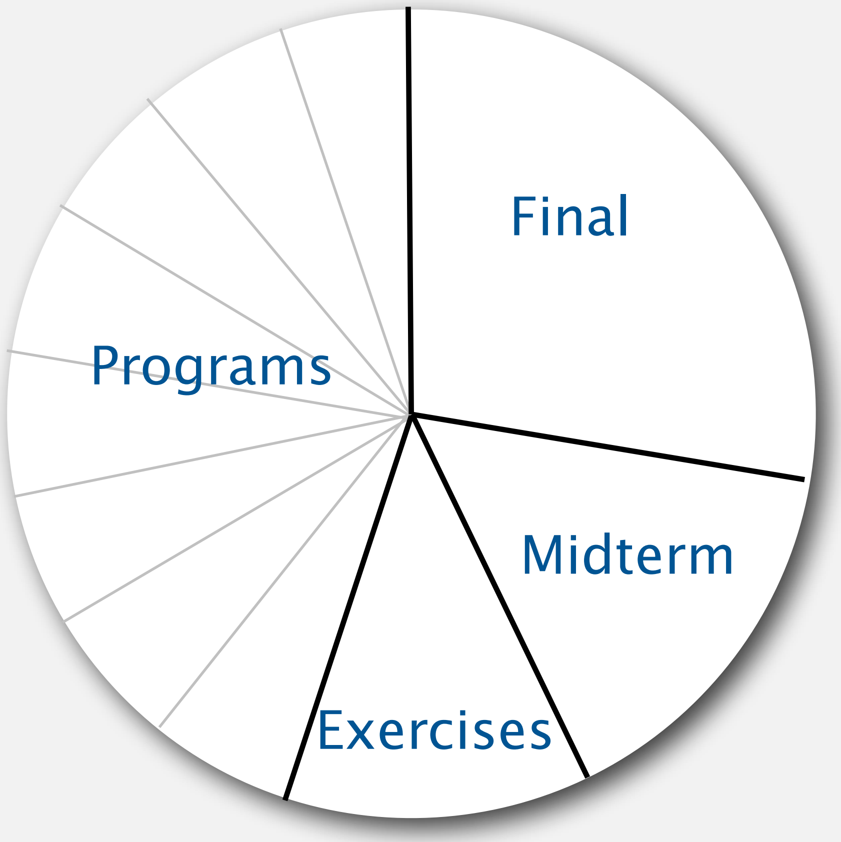
- Due on Sundays at 11pm in Blackboard.
- Collaboration/lateness policies: see web.

Exams. 15% + 30%

- Midterm (in class on Wednesday, March 12).
- Final (to be scheduled by Registrar).

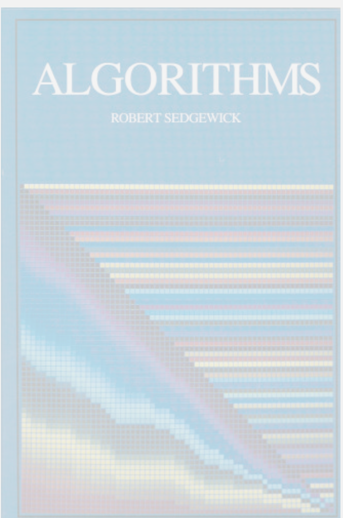
Staff discretion. [adjust borderline cases]

- Report errata.
- Contribute to Piazza discussion forum.
- Attend and participate in precept/lecture.

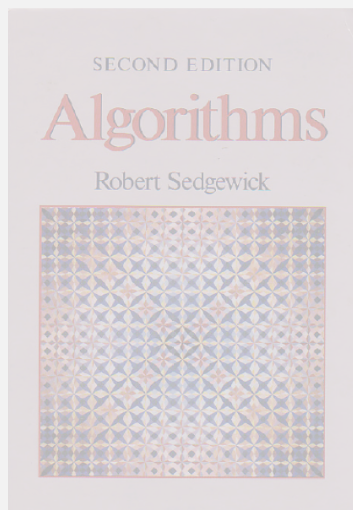


Resources (textbook)

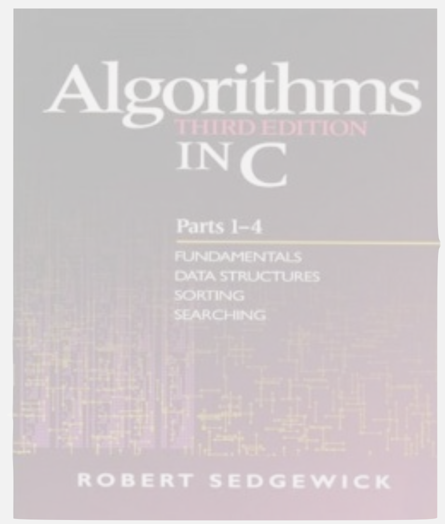
Required reading. Algorithms 4th edition by R. Sedgewick and K. Wayne, Addison-Wesley Professional, 2011, ISBN 0-321-57351-X.



1st edition (1982)

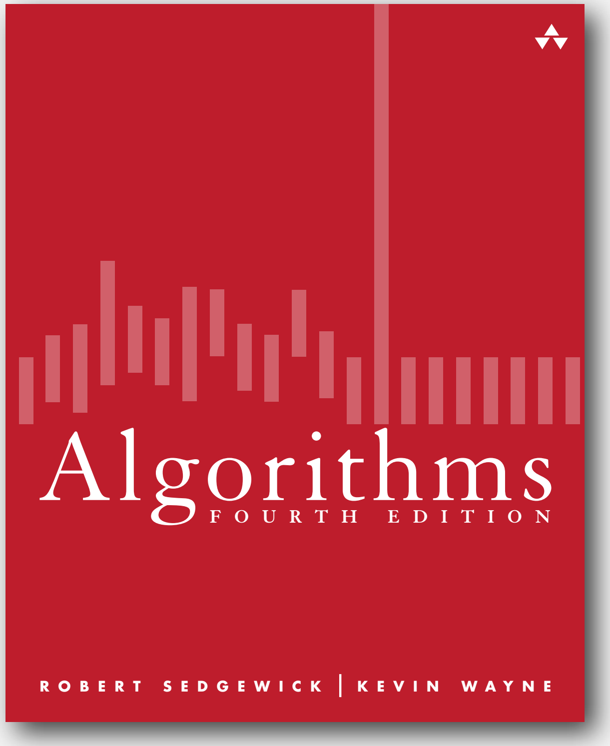


2nd edition (1988)



3rd edition (1997)

3rd book scanned by Google books



4th edition (2011)

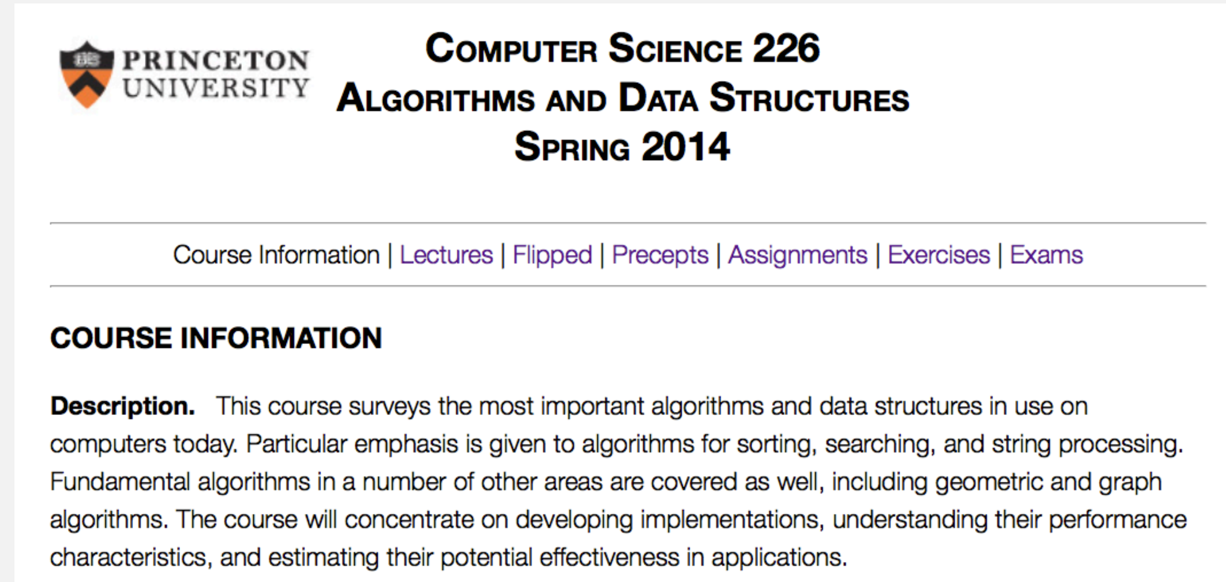
Available in hardcover and Kindle.

- Online: Amazon ($60/$35 to buy), Chegg ($25 to rent), ...
- Brick-and-mortar: Labyrinth Books (122 Nassau St).
- On reserve: Engineering library.

Resources (web)

Course content.

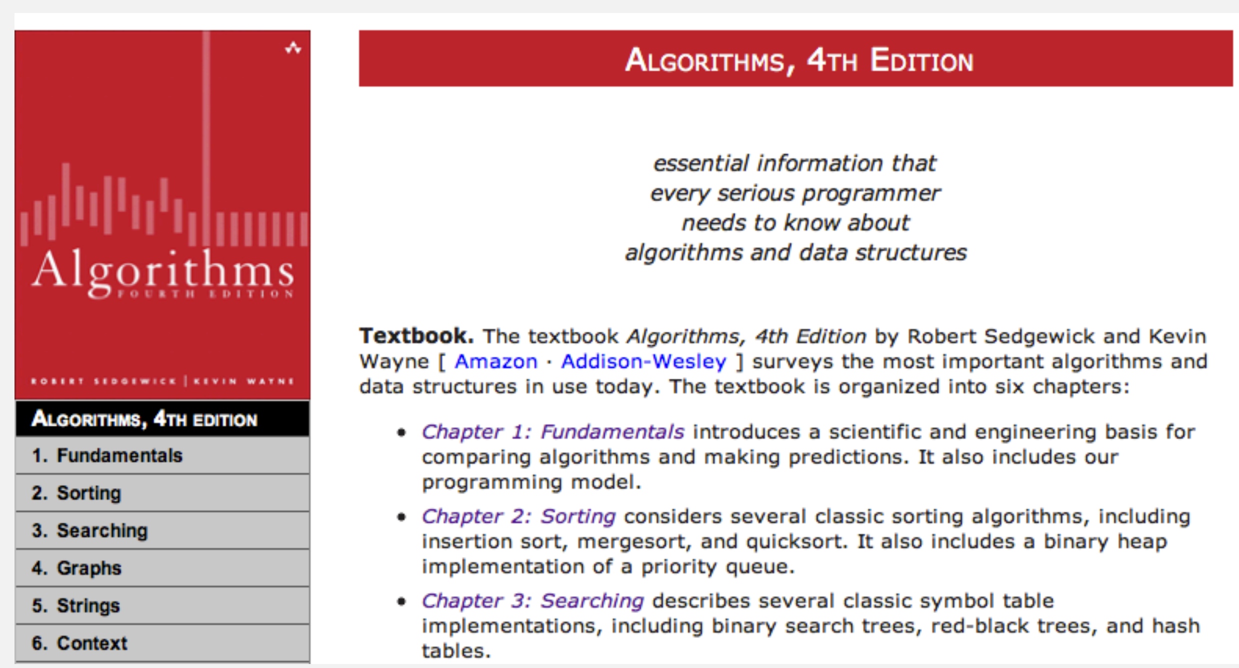
- Course info.
- Lecture slides.
- Flipped lectures.
- Programming assignments.
- Exercises.
- Exam archive.



http://www.princeton.edu/~cos226

Booksite.

- Brief summary of content.
- Download code from book.
- APIs and Javadoc.



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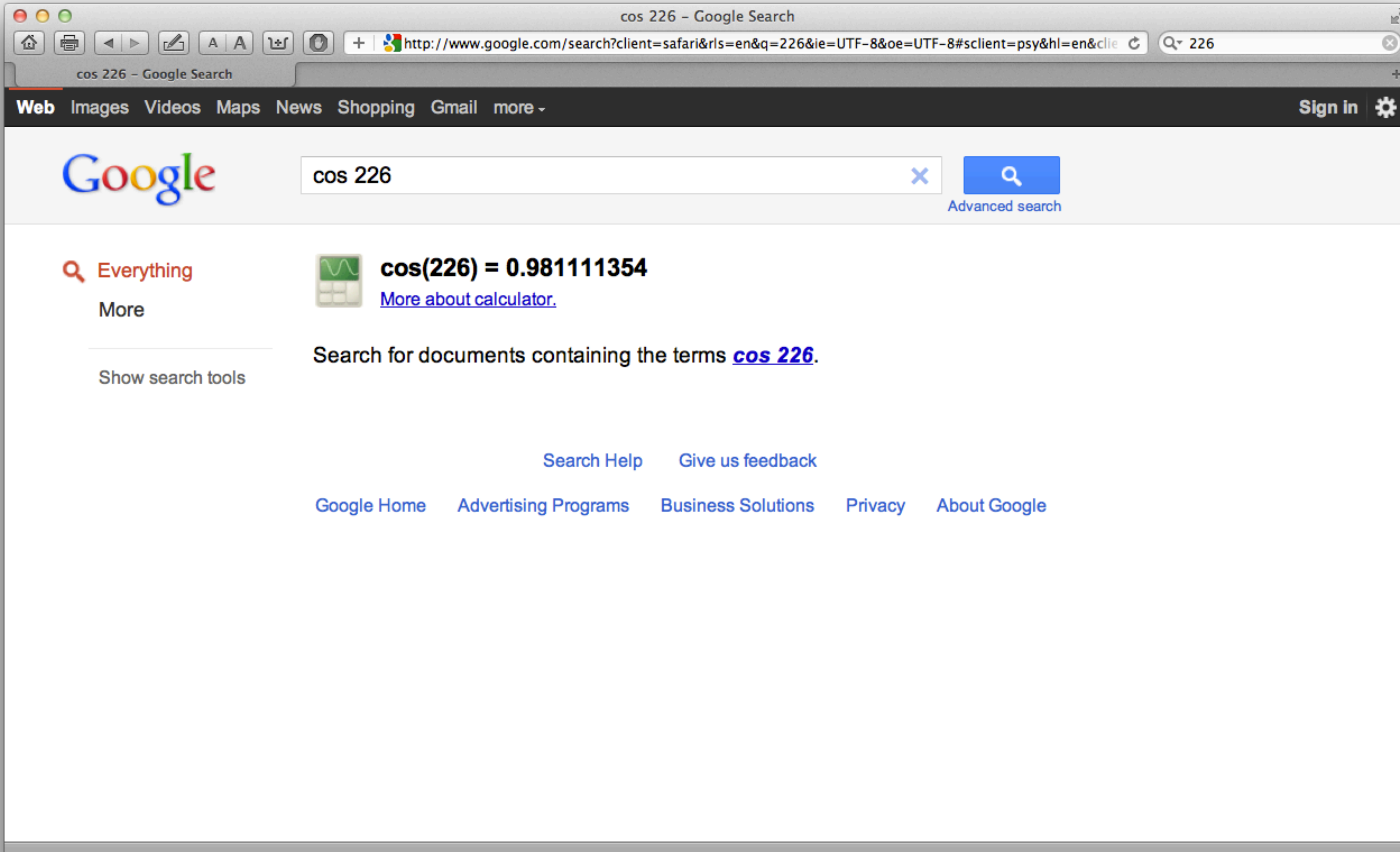
Resources (web)



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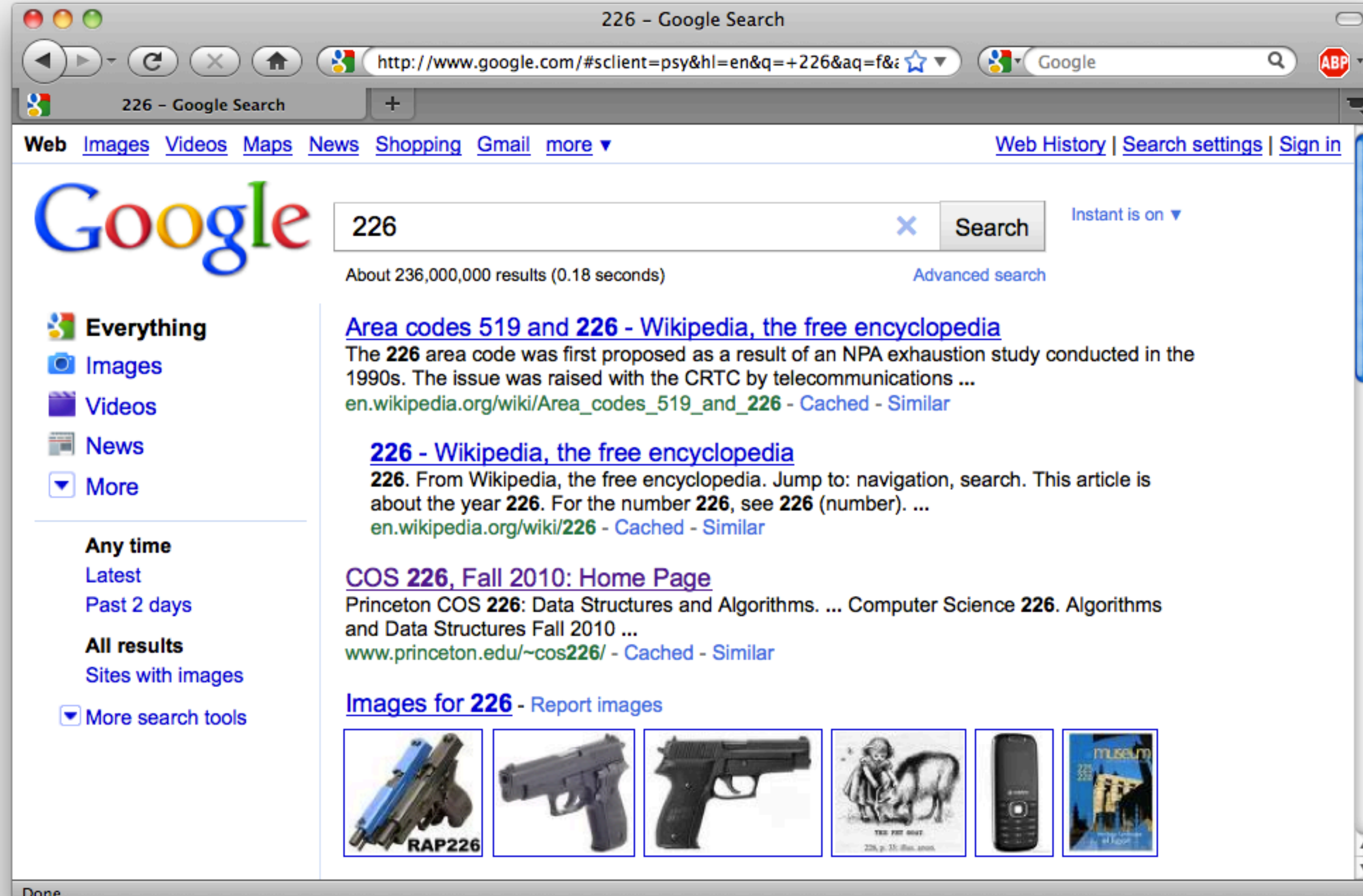
Resources (web)



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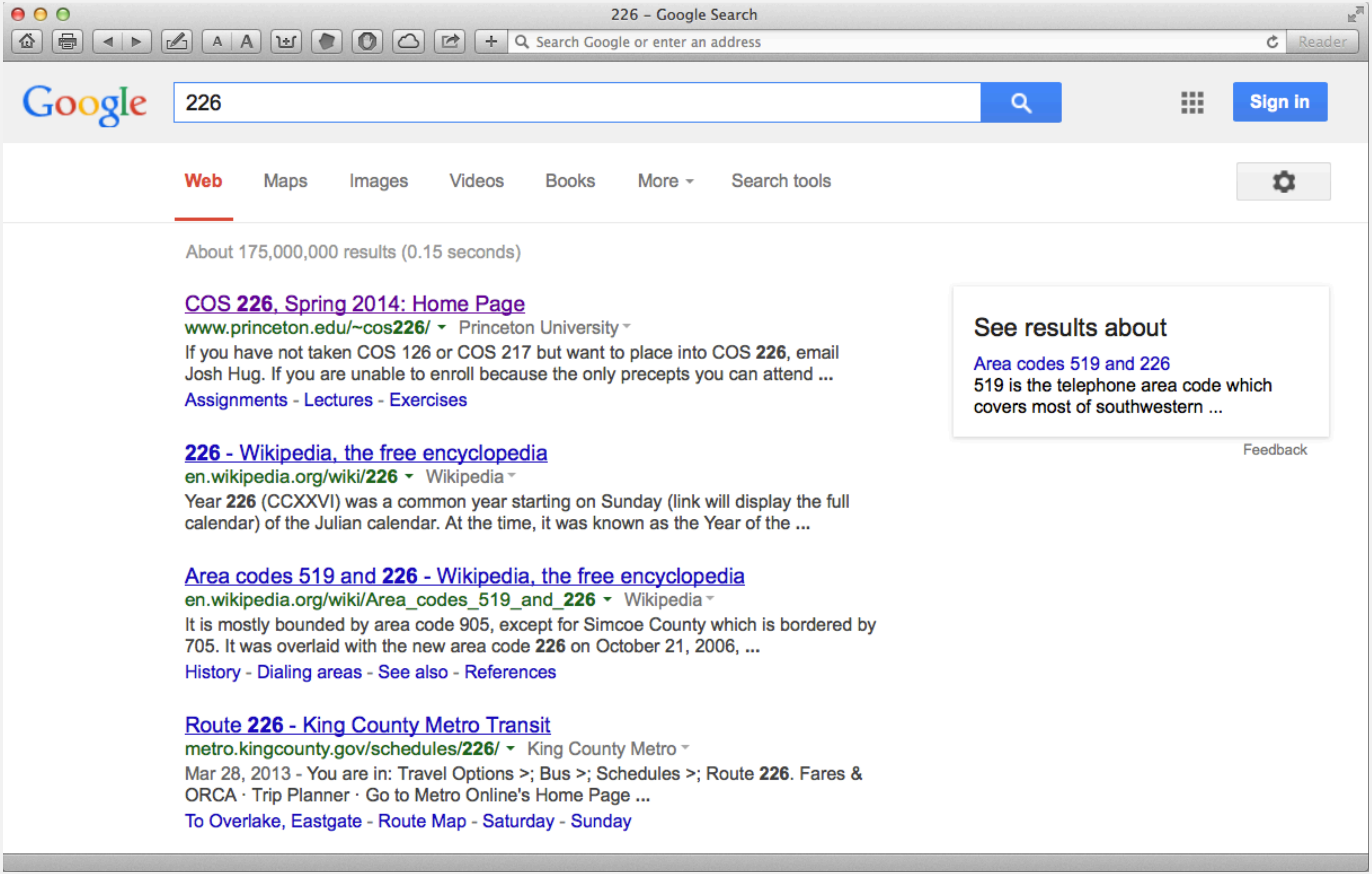
Resources (web)



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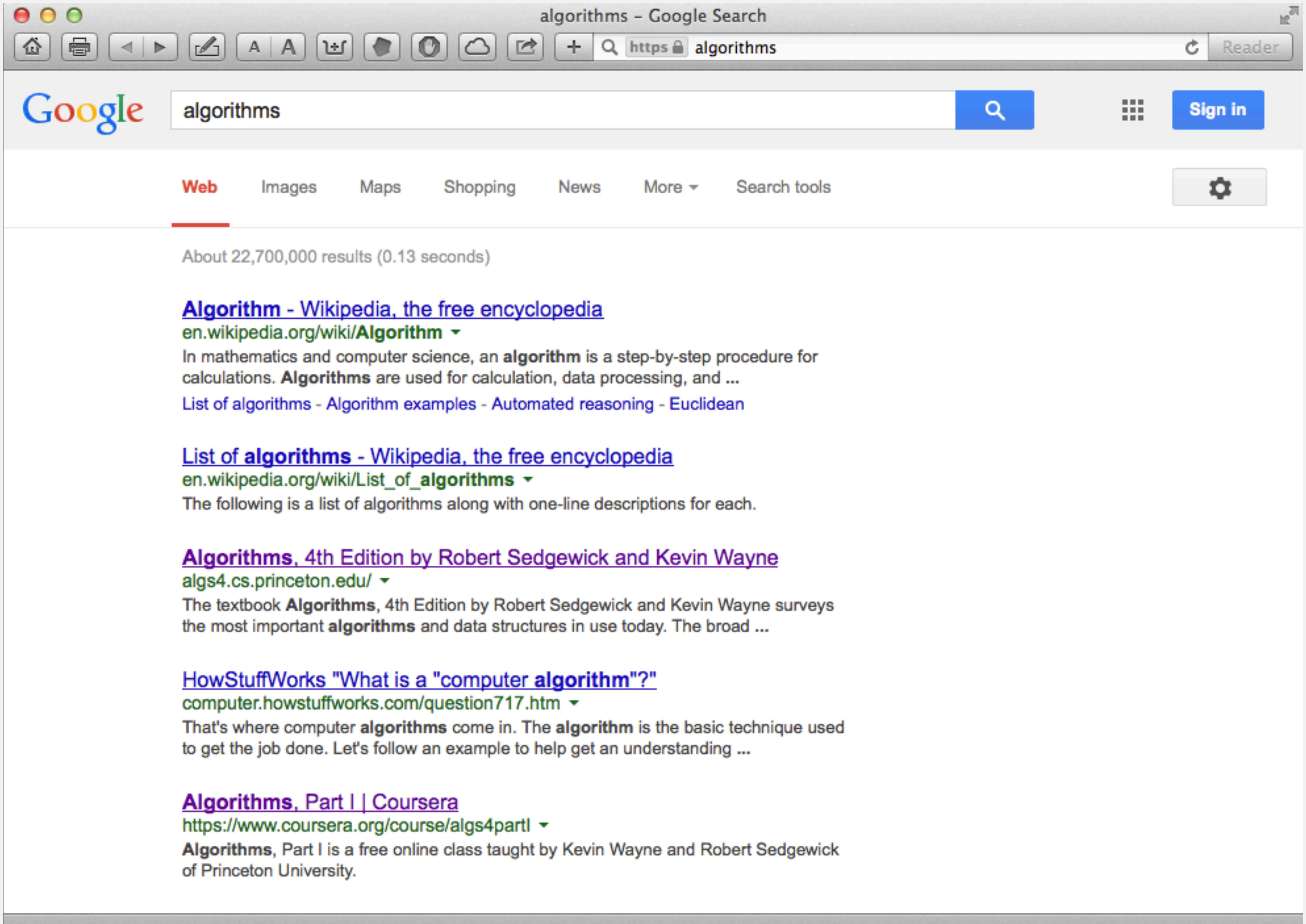
Resources (web)



Where to get help?

Piazza discussion forum.

- Low latency, low bandwidth.
- Mark solution-revealing questions as private.

piazza

http://piazza.com/princeton/spring2014/cos226

Office hours.

- High bandwidth, high latency.
- See web for schedule.

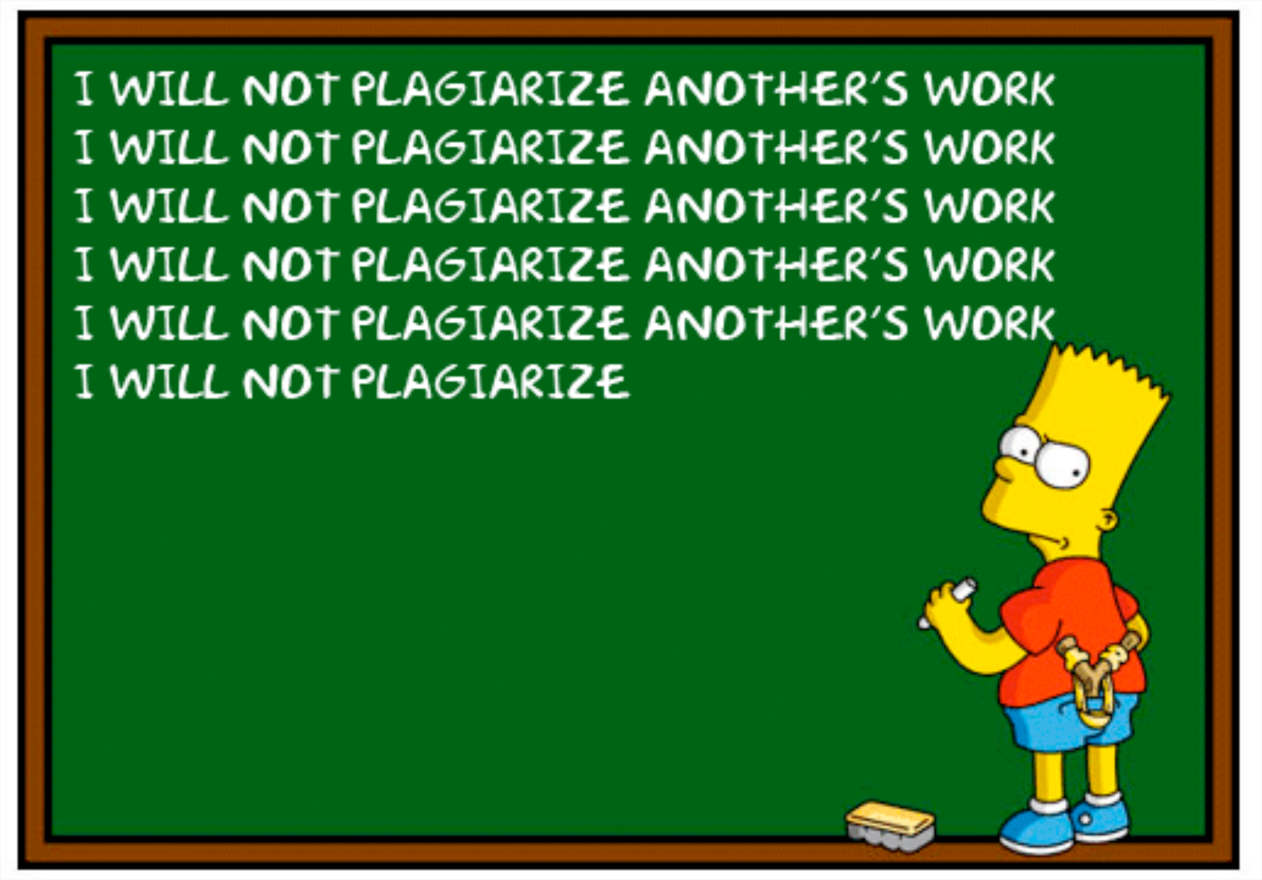
http://www.princeton.edu/~cos226

Computing laboratory.

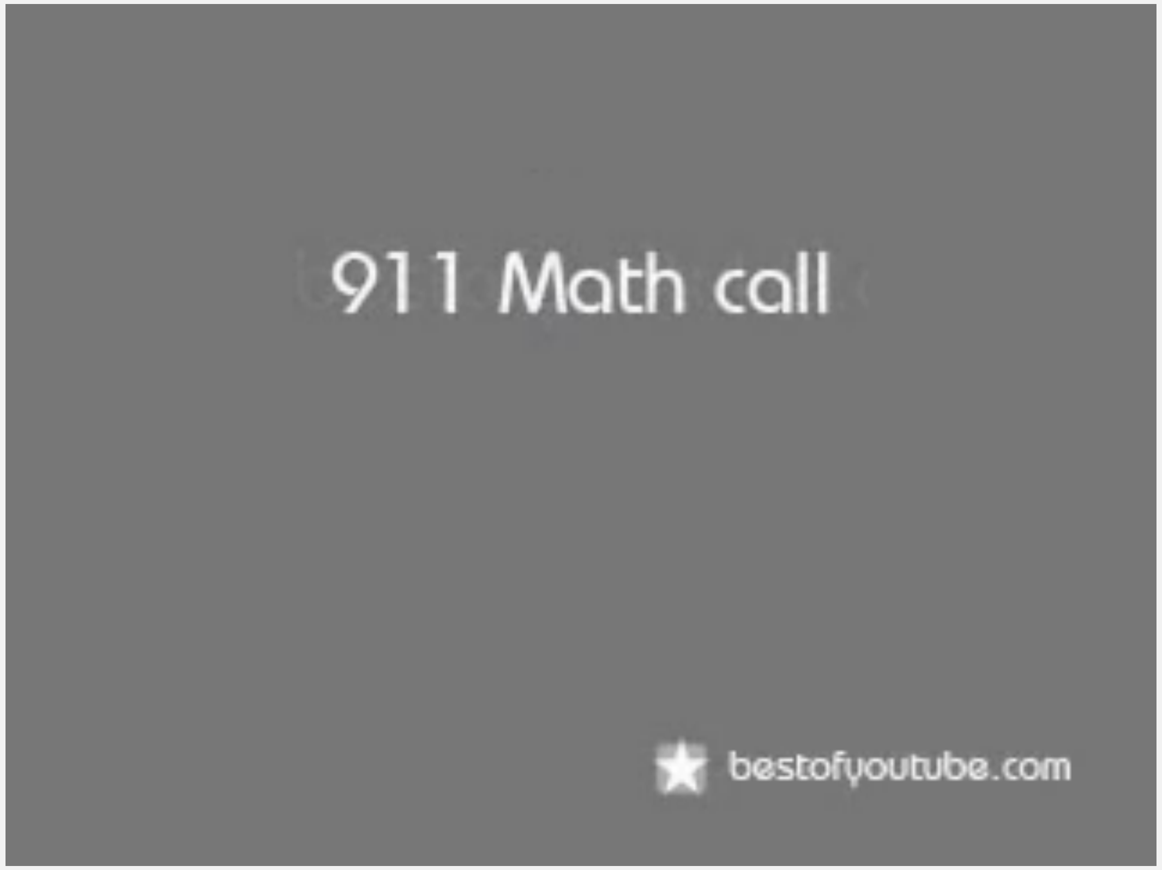
- Undergrad lab TAs in Friend 017.
- For help with debugging.
- See web for schedule.

http://www.princeton.edu/~cos226

Where not to get help?



http://world.edu/academic-plagiarism



http://www.youtube.com/watch?v=FT4NOe4vtoM

What's ahead?

Lecture 1. [today] Union find.

Lecture 2. [Wednesday] Analysis of algorithms.

Flipped lecture 1. [Wednesday] Watch video beforehand.

Precept 1. [Thursday/Friday] Meets this week.

Exercise 1. Due via Bb submission at 11pm on Sunday.

Assignment 1. Due via electronic submission at 11pm on Tuesday.

protip: start early

Right course? See me.

Placed out of COS 126? Review Sections 1.1–1.2 of Algorithms 4/e.

Not registered? Go to any precept this week.

Change precept? Use SCORE.

see Colleen Kenny-McGinley in CS 210 if the only precepts you can attend are closed

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1.5 UNION-FIND

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why not.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

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1.5 Union-Find

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Dynamic connectivity problem

Given a set of N objects, support two operation:

- Connect two objects.
- Is there a path connecting the two objects?

connect 4 and 3

connect 3 and 8

connect 6 and 5

connect 9 and 4

connect 2 and 1

are 0 and 7 connected? ✗

are 8 and 9 connected? ✔

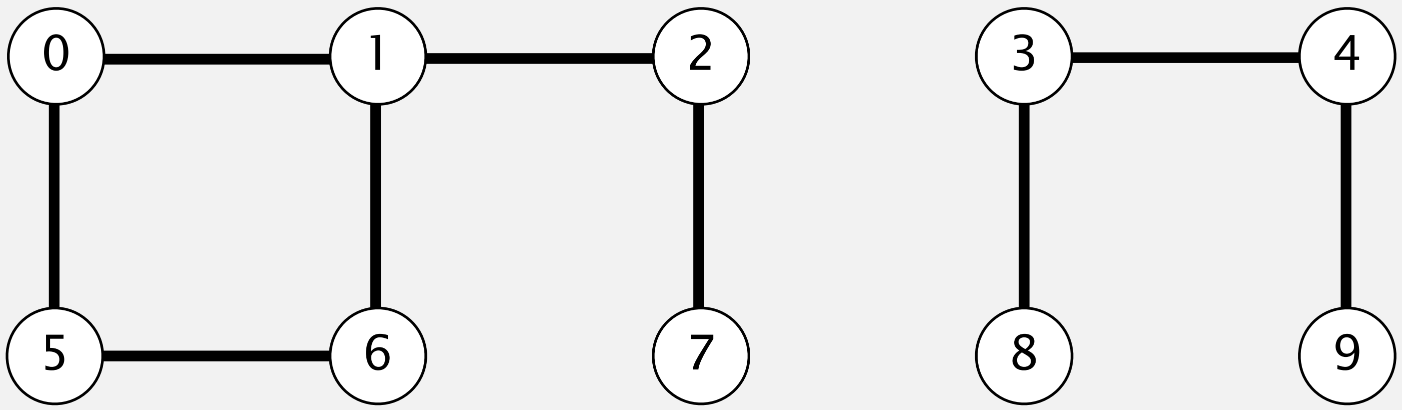
connect 5 and 0

connect 7 and 2

connect 6 and 1

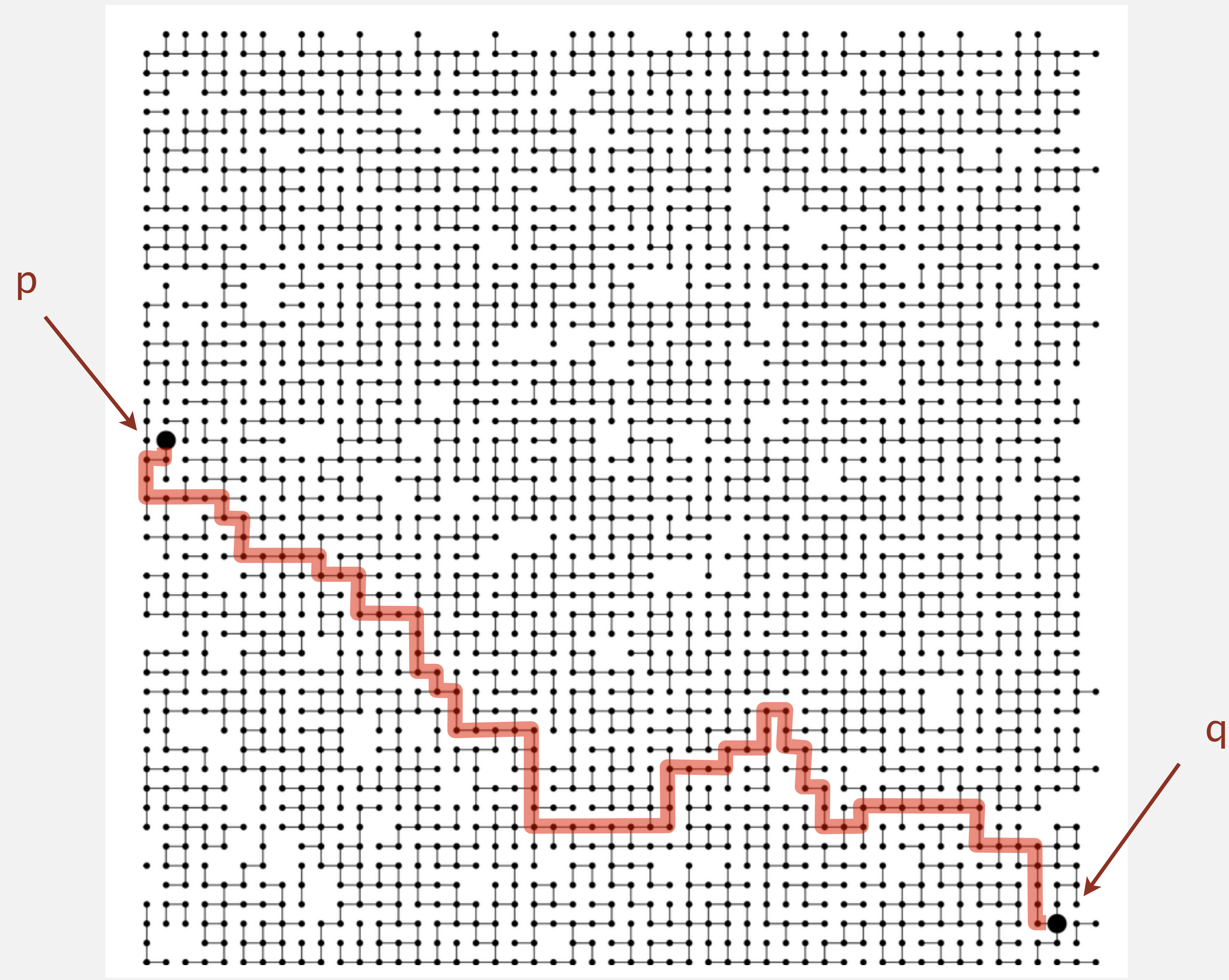
connect 1 and 0

are 0 and 7 connected? ✔



A larger connectivity example

Q. Is there a path connecting p and q ?



A. Yes.

Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to N – 1.

- Use integers as array index.
- Suppress details not relevant to union-find.

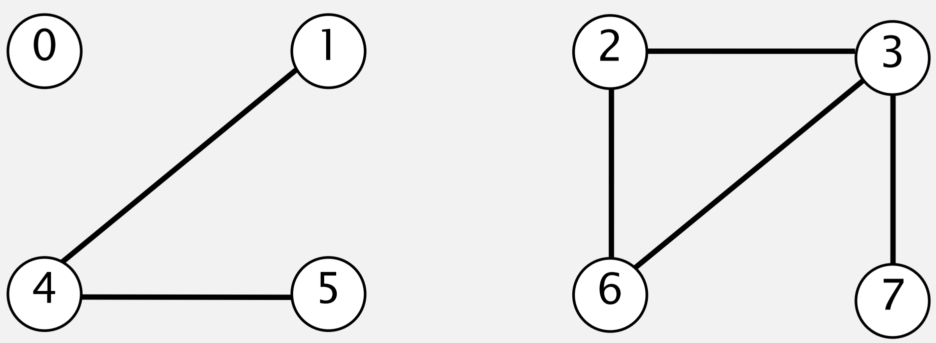
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.

Connected component. Maximal set of objects that are mutually connected.



{ 0 } { 1 4 5 } { 2 3 6 7 }

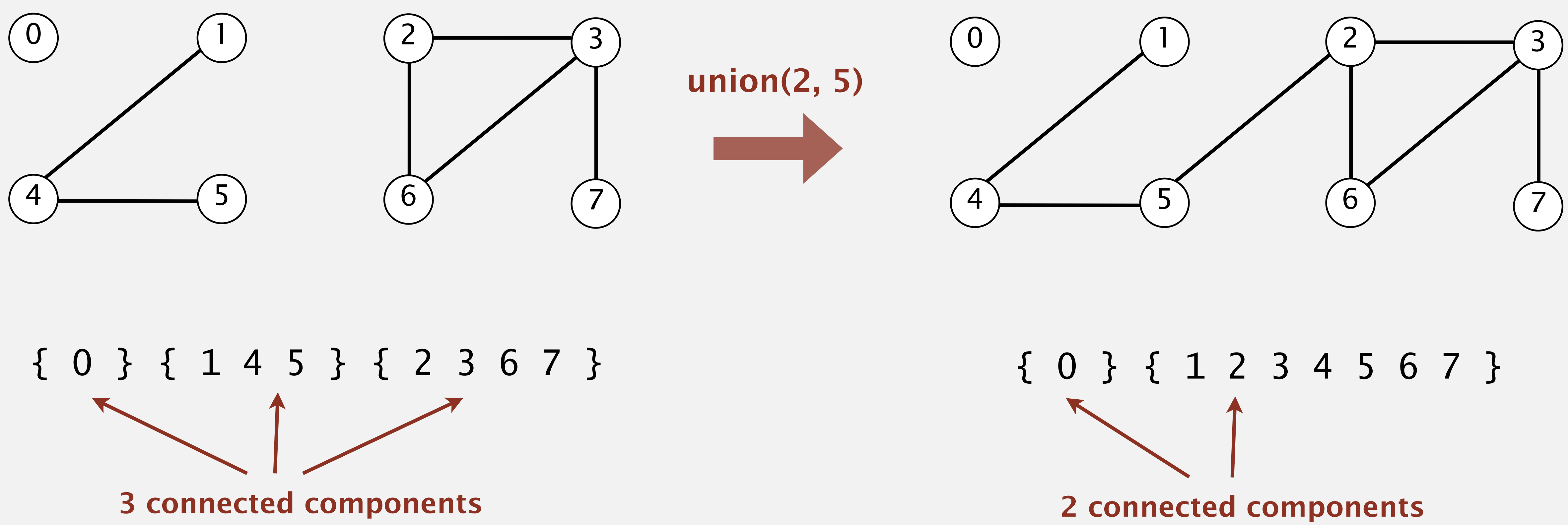
3 connected components

Implementing the operations

Find. In which component is object p ?

Connected. Are objects p and q in the same component?

Union. Replace components containing objects p and q with their union.



Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Union and find operations may be intermixed.

public class UF		
	`UF(int N)`	*initialize union-find data structure with N singleton objects (0 to N – 1)*
`void`	`union(int p, int q)`	*add connection between p and q*
`int`	`find(int p)`	*component identifier for p (0 to N – 1)*
`boolean`	`connected(int p, int q)`	*are p and q in the same component?*

```
public boolean connected(int p, int q)
{   return find(p) == find(q);   }
```

1-line implementation of connected()

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (!uf.connected(p, q))
      {
         uf.union(p, q);
         StdOut.println(p + " " + q);
      }
   }
}
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

already connected

http://algs4.cs.princeton.edu

1.5 Union-Find

- *dynamic connectivity*
- ***quick find***
- *quick union*
- *improvements*
- *applications*

Quick-find [eager approach]

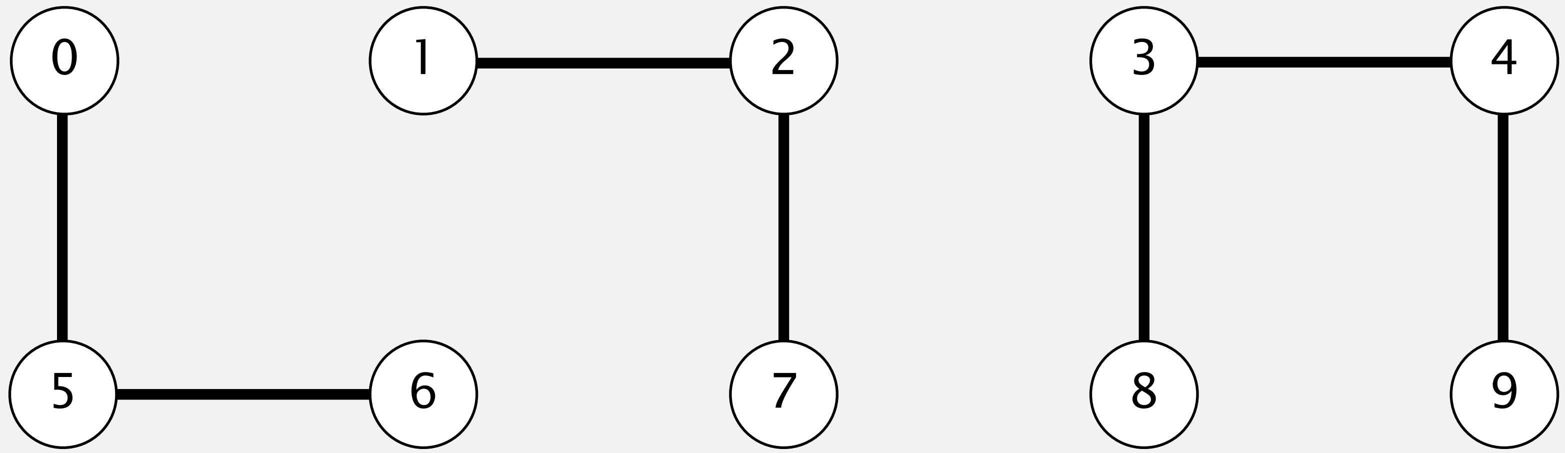
Data structure.

if and only if

- Integer array `id[]` of length `N`.
- Interpretation: `id[p]` is the id of the component containing `p`.

	0	1	2	3	4	5	6	7	8	9
`id[]`	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Quick-find [eager approach]

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[p]` is the id of the component containing `p`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Find. What is the id of `p`?

Connected. Do `p` and `q` have the same id?

`id[6] = 0; id[1] = 1`

6 and 1 are not connected

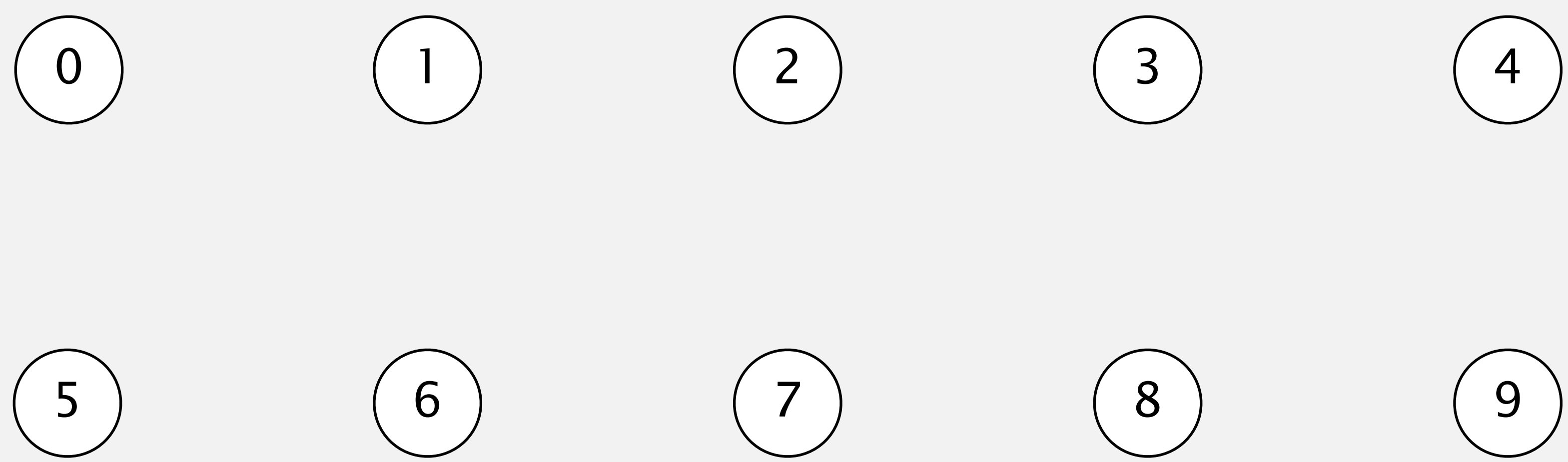
Union. To merge components containing `p` and `q`, change all entries whose id equals `id[p]` to `id[q]`.

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

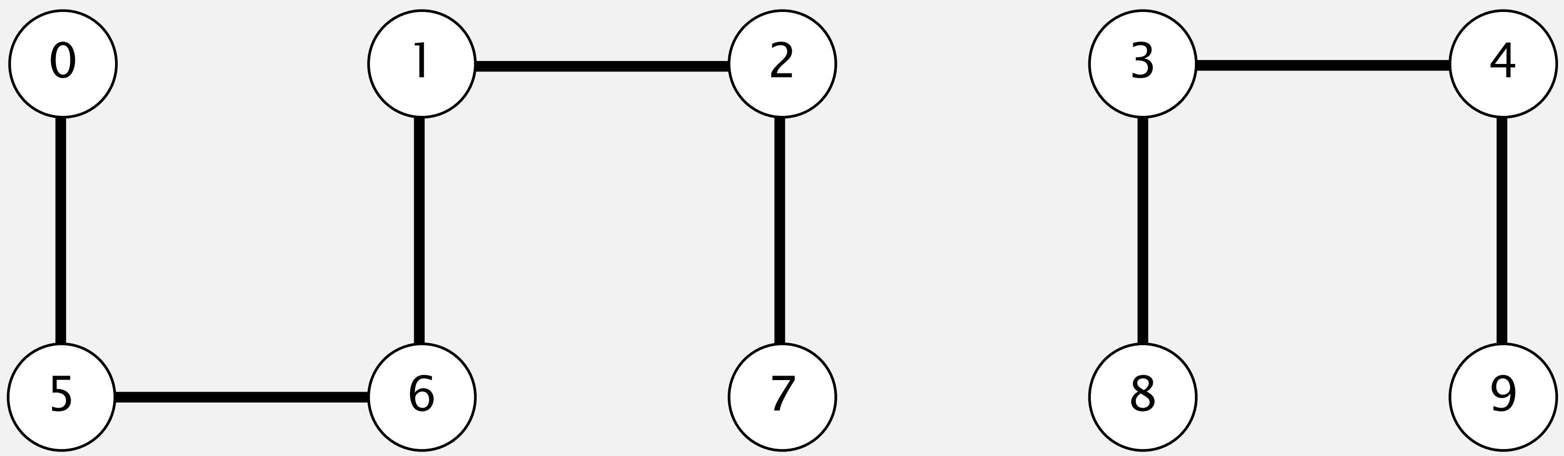
problem: many values can change

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }

   public boolean find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

return the id of p
(1 array access)

change all entries with id[p] to id[q]
(at most 2N + 2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

quadratic

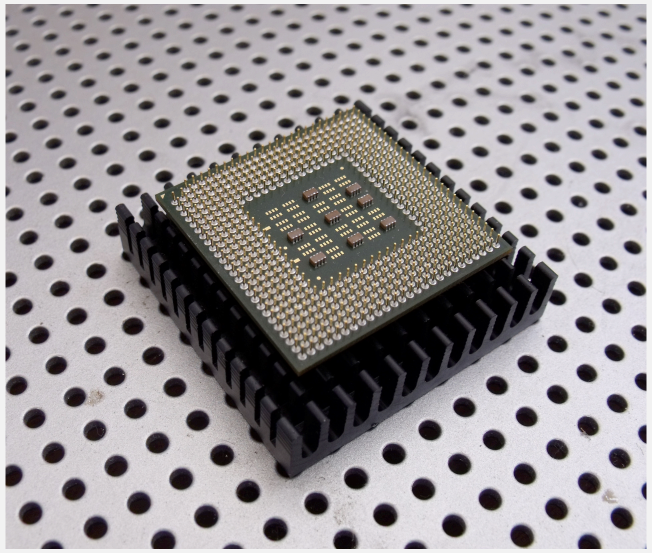
Union is too expensive. It takes N^2 array accesses to process a sequence of N union operations on N objects.

Quadratic algorithms do not scale

Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!

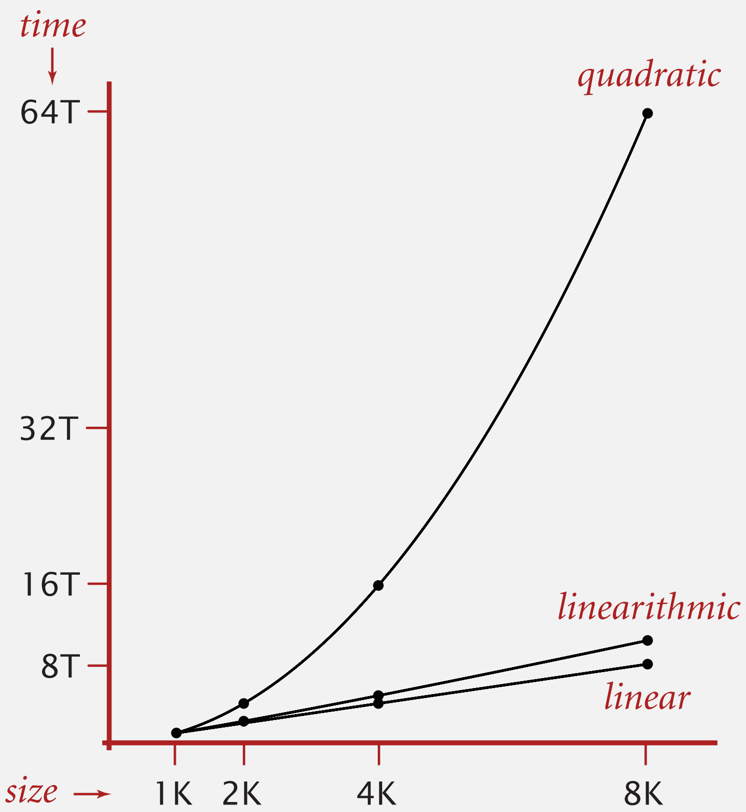


Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory $\Rightarrow$ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!



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1.5 Union-Find

- dynamic connectivity
- quick find
- **quick union**
- improvements
- applications

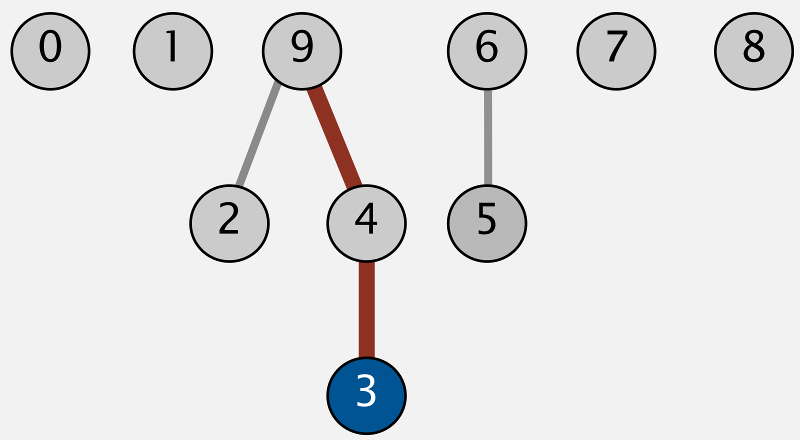
Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

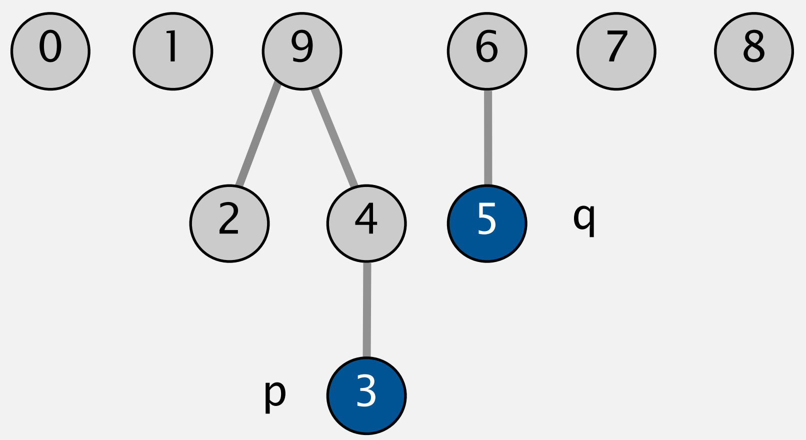
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9
root of 5 is 6
3 and 5 are not connected

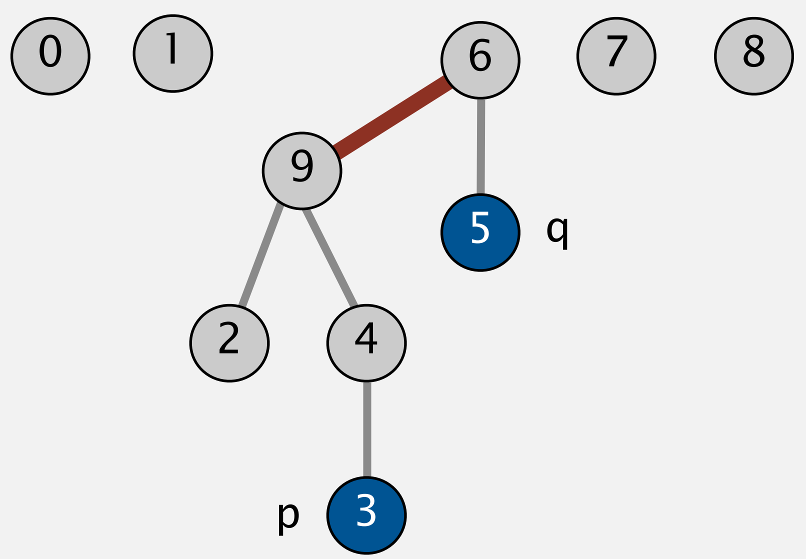
Find. What is the root of `p`?

Connected. Do `p` and `q` have the same root?

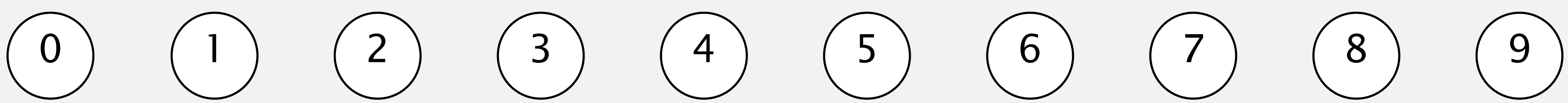
Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes

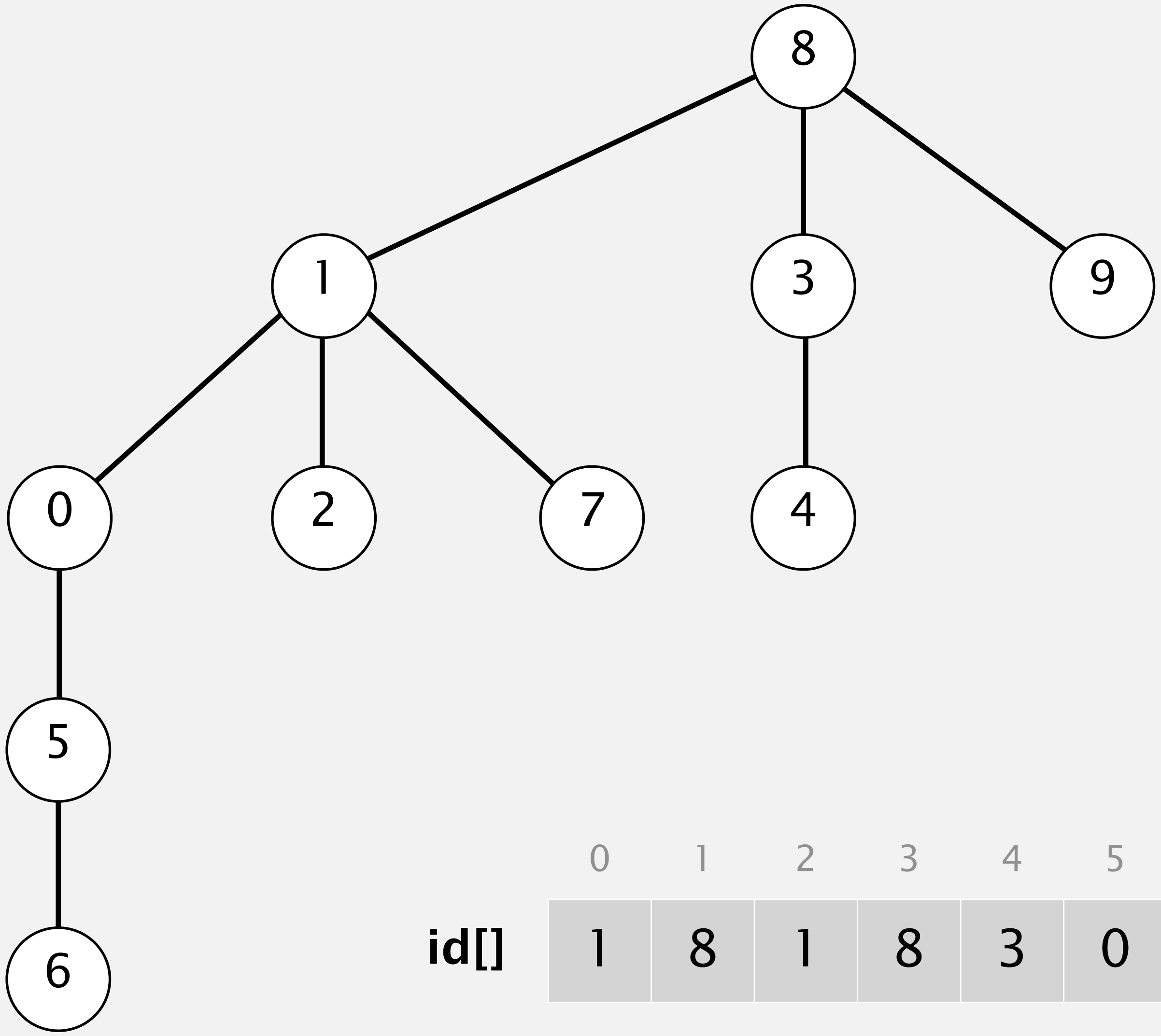


Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   public int find(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }


   public void union(int p, int q)
   {
      int i = find(p);
      int j = find(q);
      id[i] = j;
   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be N array accesses).

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1.5 Union-Find

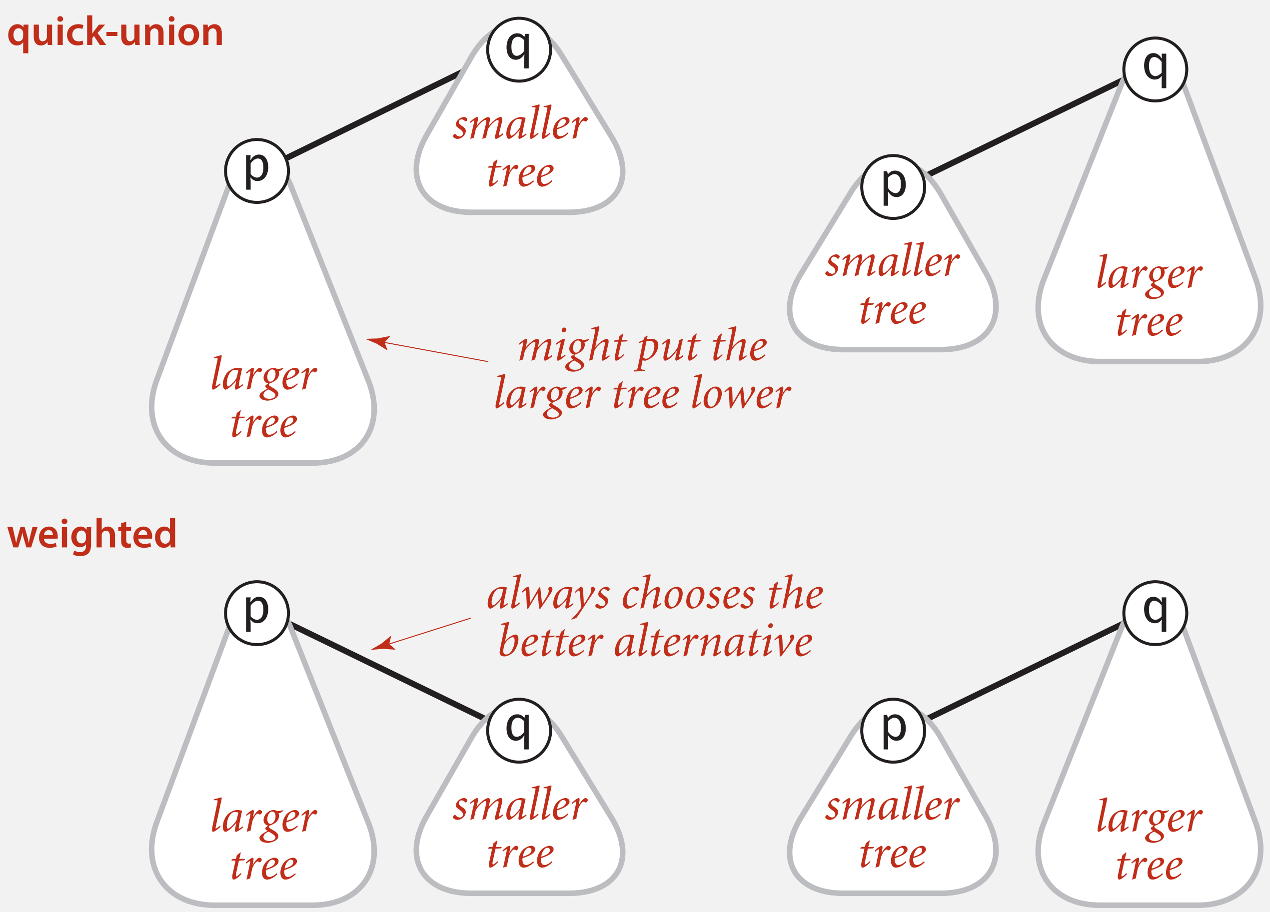
- dynamic connectivity
- quick find
- quick union
- *improvements*
- applications

Improvement 1: weighting

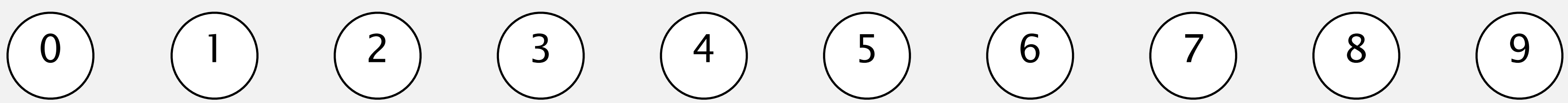
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"



Weighted quick-union demo



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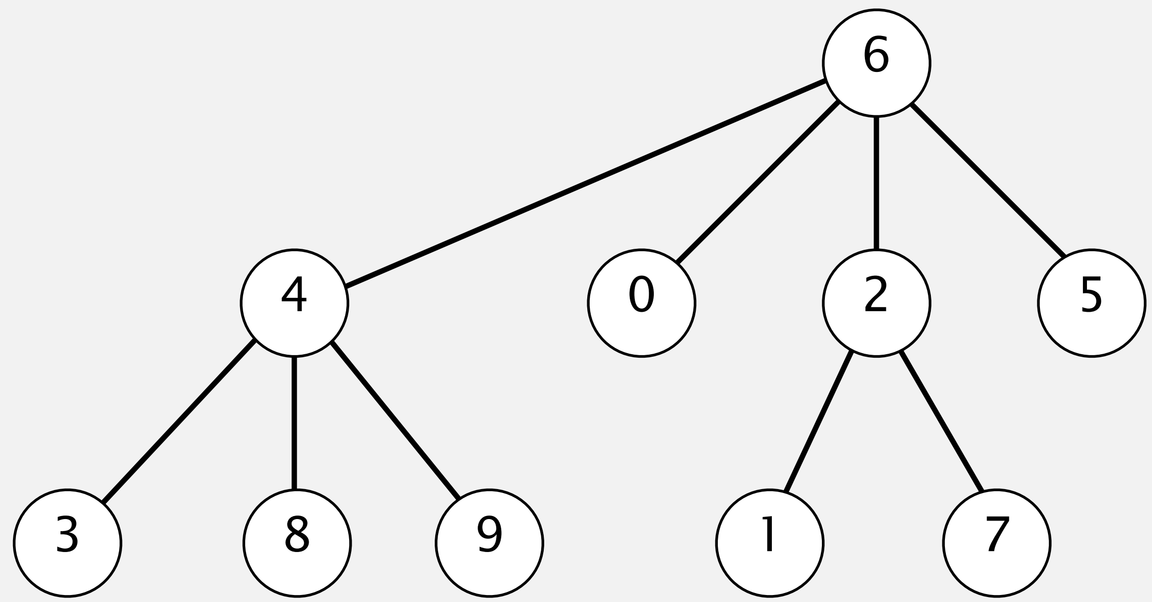
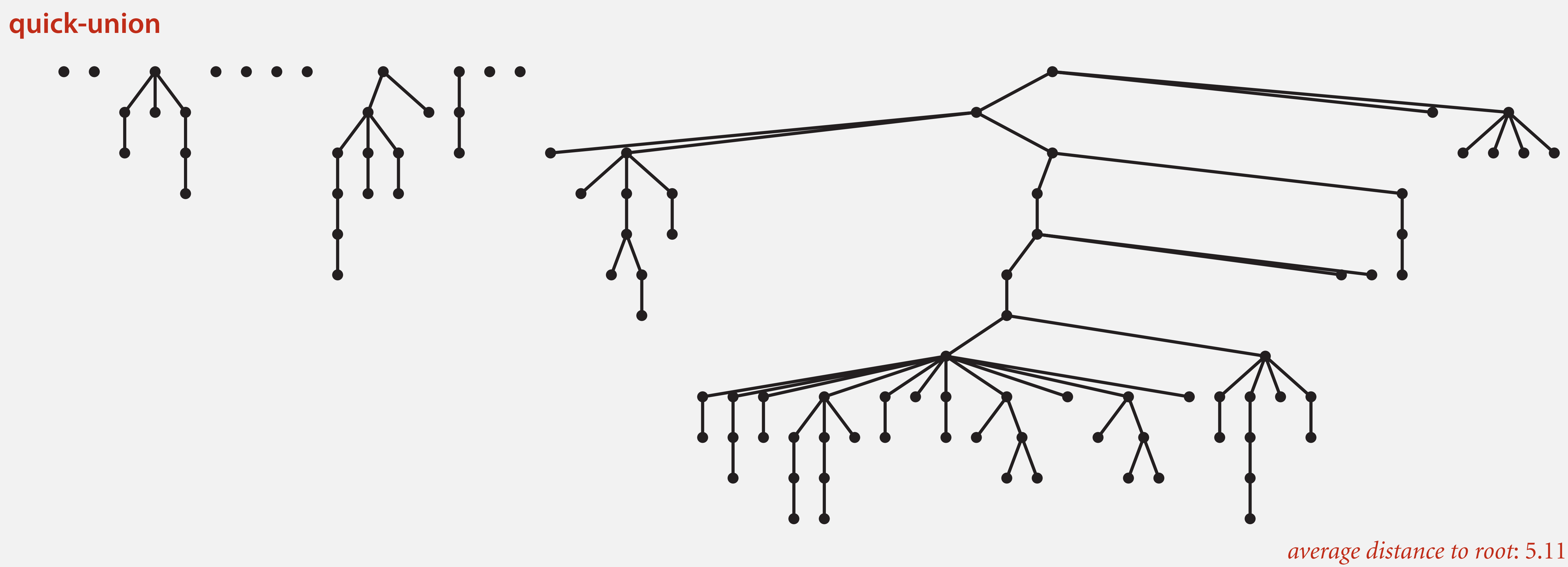
id[]	0	1	2	3	4	5	6	7	8	9

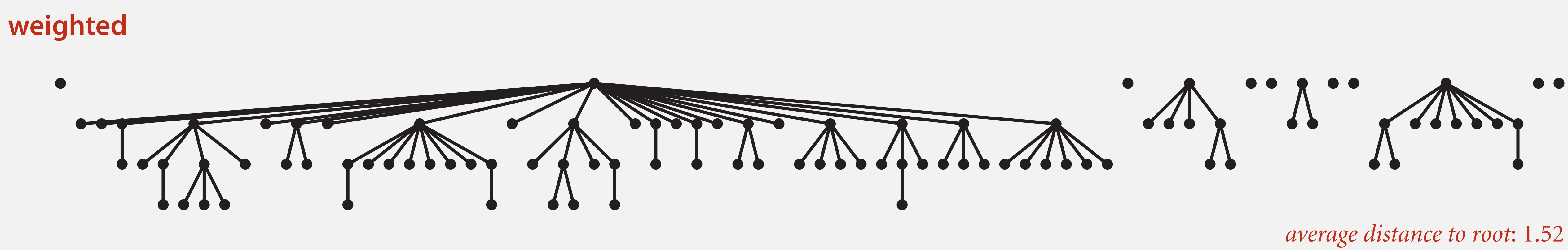
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Quick-union and weighted quick-union example





Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

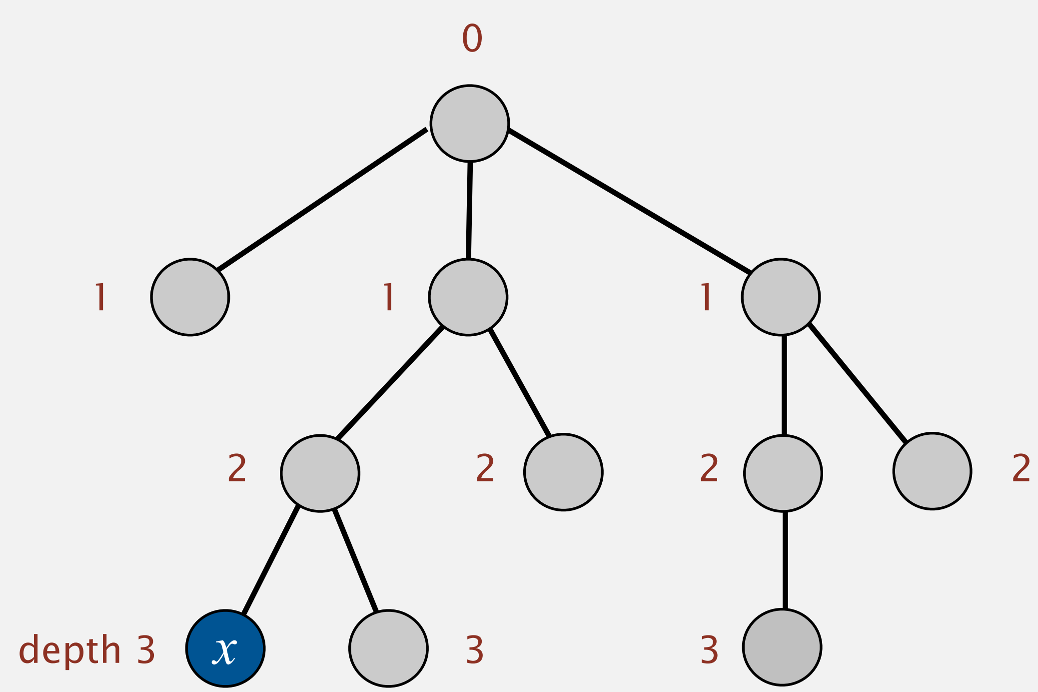
Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



N = 11
depth(x) = 3 ≤ lg N

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

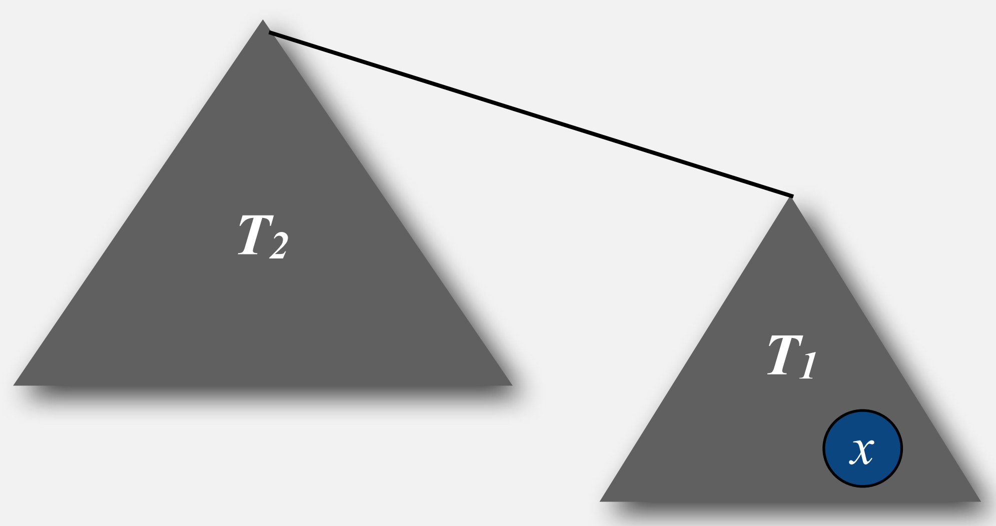
lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.

Pf. What causes the depth of object x to increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg N$ times. Why?



$$\left.\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ \vdots \\ N \end{array}\right\} \lg N$$

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

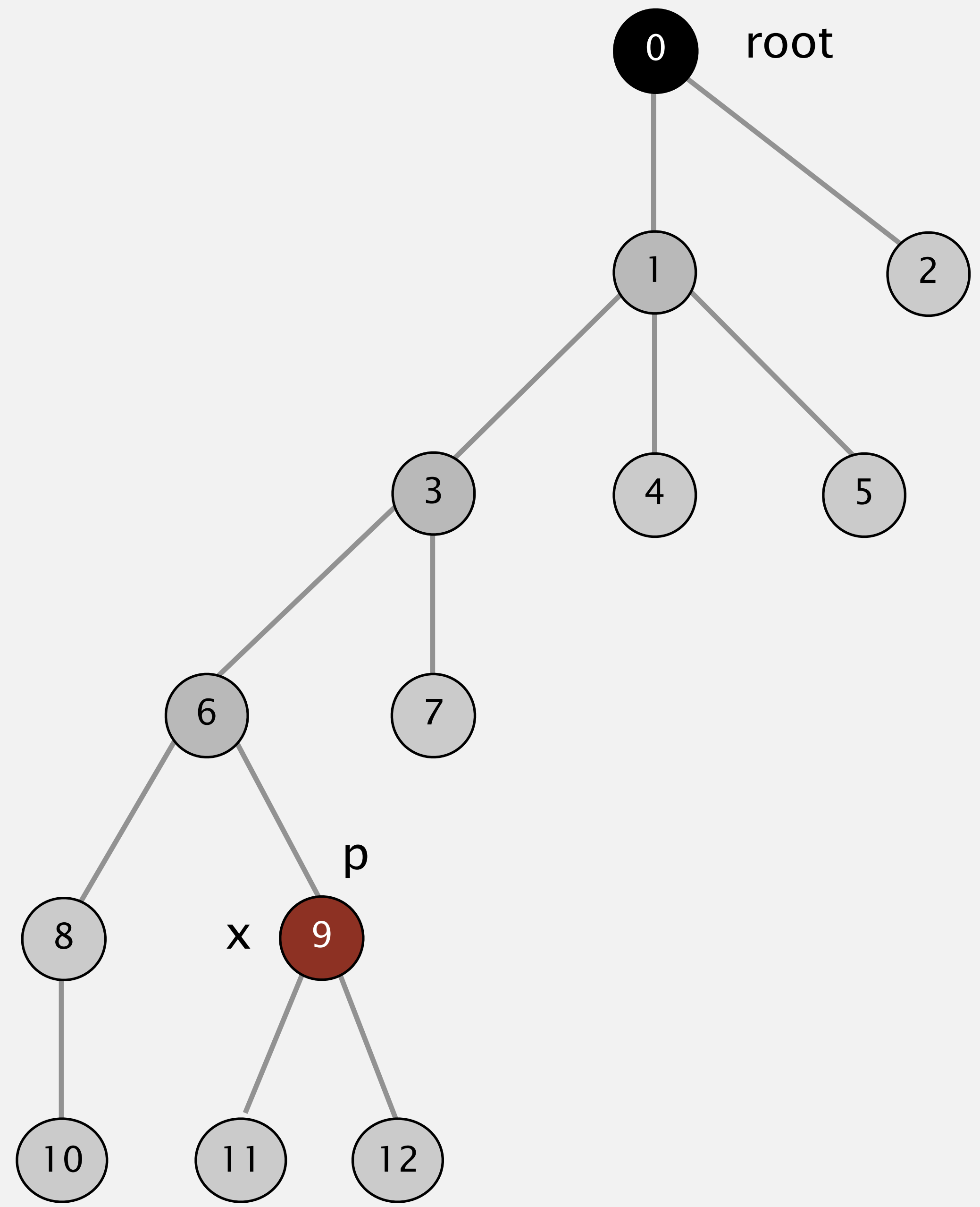
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

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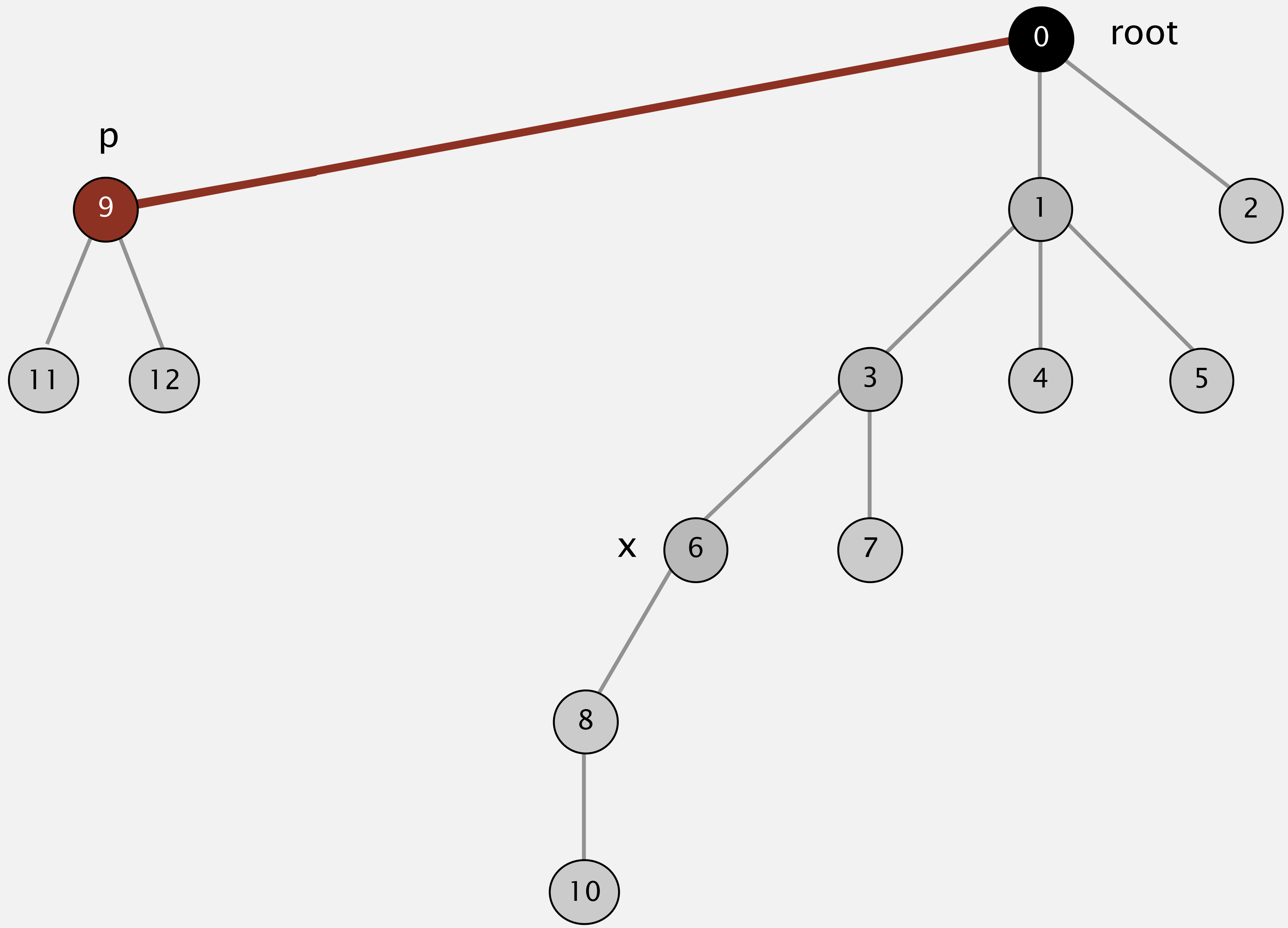
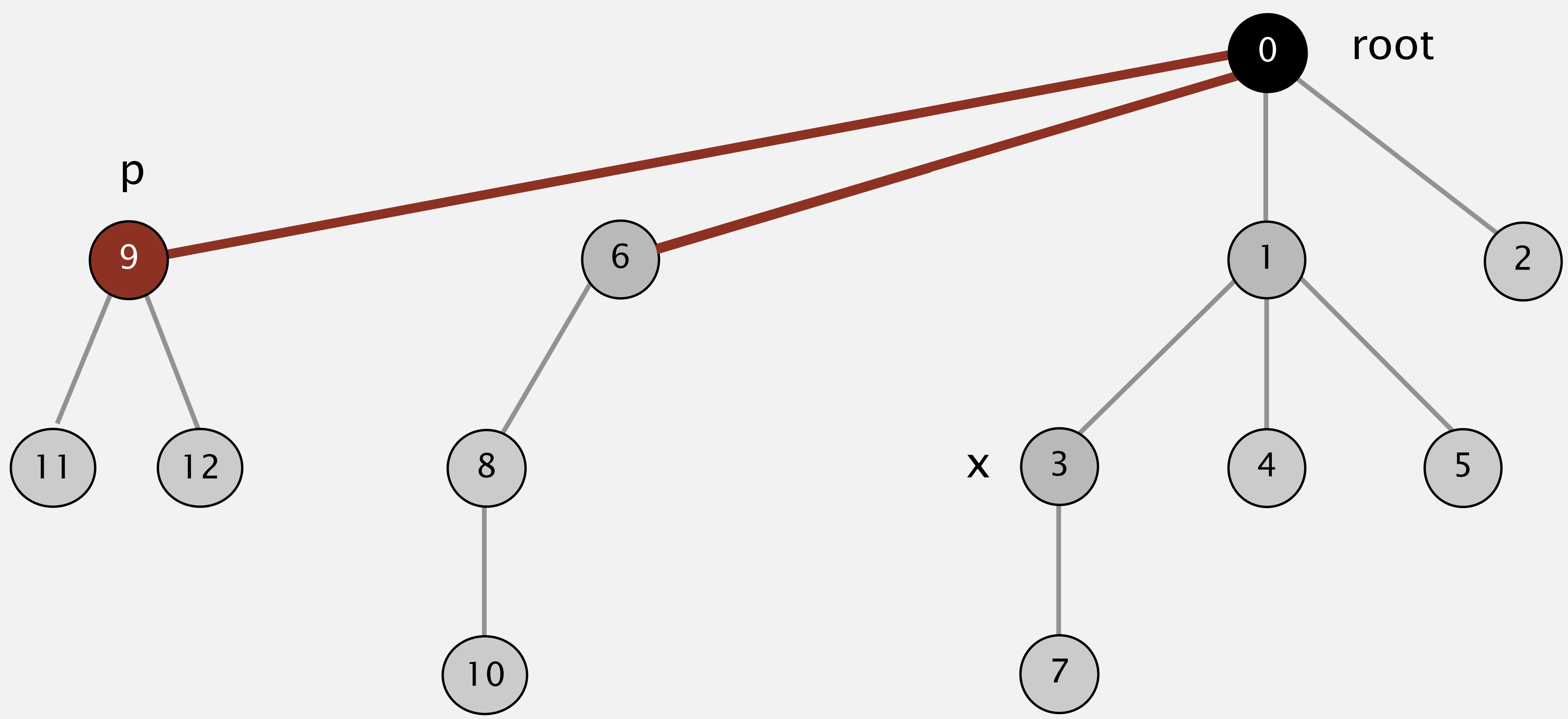


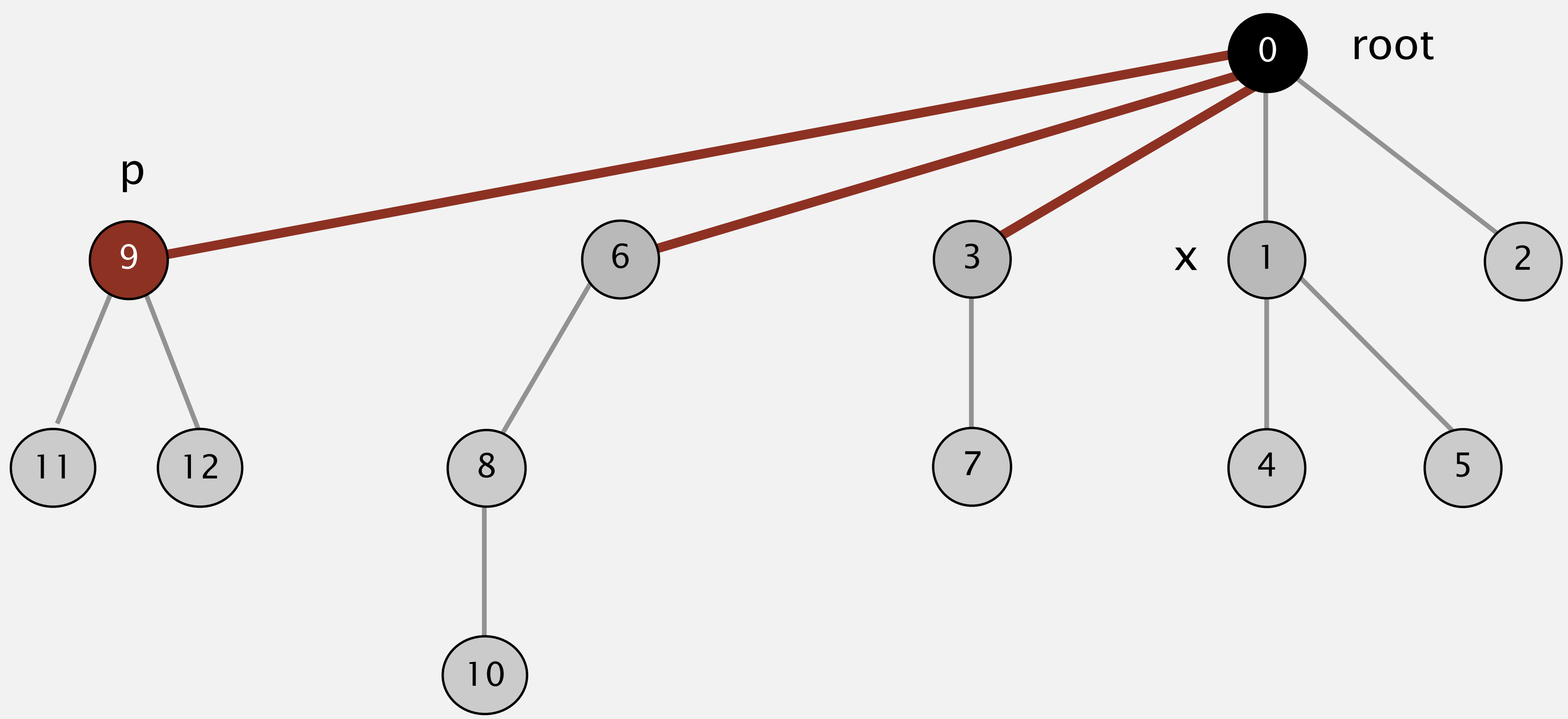
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



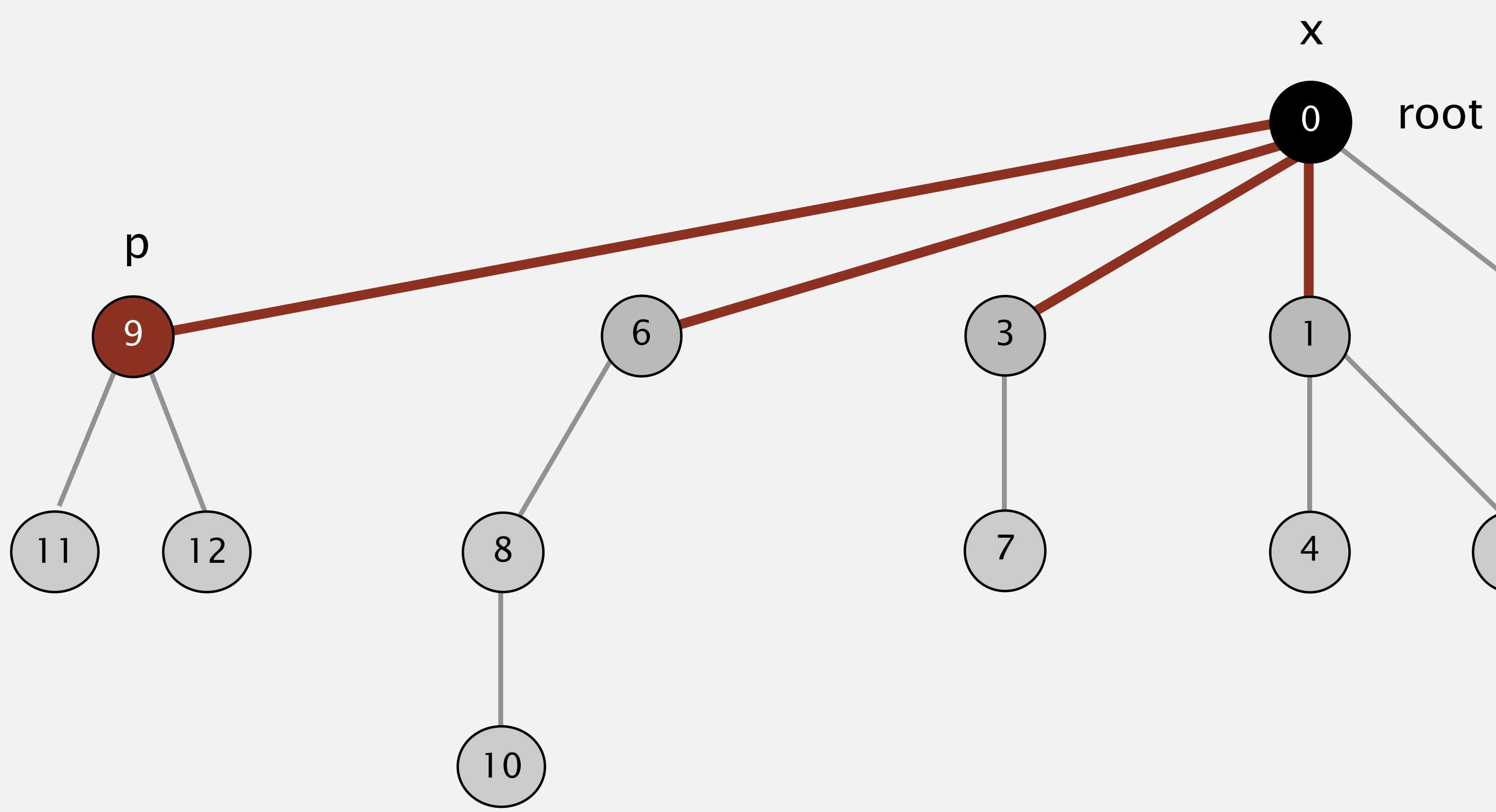
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Bottom line. Now, `find()` has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression: amortized analysis

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of M union–find ops on N objects makes $\leq\ c\,(\,N + M \lg^* N\,)$ array accesses.

- Analysis can be improved to $N + M\,\alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterated lg function

Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

order of growth for M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

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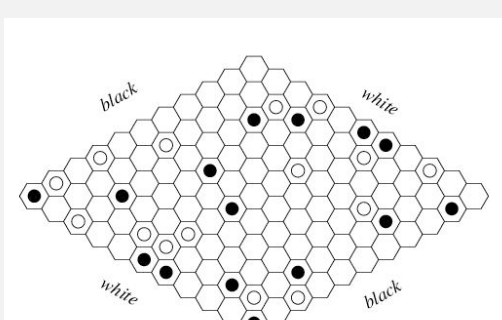
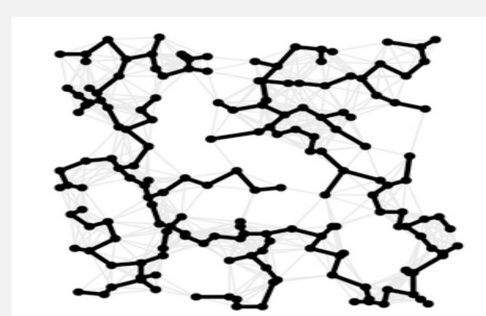
1.5 Union-Find

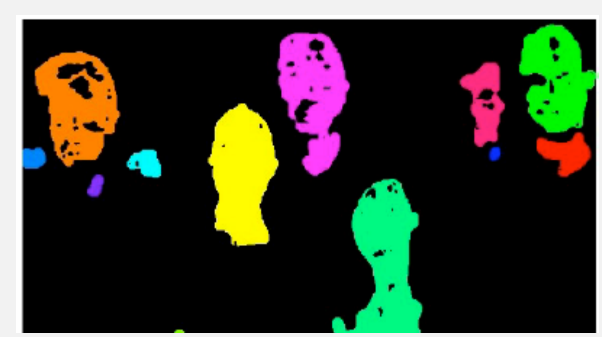
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Union-find applications

- Percolation.
- Games (Go, Hex).
- ✓ Dynamic connectivity.
- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's `bwlabel()` function in image processing.



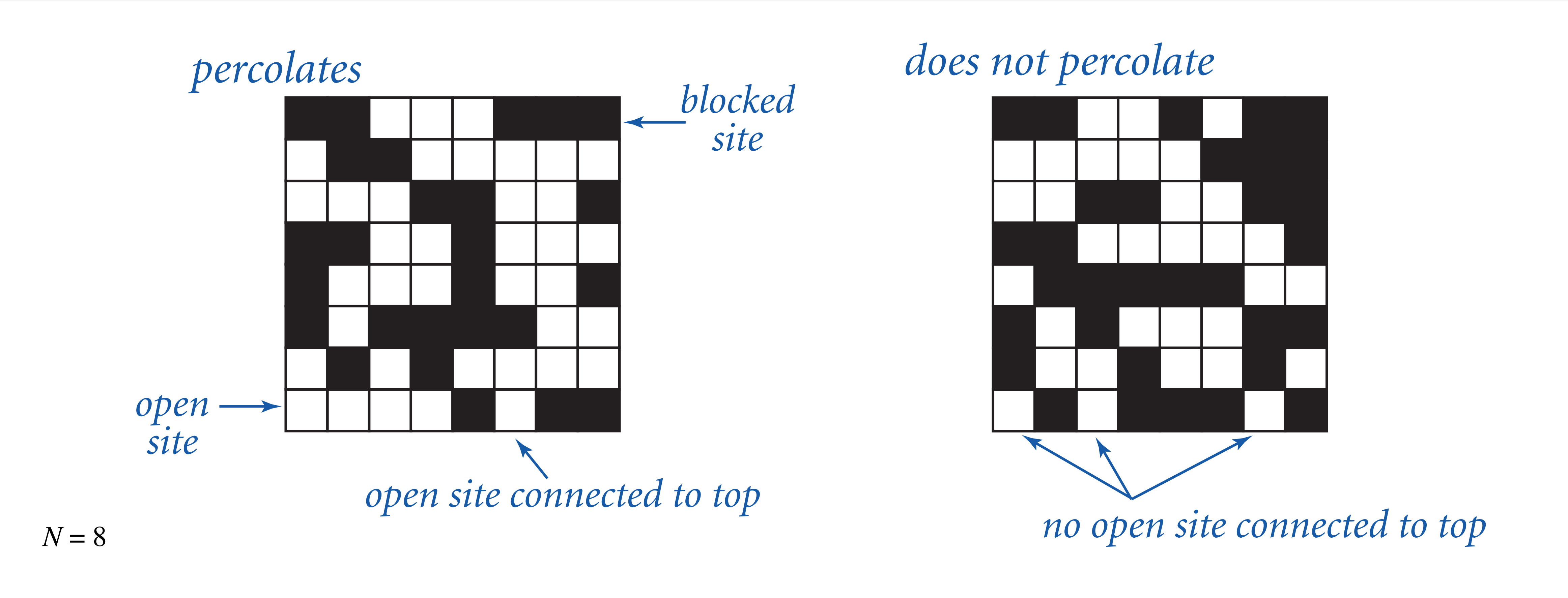




Percolation

An abstract model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.



Percolation

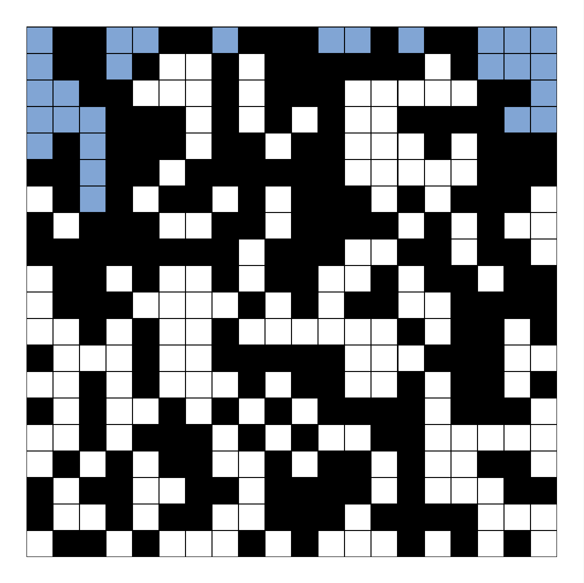
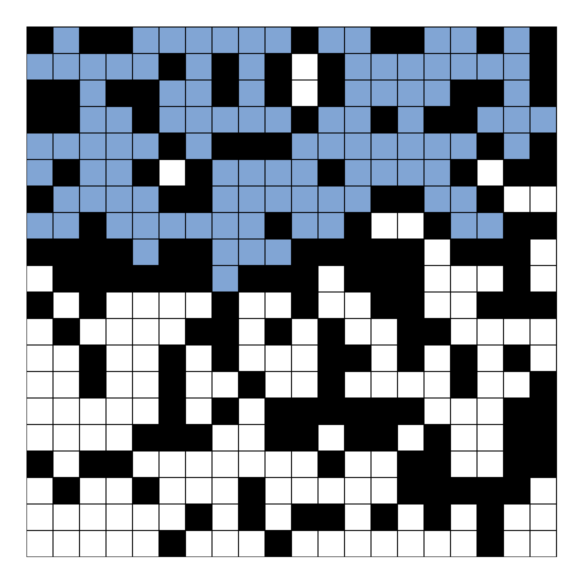
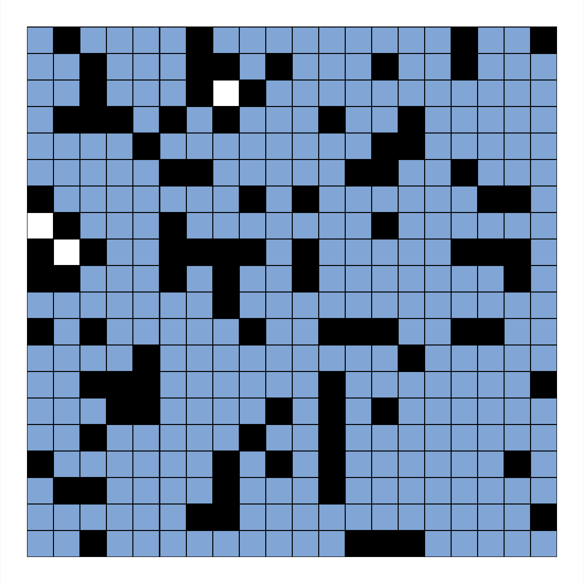
An abstract model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

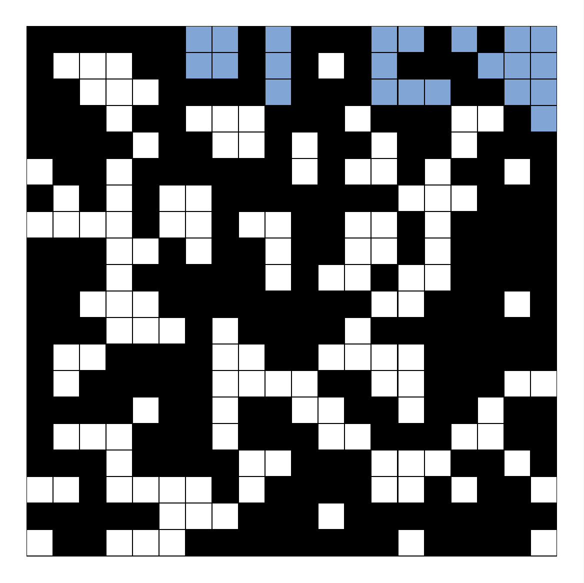
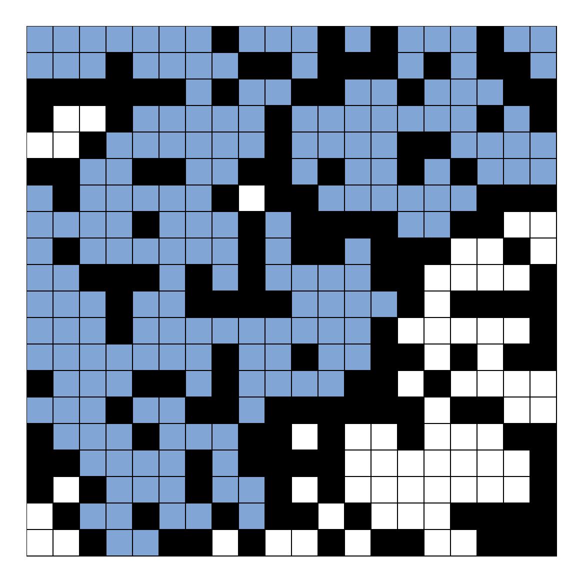
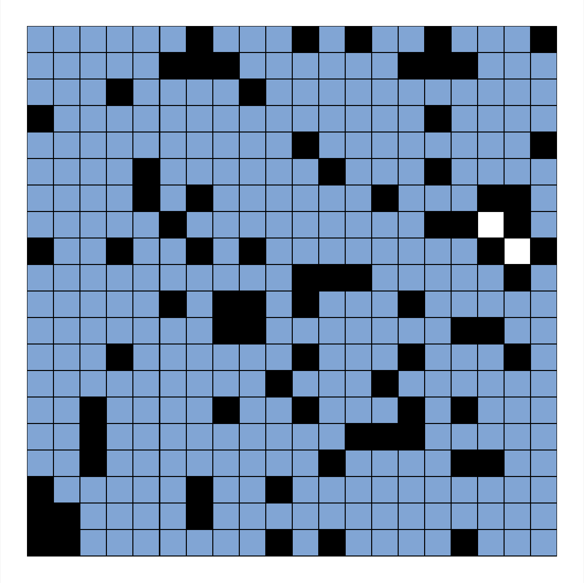
Depends on grid size N and site vacancy probability p.

p low (0.4)
does not percolate

p medium (0.6)
percolates?

p high (0.8)
percolates

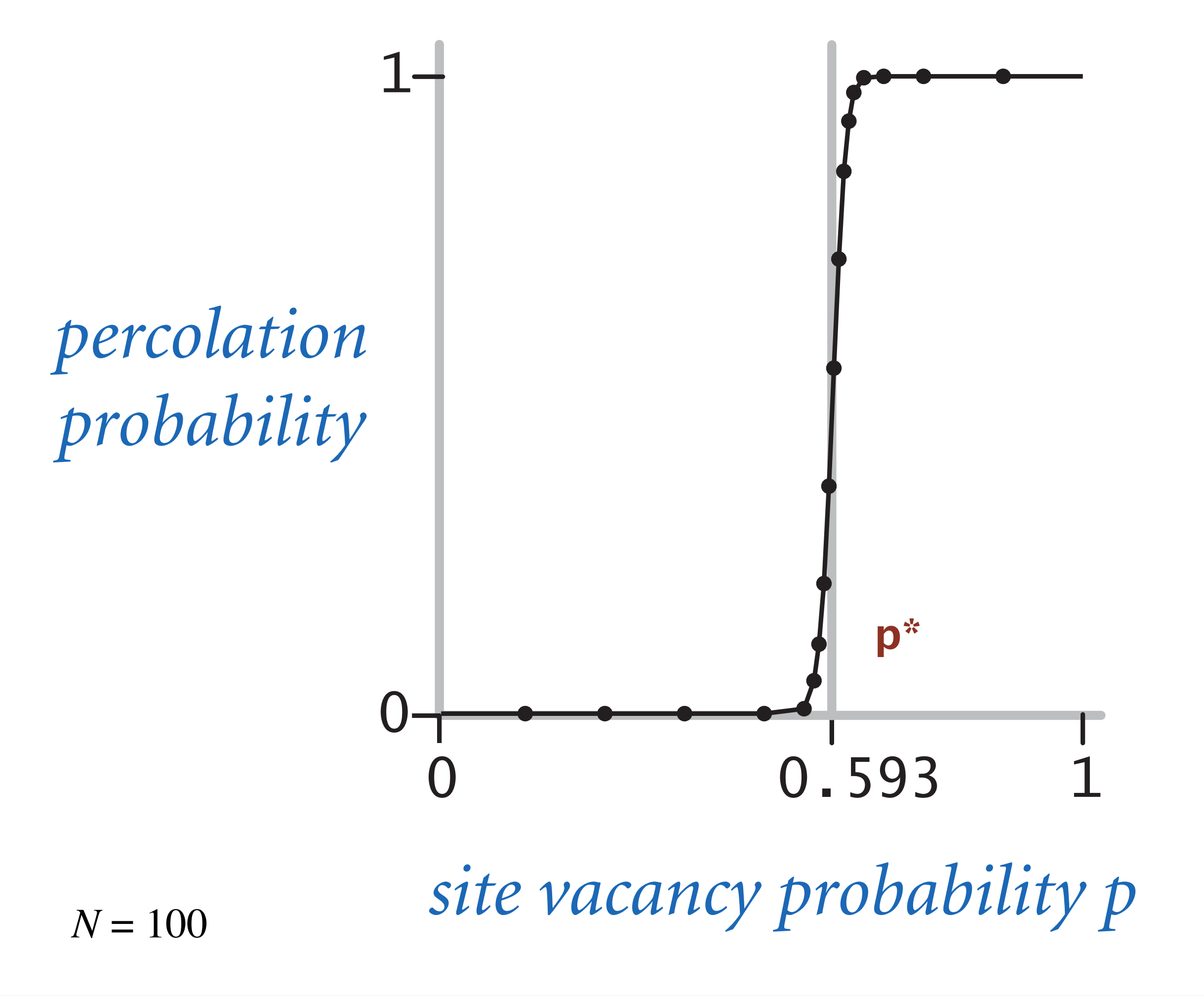




Percolation phase transition

When N is large, theory guarantees a sharp threshold p^*.

- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

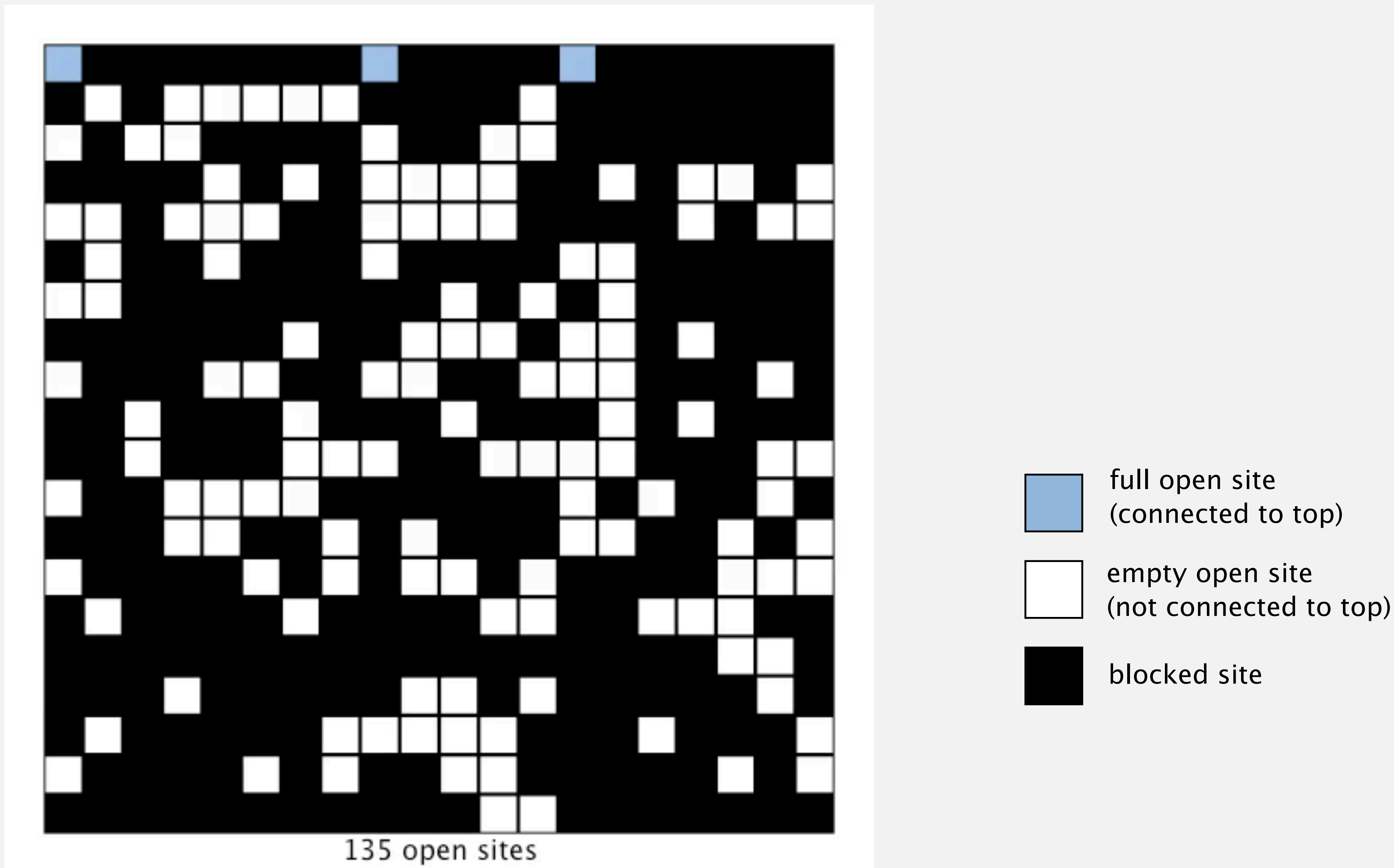
Q. What is the value of p^* ?



Monte Carlo simulation

- Initialize all sites in an N-by-N grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.

$N = 20$

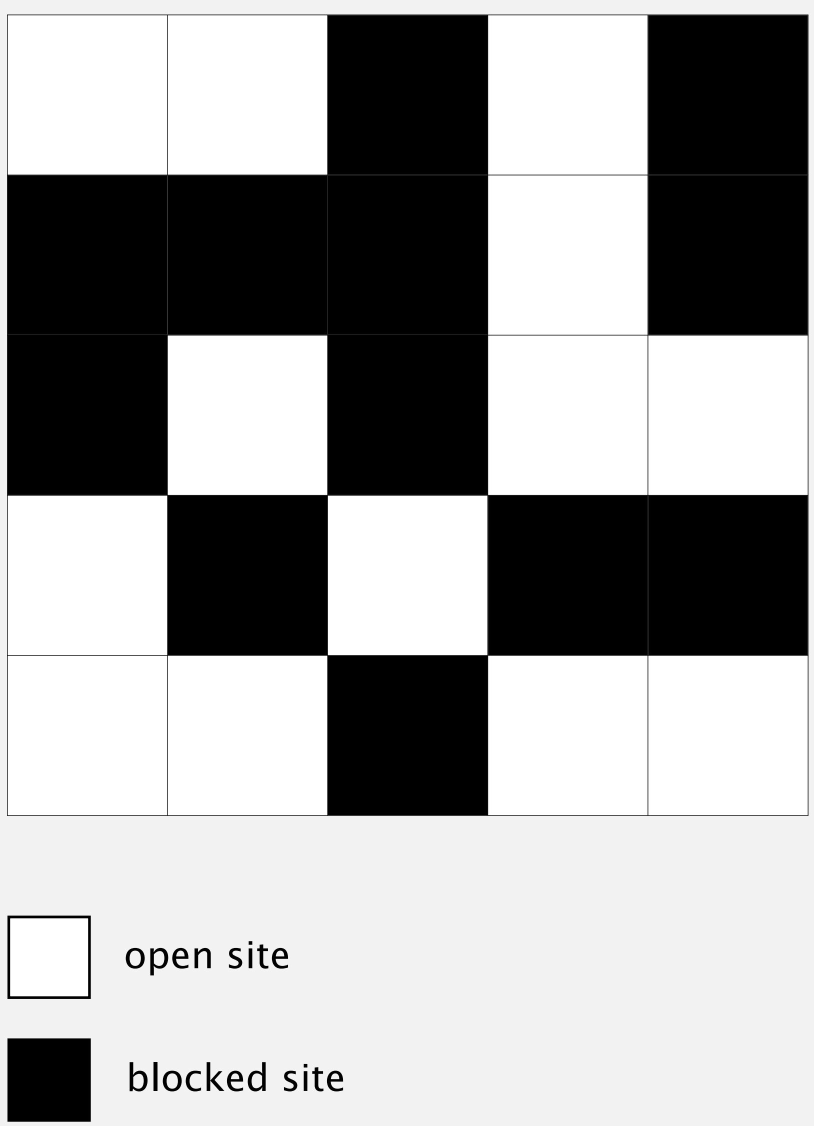


Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

A. Model as a dynamic connectivity problem and use union-find.

$N = 5$

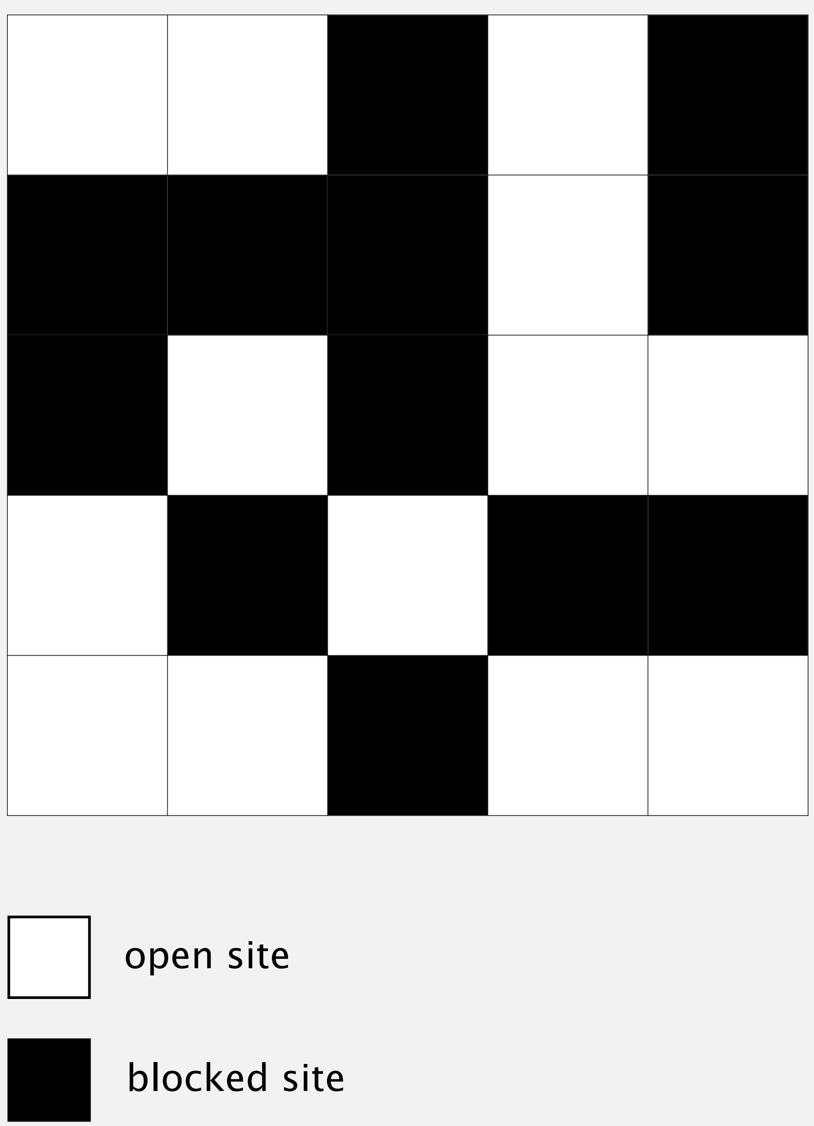


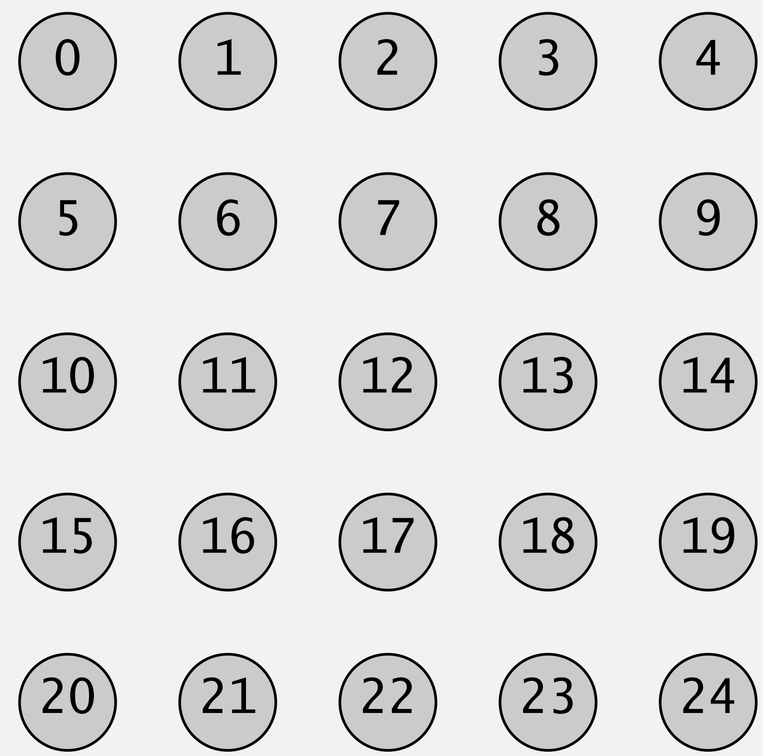
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.

$N = 5$

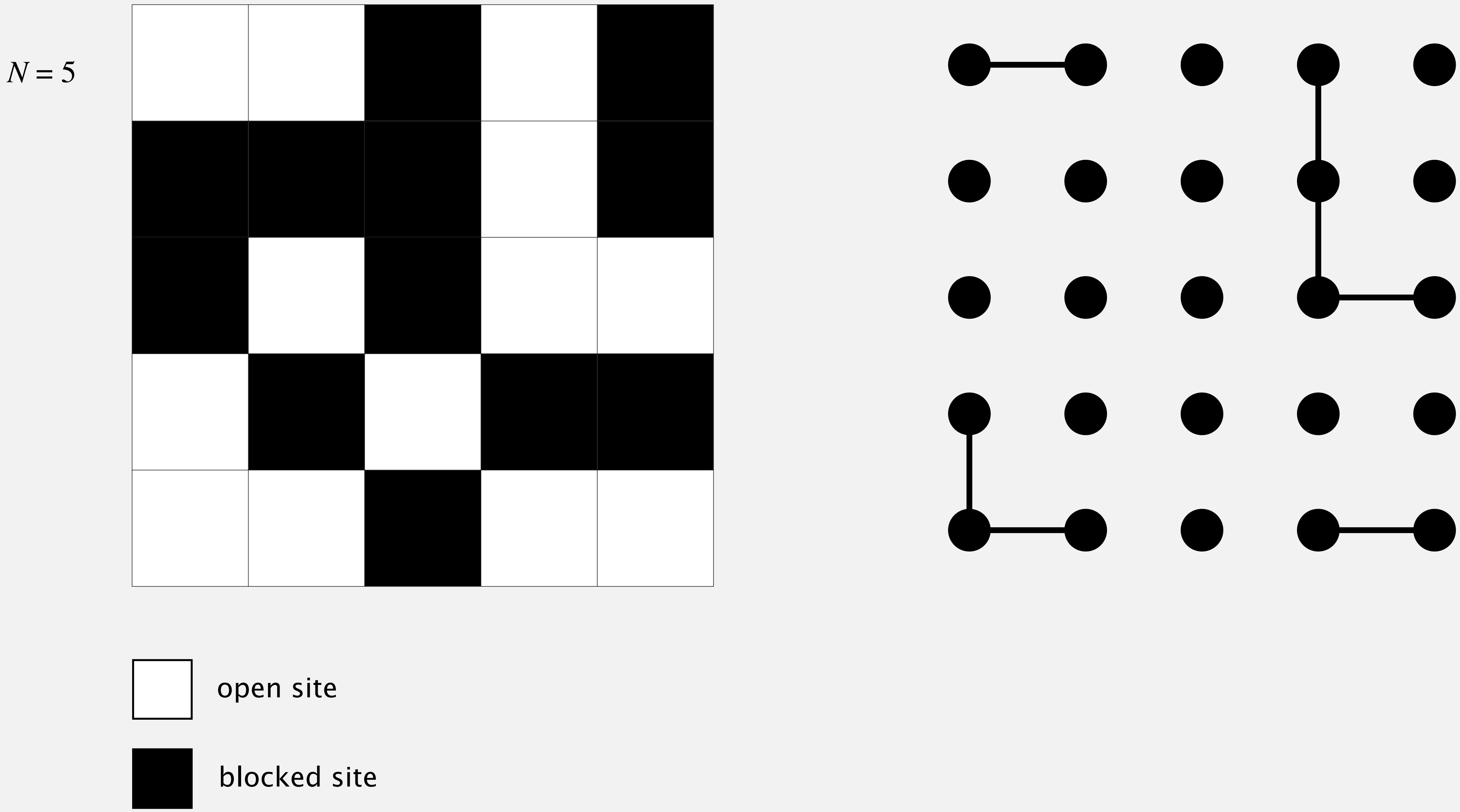




Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.



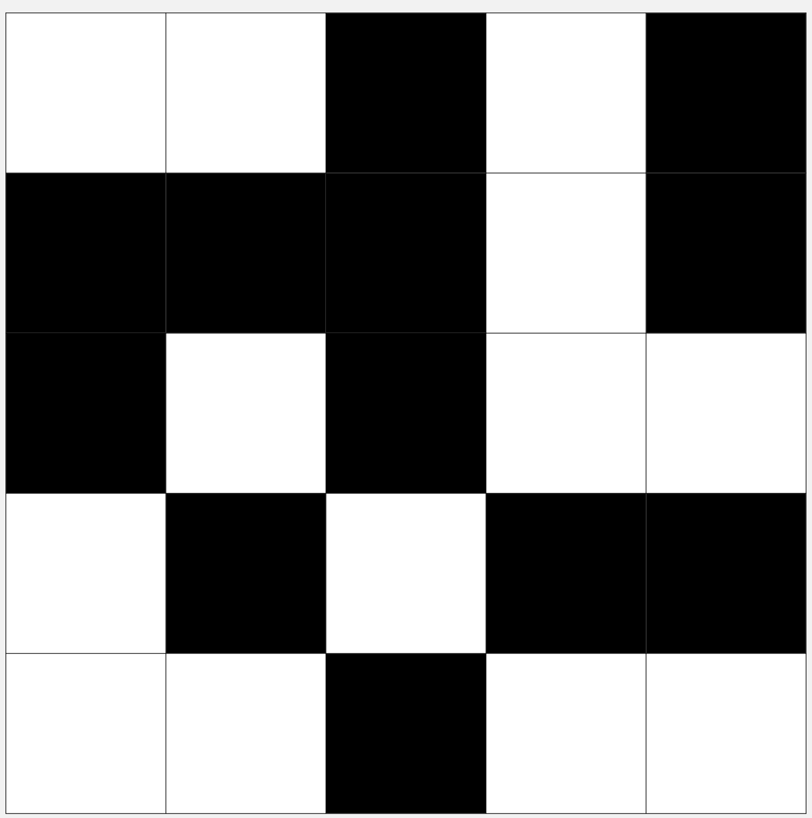
Dynamic connectivity solution to estimate percolation threshold

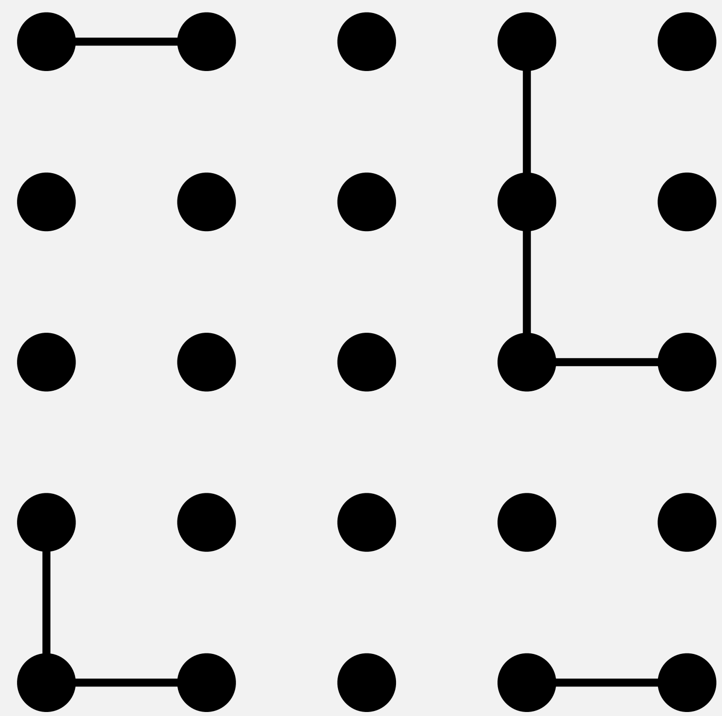
Q. How to check whether an N-by-N system percolates?

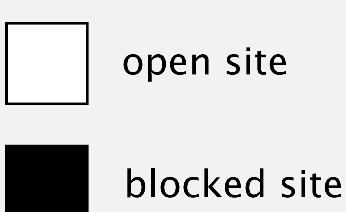
- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.
- Percolates iff any site on bottom row is connected to any site on top row.

brute-force algorithm: N^2 calls to connected()

$N = 5$







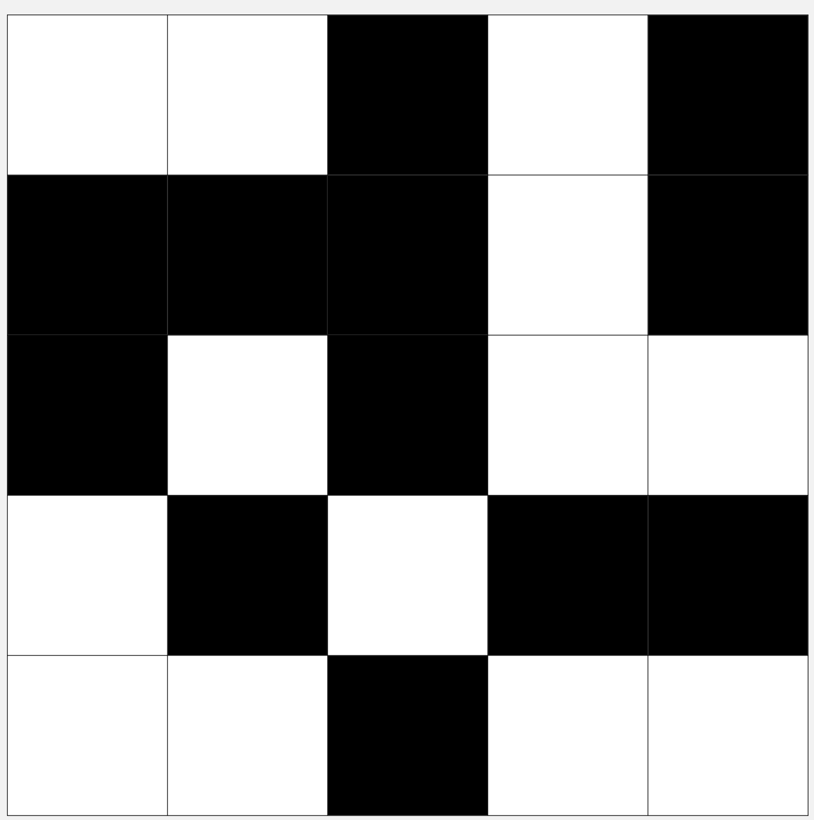
Dynamic connectivity solution to estimate percolation threshold

Clever trick. Introduce 2 virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

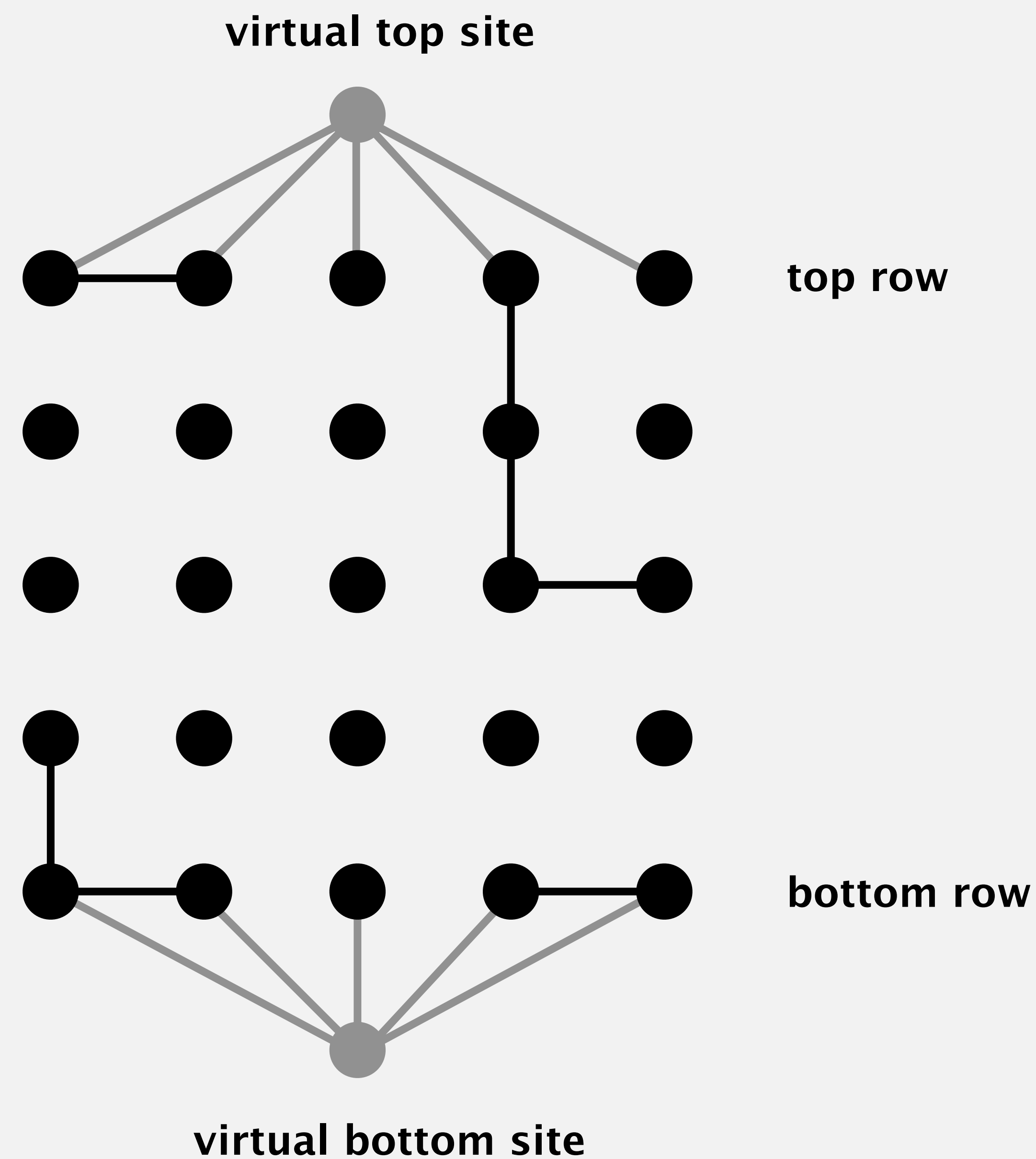
more efficient algorithm: only 1 call to connected()

$N = 5$



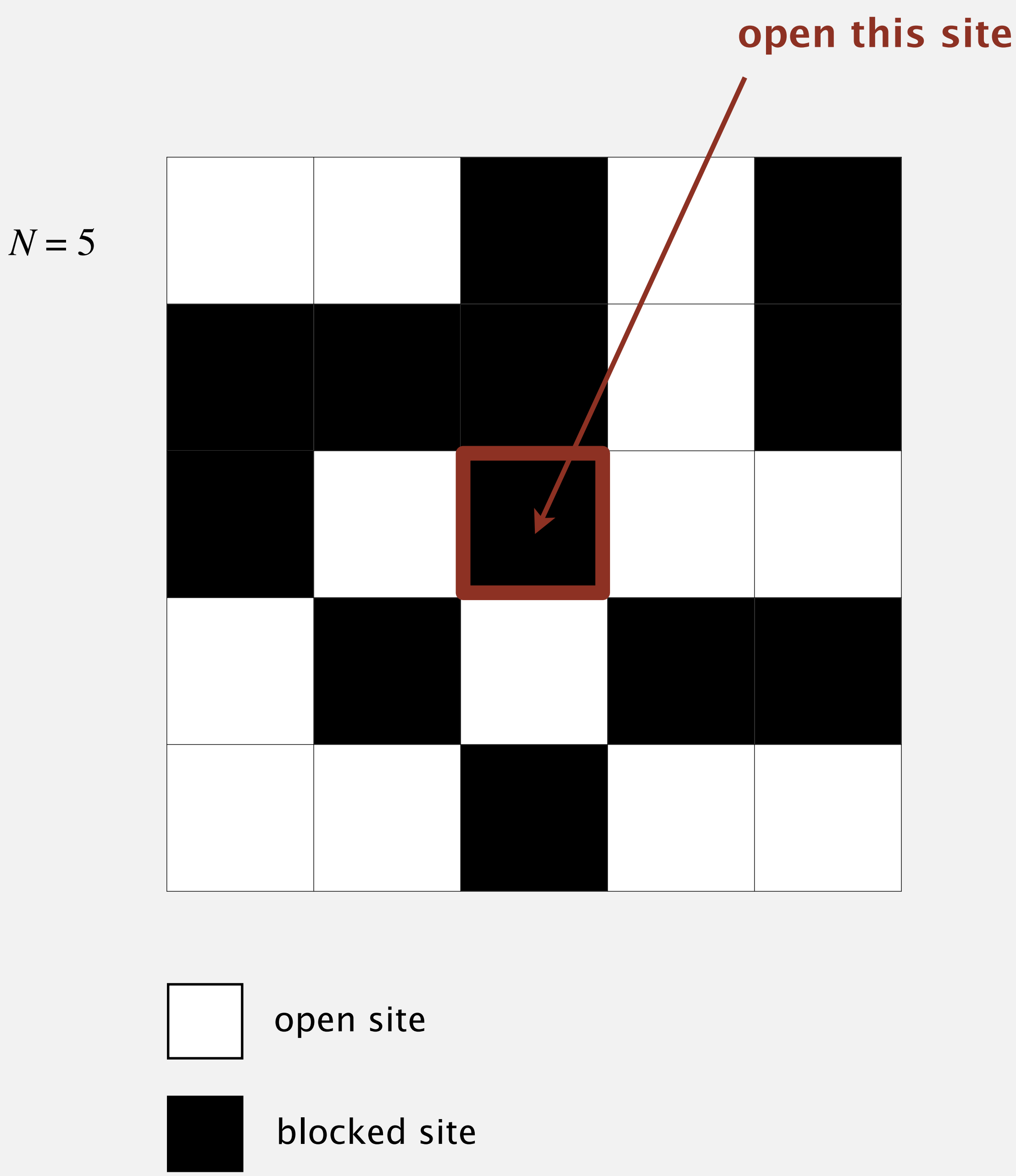
open site

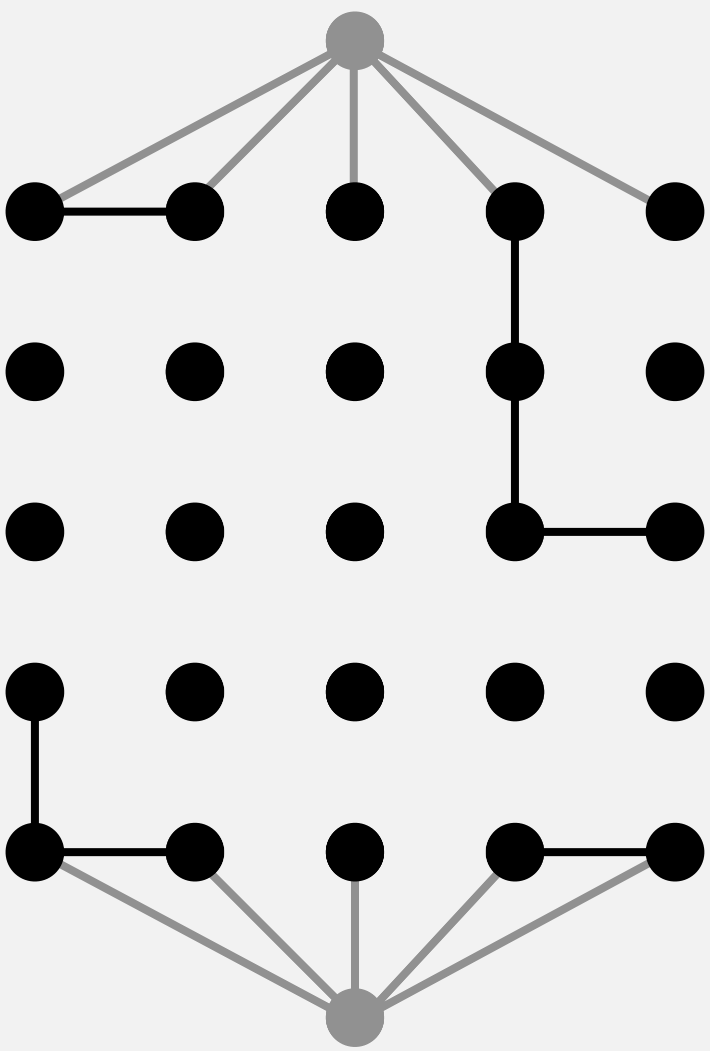
blocked site



Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?



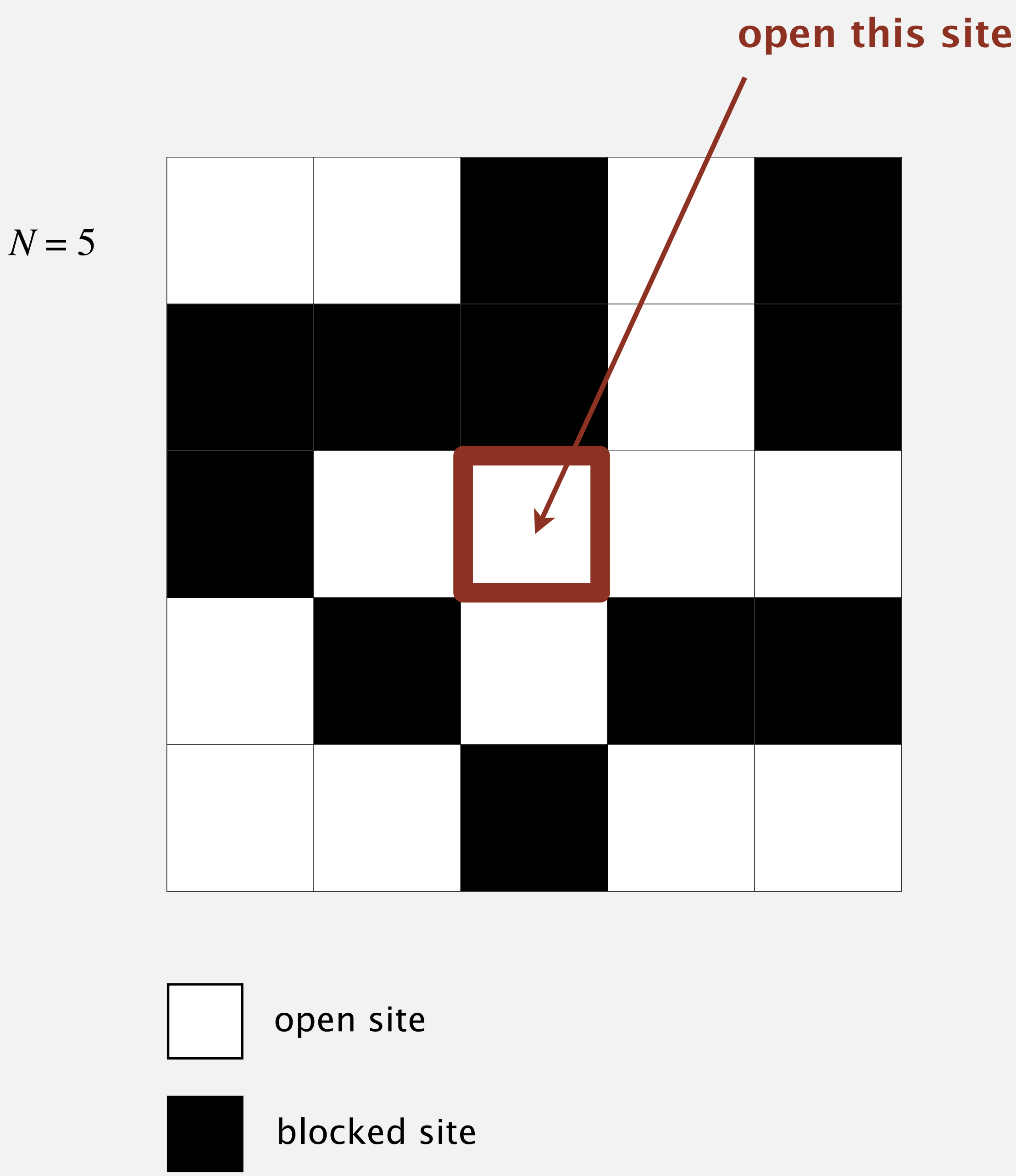


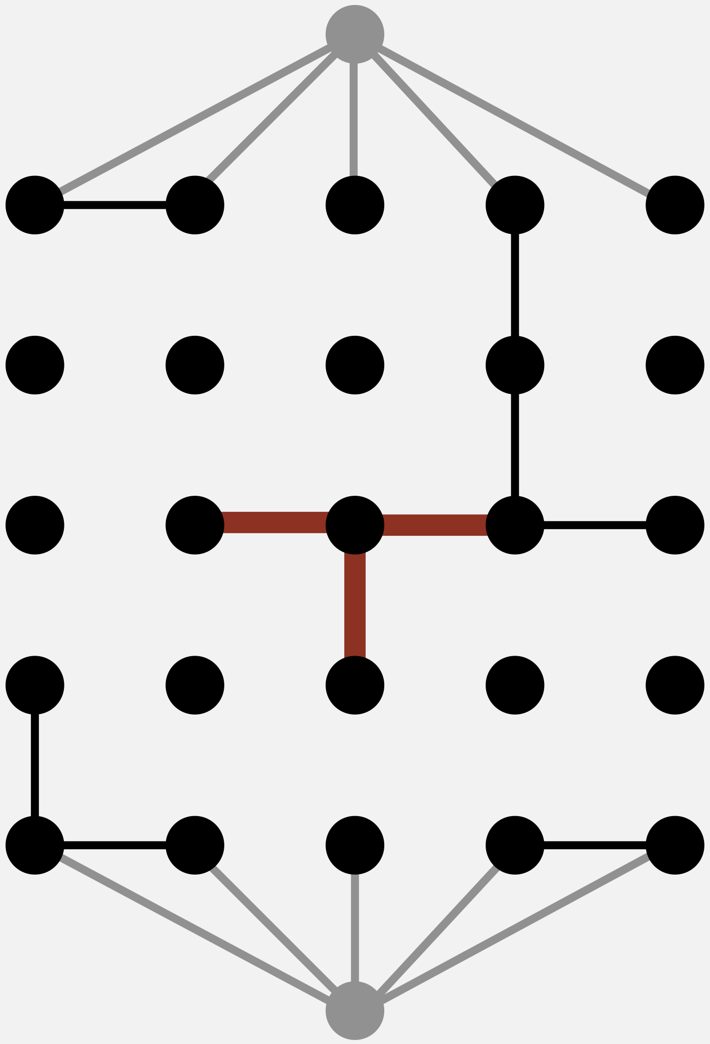
Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

A. Mark new site as open; connect it to all of its adjacent open sites.

up to 4 calls to `union()`



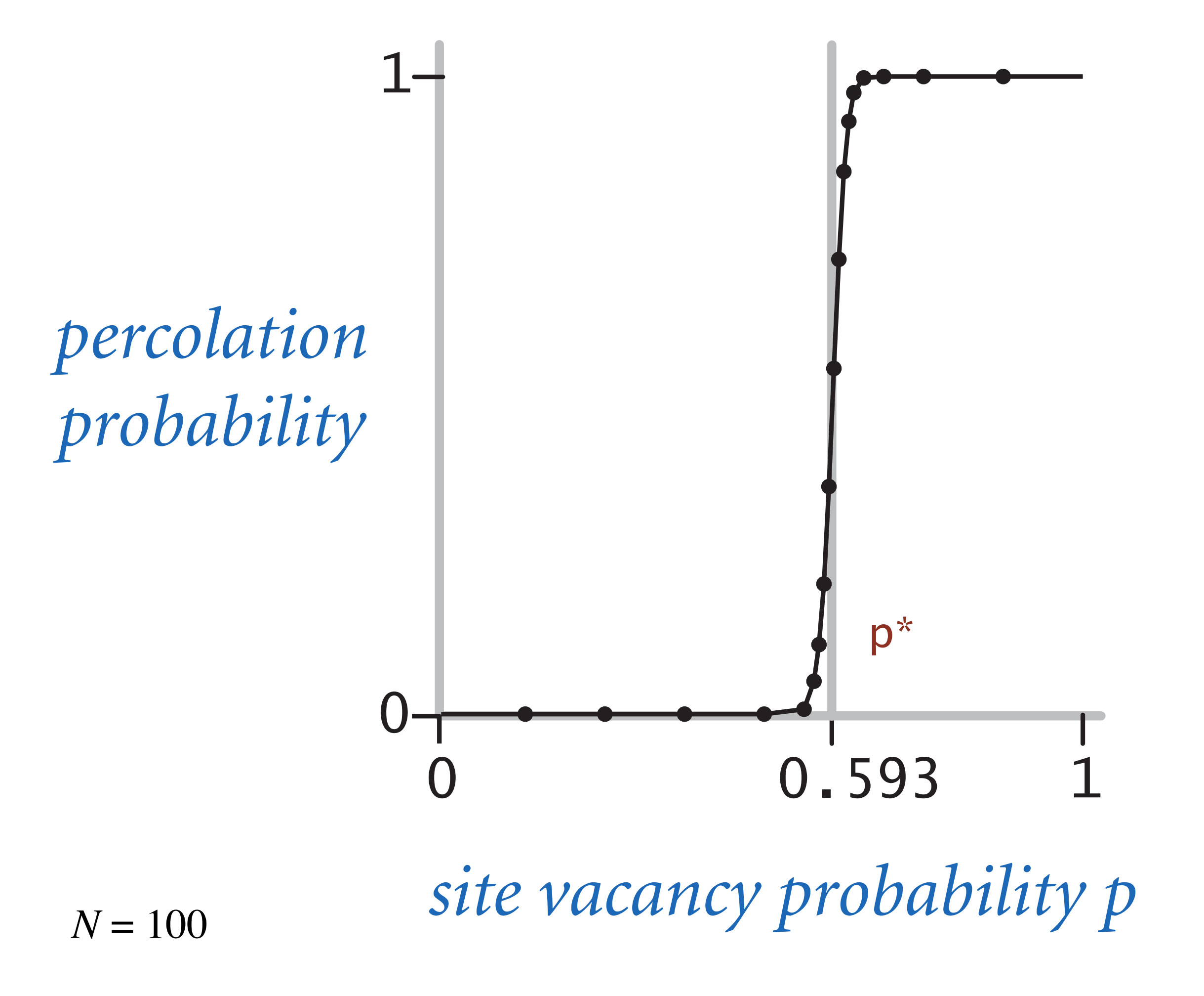


Percolation threshold

Q. What is percolation threshold p^* ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.