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Options

It is now September. You are the head of Okiahn, a major shipping company, and you have just been contracted to ship a freight-load of Christmas trees from Seattle to Hong Kong in December (3 months in the future) for \$2,000,000. Most of your costs such as labor and the costs of the ship are fixed at \$1,500,000, since they have been already set by contract or purchased. The cost of 20,000 barrels of oil that you need to fuel your ship, however, is variable, since you have not purchased it yet. You can either follow:

Plan 1 Buy all of your oil now at \$10 a barrel. Storage fees are negligible. In this case you will have zero uncertainty about the cost of oil in December.

Plan 2 Buy all of your oil in December, hoping the price of oil will not rise too much and will be lower in December than it is now. In this case you will have high uncertainty about the cost of oil.

Plan 3 Purchase a single call option in September to protect yourself in case the price of oil becomes unusually high. You will have to pay for the option, but some of the uncertainty about the cost of the oil will be reduced.

A *call option* for oil is the right to purchase from the seller of the option a barrel of oil at an agreed-upon price (the *exercise price*) at a given date in the future (the *maturity date*). The buyer of the *call option* pays a fee called the *option premium* for the option. If the *spot price* of oil (the market price of the good at the present time) rises above the exercise price, it is in the option-holder's favor to exercise the option, because she can then buy the oil at a price lower than is available on the market [1].

For example, you could buy a December oil call option for a premium of \$2, which has an exercise price of \$11. The spot price of oil is \$10 in September when you buy the option, but suppose that the spot price drastically increases to \$15 in December. In this case, you would exercise the option and buy the barrel of oil at the exercise price of \$11, thus saving yourself \$4. If the price of oil falls to \$6 instead, you will buy it at the spot price. By spending \$2 in September on the option premium, you have insured yourself against drastic increases in the price of oil, thus reducing some of your uncertainty about its price.

One detail to remember about the price of oil in December, or any other price dictated in "December dollars", is that a dollar is worth more today than in December. \$1 can be placed into a "riskless" investment today and left to collect interest at the "riskless interest rate" until December. Suppose that the quarterly riskless interest rate between today and December is incredibly high, so your \$1 invested today will be \$2 in December. Likewise having \$2 in December is equivalent to having \$1 now.

Another way to look at it is in the following example. Pretend a salesman tells you the price of a car will be \$10,000 in December. The price is dictated in December dollars, thus when it is "discounted" into September dollars, the car is worth 5,000 September dollars. If you risklessly invest \$5,000 now, you will have the money for the car in December.

The following formula is used to discount future quantities of money into their present equivalents, so that they can be compared to other quantities denominated in present dollars.

$$Value_{Present} = \frac{Value_{Future}}{e^{rT}} \tag{1}$$

- r is the annual riskless rate of interest
- T is the amount of time between now and the future point in question, expressed in a fraction of the year

Looking back at your contract to ship Christmas trees, you face the quandary of how to maximize your profit from the contract by minimizing your expenditure on oil. You can choose Plan 1, 2, or 3.

Buying options under Plan 3 can protect your company from a drastic rise in oil prices and reduce your risk, but the option premium will cut into your profits. You must find a balance point between protecting yourself and the option premium you pay for that protection. If you decide to buy options, you will ultimately have to decide which exercise price will put you at that balance point.

Note on Program Efficiency and Accuracy

As you go through each of the following steps, be sure that each program is efficient enough to get a fairly accurate job done in reasonable time. Sit down before you begin to blindly program and think about the most efficient way to make use of your algorithms and of ways to get around common problems such as integrating to infinity. Keep in mind, however, that the last few steps of this assignment will require hundreds of thousands repitions of certain code, thus any "inefficient calculations" you code will seriously cost you time-wise and could possibly make your final program impossible to run.

You should also verify your results whenever possible. For the situations in which you cannot check your answers, think of different ways to gain confidence in your solution, such as by using different algorithms to solve the same problem or decreasing the program step-sizes and looking for changes in the answer.

Note on the Length of this Assignment

As you have probably noticed, the sheer printed length of this assignment is considerably longer than the last one. The actual time it will take to complete this assignment, however, should not be considerably longer. Calculating the final step of this assignment requires the interaction of several components, hence the many steps. Other steps are not programming ones, but simply ask for a conceptual understanding of the assignment, which serve as checks for later output. If you work through the first few steps of the assignment carefully, the rest of it will fall into place.

Creating Your Integration Tools

Considering that most of the calculations in this assignment will involve integration, do not rush through this step! The rest of the assignment depends on it.

1. Integration

   (a) Using the "Trapezoidal Rule" in the function *trap*, integrate:

   $$\int_0^{\pi} sin(x)\,dx \qquad (2)$$

   What step-size do you need for accuracy to 6 decimal places? How much CPU time does it take to achieve this accuracy (use "time *program*" to time your program)?

(b) Use "Simpson's Rule" to integrate the same integral and look at the step-size and speed needed for accuracy to 6 decimal places. Would you rather use this algorithm or the Trapezoidal Rule? Why?

Laying Down the Statistical Foundation

A simple way of measuring the uncertainty of the price of oil is through a *probability density function*, $p(x)$, which gives you for a given x the "probability density". When the probability density is multiplied by a differential amount of x, dx, the result is the area of the curve under $p(x)$ bewteen x and $x + dx$. Thus $p(x)$ yields the probability that x occurs within a specified range of x when integrated over that range.

PDF's have two important statistics that "summarize" them, the mean, μ, and the standard deviation, σ.

The mean of $p(x)$ equals the expected value of x, $E(x)$, which is the addition of an infinite number of different x values, each "weighted" by its probability. $E(x)$ is calculated by the following open-form formula:

$$E(x) = \int_{-\infty}^{+\infty} x\, p(x)\, dx \tag{3}$$

Be aware that the peak of $p(x)$ is not necessarily the mean of the distribution.

The standard deviation σ is defined to be the square root of the variance $V(x)$, which is equal to the following closed-form formula:

$$V(x) = E(x^2) - [E(x)]^2 \tag{4}$$

The most famous pdf is the normal distribution function, $N(x)$, also known as the *Gaussian* or *Normal* distribution function.

$$N(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{(-(x-\mu)^2/2\sigma^2)} \tag{5}$$

Another important pdf is the *Lognormal* distribution function, $L(x)$. It is called "lognormal" because the $ln\, x$ is normally distributed.

$$L(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{(-(\ln(x)-\mu)^2/2\sigma^2)} \tag{6}$$

Note that the μ and σ used for the Gaussian and Lognormal distributions are different. In the Gaussian distribution, x is centered around μ with a standard deviation σ. In the Lognormal distribution, $\ln x$ instead of x is centered around μ.

2. Working with the Gaussian distribution: For this step, assume that μ_{gauss}=0 and σ_{gauss}=1.

   (a) Write a function, *gauss*, that will return the probability density of the Gaussian Distribution (i.e. solve the pdf) for a given x value. Use it to plot the Gaussian Distribution.

(b) Write a function C that integrates the Gaussian Distribution from $-\infty$ to a specified upper-limit, d.

$$C(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}\sigma} e^{(-(x-\mu)^2/2\sigma^2)}\, dx \tag{7}$$

This integral is commonly known as the cumulative normal distribution and you will use it later to price options.

Calculate this integral for $d = -1$ and $d = 1$. What step size of integration do you need for this distribution? Are you confident about the accuracy of your answer? Why or why not? What is an equivalent way to calculate the integrals without having to use infinite limits? Is this method preferable? Why?

3. The Lognormal Distribution: For this step, assume $\mu_{log} = 2.4$ and $\sigma_{log} = 0.4120840928$.

   (a) Write a function, *lognormal*, that will return $L(x)$. Use it to plot the Lognormal distribution.

   (b) Integrate $L(x)$ from 0 to a specified upper-limit b. Try integrating to $b = 11$. What step-size do you need for this distribution to be confident about the accuracy of your answer?

   (c) Create a function, *elogshort*, to find the solution of the following closed-form formula for calculating the expected value of the Lognormal Distribution (you will need this solution to check your result from the next step). [2]:

$$E(x) = e^{\mu_{log} + (\sigma_{log}^2/2)} \tag{8}$$

   (d) Calculate the expected value of the Lognormal distribution using the open-form formula. What step-size do you need here?

Buy Now or in December?

Now that the you have tools for integration, root finding, and statistical functions, you are ready to evaluate the first two plans for oil purchases.

4. Evaluate the Plans Conceptually

   (a) Looking at a Cartesian coordinate plane with the per-barrel price of oil, x, on the horizontal axis and probability density of x on the vertical axis, sketch the pdf of the cost of Plan 1, under which you buy all the 20,000 barrels of oil at \$10 per barrel now. Label $E(x_1)$.

   (b) Sketch the pdf of the price of oil if you purchase all the oil in December. Assume the price of oil in December will be distributed log-normally (most option-pricing formulas, such as the famous Black-Scholes formula, are based on the assumption that future prices are distributed log-normally). Label $E(x_2)$. Assume that $E(x_1) < E(x_2)$ and label $E(x_1)$ too.

   (c) Sketch the pdf of the cost of oil if you use options to reduce your risk. Again assume the price of oil in December is lognormally distributed, but consider what happens when the spot price exceeds the option's exercise price. How does $E(x_3)$ compare with $E(x_2)$? What about with $E(x_1)$?

(d) What happens to the expected cost of oil if you decrease the exercise price of the option? Do you think the premium of an option with a reduced exercise price is lower, the same, or higher? What will the premium become if the exercise price becomes zero? If the exercise price becomes infinitely high, what will the premium approach? How does Plan 3 encompass Plans 1 and 2? Sketch the strike price vs. the premium.

5. Plans 1 and 2: Buy all the oil now or in December?

   (a) What is the expected cost of Plan 1, purchasing 20,000 barrels of oil now at $10 per barrel?

   (b) Assume that the price of oil, x, in December is lognormally distributed with $\mu_{log} = 2.4$ and $\sigma_{log} = 0.4120840928$. What is the expected cost of oil in December in September dollars? Assume $r = .05$. How does this compare with Plan 1?

Using the Black-Scholes Model to price your options

Before evaluating Plan 3, you need a way of pricing options. Fortunately in 1973, Nobel prize-winners Fisher Black and Myron Scholes derived a widely used formula for calculating option premiums (see http://www.jstor.org for a postscript version of their article). For sanity's sake, assume that options are priced exactly as dictated by this model, though the pricing of options in the real world may differ. The formula is as follows:

$$w = sC(d_1) - ce^{-rT}C(d_2) \tag{9}$$

$$d_1 = \frac{\ln\frac{s}{c} + (r + \frac{v^2}{2})T}{v\sqrt{T}} \tag{10}$$

$$d_2 = d_1 - v\sqrt{T} \tag{11}$$

- s is the spot price of oil
- $C(d)$ is the Cumulative Normal distribution
- r is the annual risk-free interest rate
- T is the length of time until maturity in fraction of a year
- c is the exercise price of the option
- v is the volatility or standard deviation of the December price of oil

The derivation of the formula can be found in Black and Scholes' original paper [3].

6. Checking that the Black-Scholes model makes sense

   (a) When the exercise price is zero, what does the Black-Scholes predict the premium will be? Does this answer agree with that of step 4d?

   (b) When the exercise price becomes infinitely high, what does the Black-Scholes predict the premium will be? Does this answer agree with that of step 4d?

7. Implementing the Black-Scholes model

(a) Plot the option premiums against the exercise price using functions *w*, *d1*, *d2*, and GNUplot. Use the following values for the constants:

- $s = 10$
- $r = .05$
- $T = .25$
- $v = 0.4120840928$

Does your result check with that of the previous step?

8. Finding the expected profit, E(profit)

(a) Write a formula relating E(profit) to the following:

- Total revenue of the contract, TR
- Fixed costs of labor and ships, FC
- Expected cost of a barrel of oil in December, E(oilcost)
- The premium of an option to buy a barrel of oil, w

Remember to express everything in September dollars.

(b) Write a formula for E(oilcost) which is a function of the exercise price, c (*Hint: Refer to the sketches you drew in Step 4c*). What values does E(oilcost) approach as c approaches 0 and $+\infty$?

(c) Write a function *ecost* that returns E(oilcost) for a given exercise price and verify that it works.

(d) Create a function, *eprofit*, that outputs E(profit) for a given exercise price, and then plot *eprofit* against c.

(e) If you want to maximize your profits, you would buy options of what exercise price? Numerically how did you pinpoint the optimal exercise price?

Extra Credit

The following are suggested extra credit steps. If you can think of some other interesting aspect of this assignment that you would like to look at in greater detail, feel free to pursue it.

9. Adjusting μ_{log} and σ_{log}

(a) Halve μ. What happens to the E(profit) distribution? Intuitively explain it. Now increase μ to twice its original value. Explain what happens.

(b) Halve σ. What happens to the E(profit) distribution? Intuitively explain it. Now increase σ to twice its original value. Explain what happens.

(c) Can you make the peak of the E(profit) distribution disappear? If you can, at what exercise price would you buy options? Can accentuate the peak? Or make the optimal exercise price higher or lower?

Another consideration other than the expected profit is the amount of risk you take on undergoing this deal. Ideally you would like to minimize the variance of your expected profit, but choosing an option with an exercise price that further reduces your risk will probably reduce your expected profit too. Once again, you have to perform a balance act, but this time it is between your expected profit and the variance of that profit.

Okiahn's stockholders have ordered you to find this balance point by maximizing Okiahn's copyrighted Risk-Profit Balance Formula, which is as follows:

$$h(c) = \ln(E(P)) + \frac{1}{\ln(V(P)) + 1} \tag{12}$$

- c is the exercise price
- $E(P)$ is the expected profit (i.e. average profit) when using options with an exercise price of c
- $V(P)$ is the variance of the profit when using options with exercise price, c

10. Finding the variance of profit, V(profit). Use the original values for μ_{log} and σ_{log}.
    (a) Write a function, *vlog*, to calculate the variance of the Lognormal distribution using the open-form formula. Use the result of the closed-form formula to check your answer [2].
    $$V(x) = e^{\sigma_{log}^2 + 2\mu_{log}}(e^{\sigma_{log}^2} - 1) \tag{13}$$
    (b) What will the variance be when $c = 0$? Or when c approaches infinity?
    (c) Write the formula for V(profit), the variance of expected profit.
    (d) Create a function, *vprofit*, that returns the variance of expected profit when using options for a given exercise price. Remember:
    $$\sigma^2 = E(x^2) - [E(x)]^2 \tag{14}$$

11. Finding the optimum level of the Risk-Profit Balance Formula
    (a) Plot the RPBF for varying values of c.
    (b) Locate the peak of the RPBF. How does it differ from that of the plot of the expected profit?

References

[1] M. D. Fitzgerald. *Financial Options.* Euromoney Publications, London, 1987.

[2] Jay L. Devore. *Probability and Statistics for Engineering and the Sciences.* Brooks/Cole Publishing Co., Pacific Grove, Ca., third edition, 1991.

[3] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Political Economy*, 81:637–654, May–June 1973.