COS 522 Complexity — Homework 6.

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Total of 130 points

Exercise 1 (20 points)**.** Do Exercise 22.10

Exercise 2 (20 points)**.** Do Exercise 22.11

Exercise 3 (20 points)**.** The fact that all interesting problems seem to be either in **P** or **NP**-hard may lead one to conjecture that in fact this is true for every problem. Interestingly, this is not true, and Ladner proved in 1975 that if $\mathbf{P} \neq \mathbf{NP}$ then there is a language L in $\mathbf{NP} \setminus \mathbf{P}$ that is not **NP**-complete. This is Theorem 3.3 in pages 71–72 of the book. Please read the proof of this theorem, and then do Exercise 3.6, completing some details of that proof.

Given that theorem, one can wonder why indeed so many problems turned out to be either **P** or **NP**-hard. It turns out that many interesting families of problems have a *dichotomy property*, where each problem in the family is either **P** or **NP**-hard. There is a famous *dichotomy conjecture* that this is true for any constraint satisfaction problem (CSP). This was proven by Schaefer in 1978 for CSPs with binary alphabet, and (in a much more involved proof) extended by Bulatov in 2002 for CSPs of ternary (size 3) alphabet. An interesting new approach to proving the full conjecture (using Fourier, longcodes, and PCP !) was recently put forward by Rutgers professor Mario Szegedy with Kun. More progress was made on proving dichotomy conjectures for *counting* problems (where the goal is to count the number of satisfying assignments of a CSP). In particular, very recently Princeton postdoc Xi Chen, with Cai and Lu, proved a dichotomy for counting the number of homorphism into a given graph with complex edge weights. Assuming the Unique Game Conjecture, Raghavendra proved a kind of dichotomy theorem for approximating CSPs: for every CSP problem P and number $\rho < 1$, if there is no polynomial-time ρ-approximation algorithm for P, then for every constant $\epsilon > 0$, obtaining a $\rho + \epsilon$ approximation for P is unique-games hard.

Our next topic will be hardness vs randomness. The following exercises establish some facts that have found many uses in TCS, and in particular will be useful preparation for the proof of Impagliazzo's hardcore lemma, and Yao's XOR Lemma.

Exercise 4 (10 points)**.** Do Exercise 19.1 (we may have a typo there in some versions, just consider at the exercise as computing the probability that $X = 1$).

Exercise 5 (20 points)**.** Do Exercise 19.5 (hyperplane separating theorem)

Exercise 6 (20 points)**.** Do Exercise 19.6 (min-max theorem)

Exercise 7 (20 points)**.** Do Exercise 19.7 (every min-entropy k distribution is a convex combination of k-flat distributions)