COS 341

January 5, 2000

Final Exam

This is an open book take-home exam. Hand in your solutions by 3:00pm January 7, 2000, to Ms. Sandy Barbu, Room 323, Computer Science Building. You may use any consecutive 24-hour period between now and January 7 to work on the problems. Check periodically the course web page under "whats-new" for possible extra new information about the exam.

Do all three problems. In the following, i, j, k, n take on integer values only.

Problem 1 [20 points] Let $b_0, b_1, b_2, \cdots$ be the sequence defined by the following recurrence relation:

$$\begin{aligned} b_0 &= 1, \\ b_1 &= 2, \\ b_n &= b_{n-1} + \sum_{1 \le k \le n-1} b_k b_{n-1-k} \quad \text{for } n \ge 2. \end{aligned}$$

Let $B(x) = \sum_{k \ge 0} b_k x^k$. Derive a closed-form formula for $B(x)$.

Problem 2 [20 points] For any integer $n > 0$, let $G_n = (V, E)$ be a graph on $2n$ vertices, where $V = \{1, 2, 3, \cdots, 2n\}$, and $E = \{\{i, j\} \mid 1 \le i < j \le 2n, j \ne n + i\}$. Answer the following questions, each with a concise but rigorous justification.

(a) For what values of n are G_n Eulerian?
(b) For what values of n do G_n contain a Hamiltonian circuit?
(c) What is $\omega(G_n)$, the size of the largest clique in G_n?
(d) What is $\chi(G_n)$, the chromatic number of G_n?

Remarks In other words, G_n is obtained from the complete graph on $2n$ vertices by deleting n edges (no two of which have any endpoints in common). Thus, G_n has exactly $\binom{2n}{2} - n$ edges.

Problem 3 [20 points] For any integer $n > 0$, let $H_n = (V, E)$ be a graph on $4n + 1$ vertices, where $V = \{1, 2, 3, \cdots, 4n, 4n + 1\}$, and

$$E = \{\{i, i+1\} \mid 1 \le i \le 4n-1\} \cup \{\{1, 4n\}, \{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}, \{4n+1, 4n\}\}.$$

Thus, H_n has exactly $4n + 4$ edges. Let s_n be the number of spanning trees for H_n. Determine s_n as a closed-form expression of n.