Radix Sorting

Some of these lecture slides have been adapted from:

- *Algorithms in C, 3rd Edition*, Robert Sedgewick.

Radix Sorting

Radix sorting.

- Specialized sorting solution for strings.
- Same ideas for bits, digits, etc.

This lecture.

- LSD radix sort.
- MSD radix sort.
- Three-way radix quicksort.
- Suffix sorting.

An Application: Redundancy Detector

Longest repeated substring.

- Given a string of N characters, find the longest repeated substring.
- Example: **a a c a a g t t t a c a a g c**

Applications.

- Computational molecular biology.
- Data compression.
- Plagiarism detection.

Brute force.

- Try all indices i and j for start of possible match and check.
- $O(W N^2)$ time, where W is length of longest match.

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
↑									↑					
i									j					

A Sorting Solution

Suffix sort.

- Form N suffixes of original string.
- Sort to bring longest repeated substrings together.

```
a a c a a g t t t a c a a g c        a a c a a g t t t a c a a g c
a c a a g t t t a c a a g c          a a g c
c a a g t t t a c a a g c            a a g t t t a c a a g c
a a g t t t a c a a g c          ⇒   a c a a g c
a g t t t a c a a g c            ⇒   a c a a g t t t a c a a g c
g t t t a c a a g c                  a g c
t t t a c a a g c                    a g t t t a c a a g c
t t a c a a g c                      c
t a c a a g c                        c a a g c
a c a a g c                          c a a g t t t a c a a g c
c a a g c                            g c
a a g c                              g t t t a c a a g c
a g c                                t a c a a g c
g c                                  t t a c a a g c
c                                    t t t a c a a g c
```

Suffix Sorting

main()

```
char text[MAXN + 1];
Item suffixes[MAXN];

int main(void) {
   int c, N = 0;

   // read in text string
   while ((c = getchar()) != EOF)
      text[N++] = c;
   text[N] = '\0';


   // compute pointer to ith suffix
   for (i = 0; i < N; i++)
      suffixes[i] = text + i;

   // suffix sort and find longest repeated substring
   sort(suffixes, 0, N - 1);
   find(suffixes, 0, N - 1);
   return 0;
}
```

Implicitly form suffixes!

String Sorting Performance

	String Sort	Suffix (sec)
	Worst Case	Moby Dick
Brute	$W N^2$	36,000 §
Quicksort	W N log N †	694
Quicksort with cutoff	W N log N †	9.5

N = number of strings.
1.2 million for Moby Dick.
191 thousand for Aesop's Fables.

§ estimate
† probabilistic guarantee.

String Sorting

Notation.
- String = variable length sequence of characters.
- W = max # characters per string.
- N = # input strings.
- R = radix.
  - 256 for extended ASCII
  - 65,536 for UNICODE

C syntax.
- Array of strings: `char *a[];`
- The i^th string: `a[i]`
- The d^th character of the i^th string: `a[i][d]`
- Strings to be sorted: `a[lo], ..., a[hi]`

Key Indexed Counting

Key indexed counting.
- Count frequencies of each letter. (0^th character)
- Compute cumulative frequencies.
- Use cumulative frequencies to rearrange strings.

```
// rearrange
for (i = lo; i <= hi; i++) {
   c = a[i][d];
   temp[count[c]++] = a[i];
}
```

d = 0;

a

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

count

a	0	a	1
b	2	b	5
c	3	c	6
d	1	d	8
e	2	e	8
f	1	f	12
g	3	g	11

temp

0	a	d	d
1			
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8			
9	f	a	d
10	f	e	e
11	f	e	d

Key Indexed Counting

Key indexed counting.

- Count frequencies of each letter. (0th character)
- Compute cumulative frequencies.
- ➡ Use cumulative frequencies to rearrange strings.

```
// rearrange
for (i = lo; i <= hi; i++) {
   c = a[i][d];
   temp[count[c]++] = a[i];
}
```

d = 0;

a

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
➡ 10	e	b	b
11	a	c	e

count

a	0	a	1
b	2	b	5
c	3	c	6
d	1	d	8
e	2	e	9
f	1	f	12
g	3	g	11

temp

0	a	d	d
1			
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

Key Indexed Counting

Key indexed counting.

- Count frequencies of each letter. (0th character)
- Compute cumulative frequencies.
- ➡ Use cumulative frequencies to rearrange strings.

```
// rearrange
for (i = lo; i <= hi; i++) {
   c = a[i][d];
   temp[count[c]++] = a[i];
}
```

d = 0;

a

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
➡ 11	a	c	e

count

a	0	a	2
b	2	b	5
c	3	c	6
d	1	d	8
e	2	e	9
f	1	f	12
g	3	g	11

temp

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

Key Indexed Counting

Key indexed counting.

- Count frequencies of each letter. (0th character)
- Compute cumulative frequencies.
- ➡ Use cumulative frequencies to rearrange strings.

```
// copy back
for (i = lo; i <= hi; i++)
   a[i] = temp[i - lo];
```

a

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

count

a	0	a	2
b	2	b	5
c	3	c	6
d	1	d	8
e	2	e	9
f	1	f	12
g	3	g	11

temp

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

LSD Radix Sort

Least significant digit radix sort.

- Ancient method used for card-sorting.
- Consider digits from right to left:
  - use key-indexed counting to STABLE sort by character

0	d	a	b	0	d	a	b	0	d	a	b	0	a	c	e
1	a	d	d	1	c	a	b	1	c	a	b	1	a	d	d
2	c	a	b	2	e	b	b	2	f	a	d	2	b	a	d
3	f	a	d	3	a	d	d	3	b	a	d	3	b	e	d
4	f	e	e	4	f	a	d	4	d	a	d	4	b	e	e
5	b	a	d	5	b	a	d	5	e	b	b	5	c	a	b
6	d	a	d	6	d	a	d	6	a	c	e	6	d	a	b
7	b	e	e	7	f	e	d	7	a	d	d	7	d	a	d
8	f	e	d	8	b	e	d	8	f	e	d	8	e	b	b
9	b	e	d	9	f	e	e	9	b	e	d	9	f	a	d
10	e	b	b	10	b	e	e	10	f	e	e	10	f	e	d
11	a	c	e	11	a	c	e	11	b	e	e	11	f	e	e

LSD Radix Sort

Least significant digit radix sort.

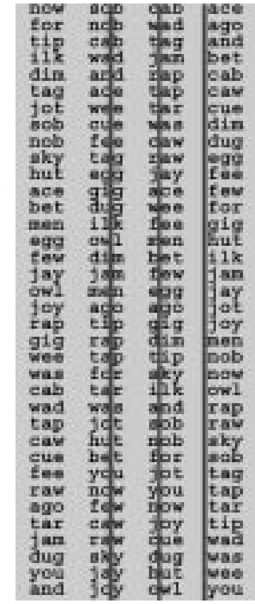
- Ancient method used for card-sorting.
- Consider digits from right to left:
  - use key-indexed counting to STABLE sort by character

LSD Radix Sort, Sedgewick Program 10.4

```
void lsd(char *a[], int lo, int hi) {
   int d;
   for (d = W-1; d >= 0; d--)
      keyindex(a, lo, hi, d);
}
```

Fixed length strings (length = W)

LSD Radix Sort Correctness



Two proofs of correctness.

- Left-to-right.
  - if two strings differ on first character, key-indexed sort puts them in proper relative order
  - if two strings agree on first character, stability keeps them in proper relative order
- Right-to-left.
  - if the characters not yet examined differ, it doesn't matter what we do now
  - if the characters not yet examined agree, later pass won't affect order

LSD Radix Sort Correctness

Running time.

- O(W(N + R)).
- Why doesn't it violate N log N lower bound?

Advantage.

- Fastest sorting method for random fixed length strings.

Disadvantages.

- Doesn't work for variable-length strings.
- Not much semblance of order until very last pass.
- Inner loop has a lot of instructions.
- Accesses memory "randomly."
- Wastes time on low-order characters.

Goal: find fast algorithm for variable length strings.

MSD Radix Sort

Most significant digit radix sort.

- Partition file into 256 pieces according to first character.
- Recursively sort all strings that start with the same character, etc.

How to sort on d^{th} character?

- Use key-indexed counting.

MSD Radix Sort Implementation

MSD Radix Sort, Sedgewick Program 10.2

```
void msdR(char *a[], int lo, int hi, int d) {
   int i;
   int count[256+1] = {0};

   if (hi <= lo) return;

   keyindex(a, lo, hi, d, count);

   // recursively sort 255 pieces
   for (i = 0; i < 255; i++)
      msdR(a, lo + count[i], lo + count[i+1] - 1, d + 1);
}

void msd(char *a[], int lo, int hi) {
   msdR(a, lo, hi, 0);
}
```

(assumes '\0'-terminated strings)

String Sorting Performance

	String Sort	Suffix (sec)
	Worst Case	Moby Dick
Brute	W N²	36,000 §
Quicksort	W N log N †	694
Quicksort with cutoff	W N log N †	9.5
LSD *	W(N + R)	-
MSD	W(N + R)	395
MSD with cutoff	W(N + R)	6.8

R = radix.
W = max length of string.
N = number of strings.

§ estimate
* assumes fixed length strings.
† probabilistic guarantee.

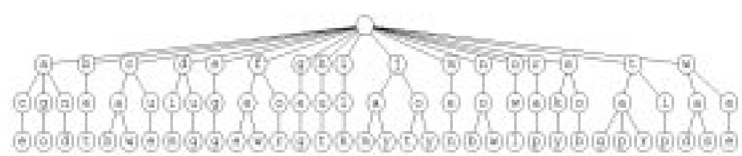
MSD Radix Sort Analysis

Disadvantages.
- Too slow for small files.
  - ASCII: 100x slower than insertion sort for N = 2
  - UNICODE: 30,000x slower for N = 2
- Huge number of recursive calls on small files.

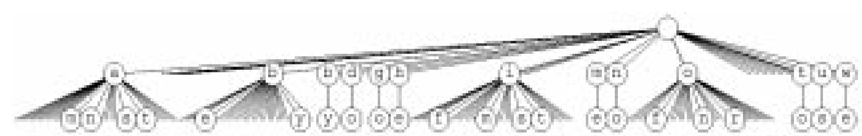
Solution: cutoff to insertion sort for small N.
- Competitive with quicksort for string keys.

Recursive Structure of MSD Radix Sort

Tree structure to describe recursive calls in MSD radix sort.



Problem: algorithm touches lots of empty nodes.
- Tree can be as much as 256 times bigger than it appears!



3-Way Radix Quicksort

Idea 1. Use dth character to "sort" into 3 pieces instead of 256!
Idea 2. Keep all duplicates together in partitioning step.
Sort each piece recursively.

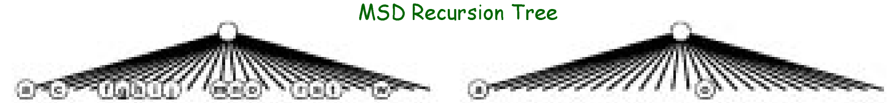
Partition


Algorithm

Recursive Structure of MSD Radix Sort vs. 3-Way Quicksort

3-way radix quicksort collapses NULL links in MSD tree.

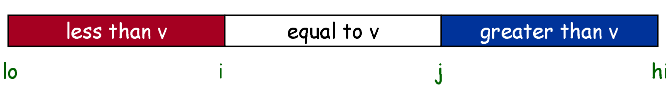
MSD Recursion Tree



3-Way Radix Quicksort Recursion Tree

3-Way Partitioning

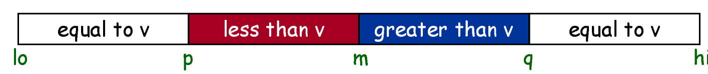
3-way partitioning.
- Natural way to deal with equal keys.
- Partition elements into 3 parts:
  - elements between i and j equal to partition element v
  - no larger elements to left of i
  - no smaller elements to right of j



Dutch national flag problem.
- Not easy to implement efficiently. (Try it!)
- Not done in practical sorts before mid-1990s.

3-Way Partitioning

Elegant solution to Dutch national flag problem.
- Partition elements into 4 parts:
  - no larger elements to left of m
  - no smaller elements to right of m
  - equal elements to left of p
  - equal elements to right of q



- Afterwards, swap equal keys into center.

All the right properties.
- Not much code.
- In-place.
- Linear if keys are all equal.
- Small overhead if no equal keys.

3-Way Radix Quicksort, Sedgewick Program 10.3

```
void quicksort3(Item a[], int lo, int hi, int d) {
    int i = lo-1, j = hi, k, p = lo-1, q = hi, v = a[hi][d];
    if (hi <= lo) return;
    while (i < j) {
        while (a[++i][d] < v)
            ;
        while (v < a[--j][d])
           if (j == lo) break;
        if (i > j) break;
        exch(a[i], a[j]);
        if (a[i][d] == v) { p++; exch(a[p], a[i]); }
        if (v == a[j][d]) { q--; exch(a[j], a[q]); }
    }
    if (p == q) {
        if (v != '\0') quicksort3(a, lo, hi, d+1);
        return;
    }
    if (a[i][d] < v) i++;
    for (k = lo; k <= p; k++, j--) exch(a[k], a[j]);
    for (k = hi; k >= q; k--, i++) exch(a[k], a[i]);
    quicksort3(a, lo, j, d);
    if ((i == hi) && (a[i][d] == v)) i++;
    if (v != '\0') quicksort3(a, j+1, i-1, d+1);
    quicksort3(a, i, hi, d);
}
```

swap equal keys to left or right

swap equal keys back to middle

Significance of 3-Way Partitioning

Equal keys omnipresent in applications when purpose of sort is to bring records with equal keys together.

- Sort population by age.
- Sort job applicants by college attended.
- Remove duplicates from mailing list.
- Line recognition problem.

Typical application.

- Huge file.
- Small number of key values.
- Randomized 3-way quicksort is LINEAR time. (Try it!)

Theorem. Quicksort with 3-way partitioning is OPTIMAL.
Proof. Ties cost to entropy. Beyond scope of 226.

Quicksort vs. 3-Way Radix Quicksort

Quicksort.

- 2N ln N STRING comparisons on average.
- Long keys are costly to compare if they differ only at the end, and this is common case!
  - absolutism
  - absolut
  - absolutely
  - absolute

3-way radix quicksort.

- Avoids re-comparing initial parts of the string.
- Uses just "enough" characters to resolve order.
- 2 N ln N CHARACTER comparisons on average.
  - independent of word length W for random strings
- Sublinear sort for large W since input is of size NW.

String Sorting Performance

	String Sort	Suffix (sec)
	Worst Case	Moby Dick
Brute	$W N^2$	36,000 §
Quicksort	W N log N †	694
Quicksort with cutoff	W N log N †	9.5
LSD *	W(N + R)	-
MSD	W(N + R)	395
MSD with cutoff	W(N + R)	6.8
3-Way Radix Qsort	W N log N †	2.8

R = radix.
W = max length of string.
N = number of strings.

§ estimate
* assumes fixed length strings.
† probabilistic guarantee.

Suffix Sorting: Worst Case Input

Length of longest match small.

- 3-way radix quicksort rules!

Length of longest match very long.

- 3-way radix quicksort is quadratic.
- Two copies of Moby Dick.

Can we do better?

- N log N ?
- Linear time ?

Observation. Must find longest repeated substring WHILE suffix sorting to beat quadratic worst case.

```
abcdefghi
abcdefghiabcdefghi
bcdefghi
bcdefghiabcdefghi
cdefghi
cdefghiabcdefgh
defghi
efghiabcdefghi
efghi
fghiabcdefghi
fghi
ghiabcdefghi
fhi
hiabcdefghi
hi
iabcdefghi
i
```

Input: "abcdeghiabcdefghi"

Suffix Sorting in N log N Time: Key Idea

```
 0 babaaaabcbabaaaaa0        17 0babaaaabcbabaaaaa
 1 abaaaabcbabaaaaa0b        16 a0babaaaabcbabaaaa
 2 baaaabcbabaaaaa0ba        15 aa0babaaaabcbabaaa
 3 aaaabcbabaaaaa0bab        14 aaa0babaaaabcbabaa
 4 aaabcbabaaaaa0baba         3 aaaabcbabaaaaa0bab
 5 aabcbabaaaaa0babaa        12 aaaaa0babaaaabcbab
 6 abcbabaaaaa0babaaa     ⇒  13 aaaa0babaaaabcbaba
 7 bcbabaaaaa0babaaaa     ⇒   4 aaabcbabaaaaa0baba
 8 cbabaaaaa0babaaaab         5 aabcbabaaaaa0babaa
 9 babaaaaa0babaaaabc         1 abaaaabcbabaaaaa0b
10 abaaaaa0babaaaabcb        10 abaaaaa0babaaaabcb
11 baaaaa0babaaaabcba         6 abcbabaaaaa0babaaa
12 aaaaa0babaaaabcbab         2 baaaabcbabaaaaa0ba
13 aaaa0babaaaabcbaba        11 baaaaa0babaaaabcba
14 aaa0babaaaabcbabaa         0 babaaaabcbabaaaaa0
15 aa0babaaaabcbabaaa         9 babaaaaa0babaaaabc
16 a0babaaaabcbabaaaa         7 bcbabaaaaa0babaaaa
17 0babaaaabcbabaaaaa         8 cbabaaaaa0babaaaab
```

Input: "babaaaabcbabaaaaa"

Suffix Sorting in N log N Time

Manber's MSD algorithm.

- Phase 0.
  - sort on first character using key-indexed sorting
- Phase n.
  - given list of suffixes sorted on first n characters, create list of suffixes sorted on first 2n characters
- Finishes after lg N phases.

Manber's LSD algorithm.

- Same idea but go from right to left.
- O(N log N) guaranteed running time.
- O(N) extra space.

String Sorting Performance

	String Sort	Suffix Sort (seconds)	
	Worst Case	Moby Dick	AesopAesop
Brute	$W N^2$	36,000 §	3,990 §
Quicksort	W N log N †	694	320
Quicksort with cutoff	W N log N †	9.5	167
LSD *	W(N + R)	-	-
MSD	W(N + R)	395	crash (!)
MSD with cutoff	W(N + R)	6.8	162
3-Way Radix Qsort	W N log N †	2.8	400
Manber LSD	N log N ‡	17	8.5

R = radix.
W = max length of string.
N = number of strings.

§ estimate
* assumes fixed length strings.
† probabilistic guarantee.
‡ suffix sorting only.