COS 341
January 7, 2000

Solutions to Final Exam

Problem 1

$$\begin{aligned}\sum_{n\geq 2} b_n x^n &= \sum_{n\geq 2} b_{n-1}x^n + \sum_{n\geq 2} x^n \sum_{1\leq k\leq n-1} b_k b_{n-1-k} \\ &= x\sum_{n\geq 2} b_{n-1}x^{n-1} + x\sum_{m\geq 1} x^m \sum_{1\leq k\leq m} b_k b_{m-k}.\end{aligned}$$

This implies

$$B(x) - b_0 - b_1 x = x(B(x) - b_0) + x(b_1 x + b_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots),$$

ie,

$$B(x) - 1 - 2x = x(B(x) - 1) + x(B(x) - 1)B(x).$$

This leads to $xB(x)^2 - B(x) + (1 + x) = 0$, and hence using $B(0) = b_0 = 1$ we have

$$B(x) = \frac{1 - \sqrt{1 - 4x(1+x)}}{2x}.$$

Problem 2

(a) For $n = 1$, G_n consists of two isolated vertices and is thus by definition Eulerian. For $n > 1$, G_n is Eulerian since (A) it is connected (vertex 1 is connected to vertex $n+1$ through $1 - 2 - (n+1)$, and vertex 1 has an edge to each of the remaining vertices) and (B) every vertex has even degree (in fact $2n - 2$).

(b) For $n = 1$, G_n consists of two isolated vertices and has no Hamiltonian circuit. For $n > 1$, G_n has the following Hamiltonian circuit $1, 2, 3, \cdots, n-1, n, n+1, n+2, \cdots, 2n, 1$.

(c) The answer is $\omega(G_n) = n$. Note that $\omega(G_n) \geq n$ since $\{1, 2, \cdots, n\}$ is a clique; $\omega(G_n) < n + 1$ since any clique can contain at most one of the vertices $i, n+i$ for each $1 \leq i \leq n$.

(d) The answer is $\chi(G_n) = n$. Note that $\chi(G_n) \geq n$ since $\{1, 2, \cdots, n\}$ is a clique and thus each vertex in it has to be painted with a different color; $\chi(G_n) \leq n$ since we can just paint both vertices $i, n+i$ with color i, for each $1 \leq i \leq n$.

Problem 3 Let $E_0 = \{\{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}, \{4n+1, 4n\}\}$, and

$$E_1 = \{\{4n, 1\}, \{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}\},$$

$$
\begin{aligned}
E_2 &= \{\{n, n+1\}, \{n+1, n+2\}, \{n+2, n+3\}, \cdots, \{2n-1, 2n\}\}, \\
E_3 &= \{\{2n, 2n+1\}, \{2n+1, 2n+2\}, \{2n+2, 2n+3\}, \cdots, \{3n-1, 3n\}\}, \\
E_4 &= \{\{3n, 3n+1\}, \{3n+1, 3n+2\}, \{3n+2, 3n+3\}, \cdots, \{4n-1, 4n\}\}.
\end{aligned}
$$

Then $E = \cup_{0 \le i \le 4} E_i$.

A spanning tree of H_n has $4n$ edges, and can be specified by the 4 edges missing from E. For $\alpha \in \{0, 1, 2, 3, 4\}$, let $s_{n,\alpha}$ be the number of spanning trees of H_n for which α of the missing edges are from E_0. Then

$$
s_n = \sum_{0 \le \alpha \le 4} s_{n,\alpha}.
$$

Clearly, $s_{n,4} = 0$ since at least one edge from E_0 is needed to keep vertex $4n+1$ from being isolated.

To calculate $s_{n,3}$, we count first how many spanning trees there are that contain $\{4n+1, n\}$ but no other edge from E_0. A spanning tree is now specified by the one missing edge from $\cup_{1 \le i \le 4} E_i$, so that number is $| \cup_{1 \le i \le 4} E_i| = 4n$. We can prove the same result if we count the number of spanning trees that contain any one specific edge but no other edges in E_0. Thus,

$$
s_{n,3} = 4 \cdot 4n = 16n.
$$

To calculate $s_{n,2}$, let a_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 2n\}$ but no other edges in E_0; let b_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 3n\}$ but no other edges in E_0. Clearly,

$$
s_{n,2} = 4a_n + 2b_n.
$$

We compute a_n. A spanning tree of this type is specified by a missing edge chosen from E_2, and a missing edge from $E_1 \cup E_3 \cup E_4$. Thus,

$$
a_n = |E_2| \cdot |E_1 \cup E_3 \cup E_4| = 3n^2.
$$

Similarly,

$$
b_n = |E_2 \cup E_3| \cdot |E_1 \cup E_4| = 4n^2.
$$

This leads to

$$
s_{n,2} = 4 \cdot 3n^2 + 2 \cdot 4n^2 = 20n^2.
$$

To calculate $s_{n,1}$, let c_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}$ but no other edges in E_0. Then $s_{n,1} = 4c_n$. To compute c_n, note that such a spanning tree is specified by a missing edge from each of the sets $E_2, E_3, E_4 \cup E_1$. Thus, $c_n = |E_2| \cdot |E_3| \cdot |E_4 \cup E_1| = 2n^3$. Hence,

$$s_{n,1} = 4 \cdot 2n^3 = 8n^3.$$

To calculate $s_{n,0}$, note that such a spanning tree is specified by a missing edge from each of the sets E_1, E_2, E_3, E_4. Thus,

$$s_{n,0} = |E_1| \cdot |E_2| \cdot |E_3| \cdot |E_4| = n^4.$$

Putting everything together, we have

$$s_n = \sum_{0 \leq \alpha \leq 4} s_{n,\alpha} = n^4 + 8n^3 + 20n^2 + 16n.$$