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Intractability

- *introduction*
- *P vs. NP*
- *poly-time reductions*
- *NP-completeness*
- *Dealing with intractability*
- *Leveraging intractability*

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- ‣ *Leveraging intractability*

Fundamental questions

What is an algorithm?

What is an efficient algorithm?

Which problems can be solved efficiently and which are intractable?

How can we prove that a problem is intractable?

How can we cope with intractability?

How can we benefit from intractability?

Multiplication

$$37 \cdot 79 = {\color{red}?}$$

Factorization

$$? \cdot ? = 2881$$

Slightly bigger multiplication

$$\begin{array}{l} 33478071698956898 \\ 78604416984821269 \\ 08177047949837137 \\ 68568912431388982 \\ 88379387800228761 \\ 47116525317430877 \\ 37814467999489 \end{array} \cdot \begin{array}{l} 36746043666799590 \\ 28244633799627952 \\ 63227915816434308 \\ 76426760322838157 \\ 39666511279233373 \\ 41714339681027009 \\ 2798736308917 \end{array} = ?$$

Computed in a split second by a standard laptop!

Slightly bigger factorization

? · ? =

1230186684530117
7551304949583849
6272077285356959
5334792197322452
1517264005072636
5751874520219978
6469389956474942
7740638459251925
5732630345373154
8268507917026122
1429134616704292
1431160222124047
9274737794080665
3514195974598569
02143413

RSA-768, 232 digits

$50,000
RSA factoring challenge
2 years, team of mathematicians

Multiplication (computationally easy)

Multiplication. Given integers x, y, return xy.

Algorithm. Grade-school multiplication runs in time $\Theta(n^2)$, where n is the number of digits in x, y.

Integer factorization (computationally hard?)

Factorization (search). Given an integer x, find a nontrivial factor. ← *or report that no such factor exists*

neither 1 nor x

Applications. Cryptography. [stay tuned]

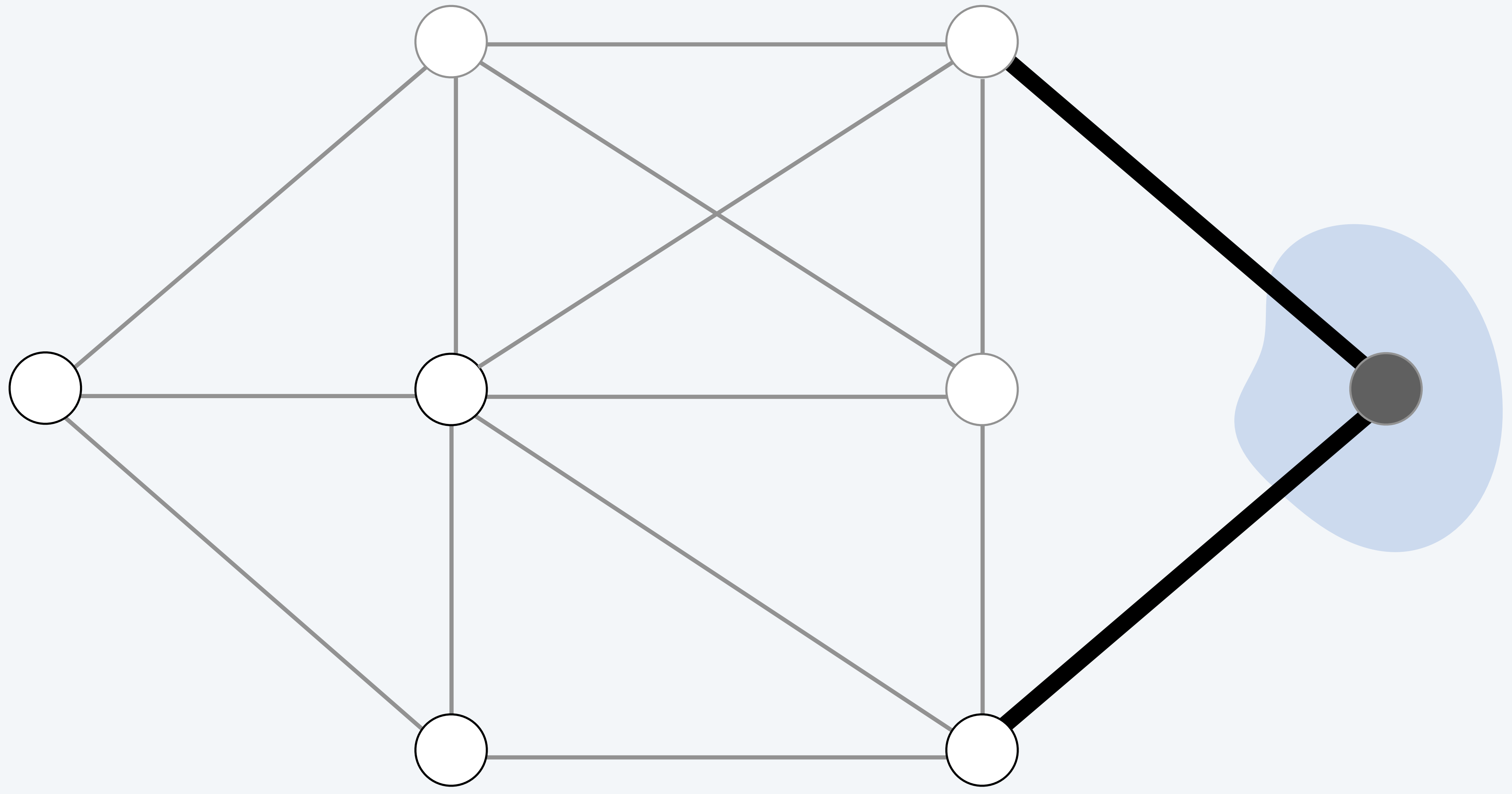
Brute-force search. Try all possible divisors between 2 and $\sqrt{x}$.

Can we do anything substantially more clever?

if there's a nontrivial factor larger than $\sqrt{x}$, there is one smaller than $\sqrt{x}$

Mincut (computationally easy)

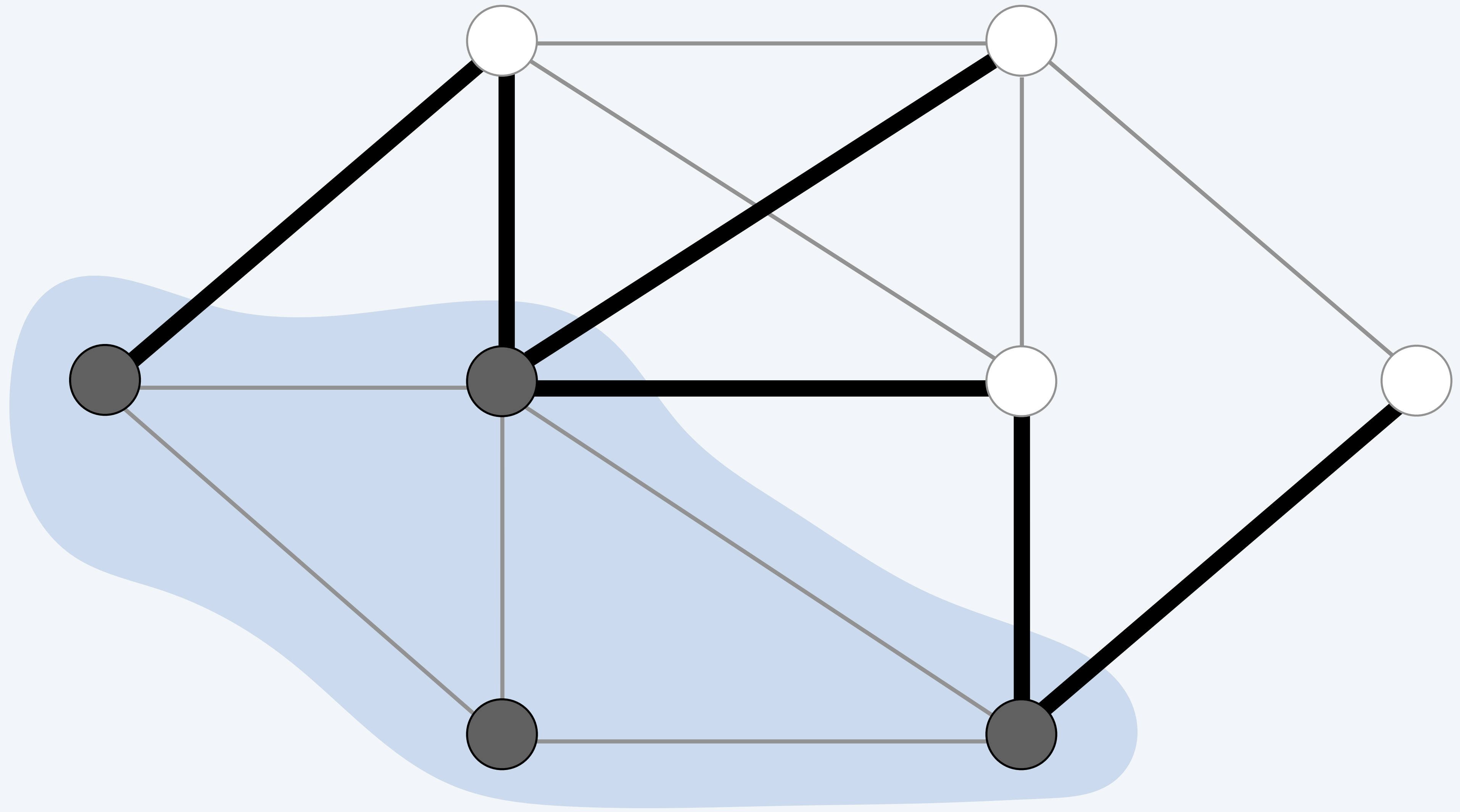
Mincut (search). Given a graph G, return a cut that minimizes the number of crossing edges.



Algorithm. Ford–Fulkerson-based algorithm runs in time $\Theta(VE^2)$.

Maxcut (computationally hard?)

Maxcut (search). Given a graph G, return a cut that maximizes the number of crossing edges.



Brute-force search. Try all $2^V - 2$ possible cuts.

Can we do anything substantially more clever?

Probably not. [stay tuned]

boolean satisfiability with 2 vars (computationally easy)

2-SAT (search). Given m boolean equations over the variables $x_1 \ldots x_n$ in the form "y_i *or* y_j = *true*", where y_i is either x_i or $\neg x_i$, return a truth assignment that satisfies all equations.

CNF, conjunctive normal form

or report that no such assignment is possible

Example.

$\neg x_1$	*or*	x_2	=	*true*
x_1	*or*	x_3	=	*true*
$\neg x_2$	*or*	$\neg x_3$	=	*true*
$\neg x_2$	*or*	x_4	=	*true*
x_3	*or*	$\neg x_4$	=	*true*

2-SAT instance

x_1	=	*false*
x_2	=	*false*
x_3	=	*true*
x_4	=	*true*

satisfying assignment

SAT applications.

- Automatic verification systems for software.
- Mean field diluted spin glass model in physics.
- Electronic design automation (EDA) for hardware.

boolean satisfiability with 3 vars (computationally hard?)

3-SAT (search). Same as 2-SAT, but every equation has 3 variables instead of 2.

Example.

$\neg x_1$	*or*	x_2	*or*	x_3	=	*true*
x_1	*or*	$\neg x_3$	*or*	x_4	=	*true*
x_2	*or*	$\neg x_3$	*or*	$\neg x_1$	=	*true*
$\neg x_2$	*or*	x_4	*or*	x_3	=	*true*
$\neg x_3$	*or*	$\neg x_4$	*or*	$\neg x_2$	=	*true*

3-SAT instance

x_1	=	*false*
x_2	=	*false*
x_3	=	*true*
x_4	=	*true*

satisfying assignment

Brute-force search. Try all 2^n possible assignments (n = # variables).

Can we do anything substantially more clever?

Probably not. [stay tuned]

How difficult can it be?

Imagine a galactic computer…

- With as many processors as electrons in the universe.
- Each processor having the power of today's supercomputers.
- Each processor working for the lifetime of the universe.

quantity	estimate
electrons in universe	10^{79}
instructions per second	10^{13}
age of universe in seconds	10^{17}

Could galactic computer solve satisfiability instance with $1{,}000$ variables using brute-force search?

Not even close: $2^{1000} > 10^{300} \gg 10^{79} \cdot 10^{13} \cdot 10^{17} = 10^{109}$.

Lesson. Exponential growth dwarfs technological change!

Efficient algorithms

What is an efficient algorithm?

# of bits in the input's representation

Algorithm whose running time is at most polynomial *in the size of the input.*

What is an algorithm?

A Turing Machine! Equivalently, a program in Java/Python/C++/…

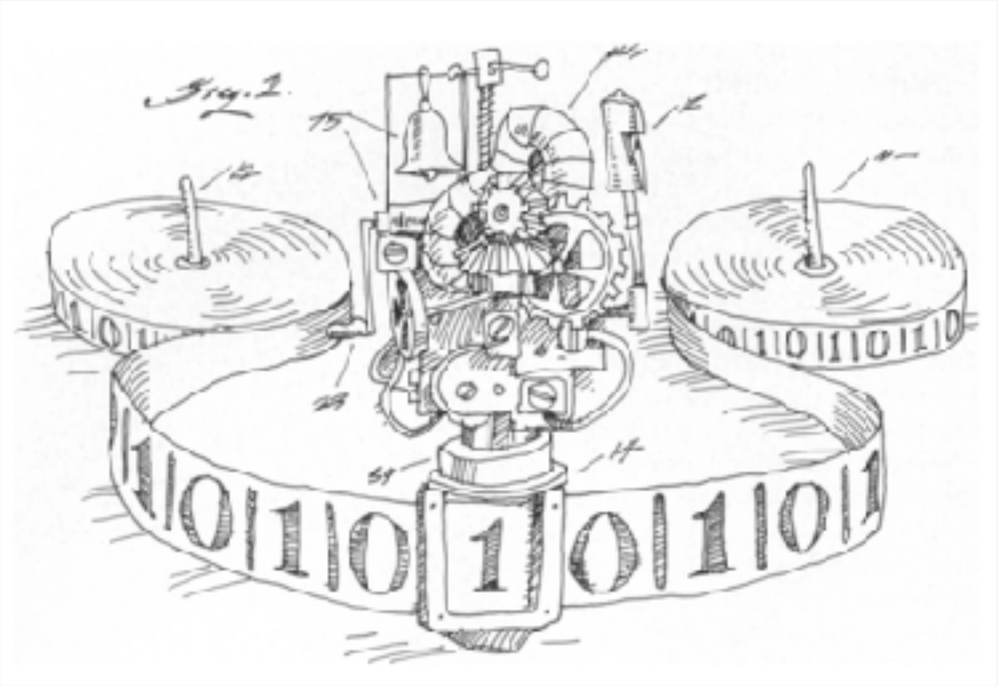
Extended Church–Turing thesis. Any problem the can be efficiently solved by a physical system can also be efficiently solved by a Turing machine.

falsifiable thesis. believed to be false — quantum computers

is $n^{billion}$ better than $2^{n/billion}$?

Why is polynomial time considered efficient?

robust across models, closed under composition, most poly-time algos have small exponents.



A Turing machine

order	emoji	name	today
$\Theta(1)$	😍	*constant*	🙂
$\Theta(\log n)$	😎	*logarithmic*	🙂
$\Theta(n)$	😁	*linear*	🙂
$\Theta(n \log n)$	😀	*linearithmic*	🙂
$\Theta(n^2)$	😕	*quadratic*	🙂
$\Theta(n^3)$	🙁	*cubic*	🙂
$\Theta(n^{\log n})$	😨	*quasipolynomial*	👿
$\Theta(1.1^n)$	😭	*exponential*	👿
$\Theta(2^n)$	😈	*exponential*	👿
$\Theta(n!)$	👿	*factorial*	👿

Which of the following are poly-time algorithms?

A. Brute-force search for 3-SAT.

B. Brute-force search for maxcut.

C. Brute-force search for factorization.

D. All of the above.

E. None of the above.

Intractable problems

A problem is tractable if there exists an efficient (poly-time) algorithm that solves it.

Otherwise, it is intractable.

How can we tell which problems are tractable?

Generally no easy way. 😭

Seemingly similar problems can behave differently & efficient algorithms are often clever and complex.

Focus of today's lecture!

tractable	**intractable?**
primality	*factorization*
shortest path	*longest path*
min cut	*max cut*
Euler cycle	*Hamiltonian cycle*
2-SAT	3-SAT
⋮	⋮

require math insights

Intractable problems

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The P complexity class

A decision problem is a Boolean function that, given an input, answers YES/NO.

Def. **P** is the set of all decision problems that can be solved in polynomial time.

Examples.

2-SAT (decision): Given a system of equations, is there an assignment that satisfies all equations?

mincut (decision): Given a graph G and integer k, is there a cut in G with more than k crossing edges?

multiplication (decision): Given integers x, y, k, is $xy \geq k$?

primality (decision): Given an integer x, is x prime? ⟵ *first poly-time algorithm in 2002!*

Are all "interesting" problems in **P**? Perhaps there is always a clever algorithm…

The NP complexity class

Def. **NP** is the set of all decision problems for which a YES answer can be verified in polynomial time provided a "witness" (a.k.a "proof", "certificate").

$x = 2881$	$k = 50$

factorization instance

Examples.

Factorization (decision): Given integers x, k, does x have a nontrivial factor $\leq k$?

- Witness. A nontrivial factor $f \leq k$ of x.
- Verification. Output YES if $1 < f \leq k$ and f divides x. ⟵ *quadratic time using long division*

43

witness

3-SAT (decision): Given a system of equations, is there an assignment that satisfies all equations?

- Witness. A satisfying assignment.
- Verification. Output YES if the assignment satisfies all equations.

x_1	=	*false*
x_2	=	*false*
x_3	=	*true*
x_4	=	*true*

satisfying assignment

Note. A problem is in **NP** if a *purported* witness for a *YES* answer can be verified in poly time:

- It does not require *finding* the witness (e.g., the candidate factor is provided).
- It does not require verifying a *NO* answer (e.g., no factor $\leq k$).

Which decision version of maxcut is in NP?

A. Given a graph G and integer k, does the maximum cut in G has *at most* k crossing edges.

B. Given a graph G and integer k, does the maximum cut in G has *at least* k crossing edges.

C. Both A and B.

D. Neither A nor B.

P vs. NP

P = set of decision problems whose solution can be *computed* efficiently (in poly-time).

NP = set of decision problems whose solution can be *verified* efficiently (in poly-time).

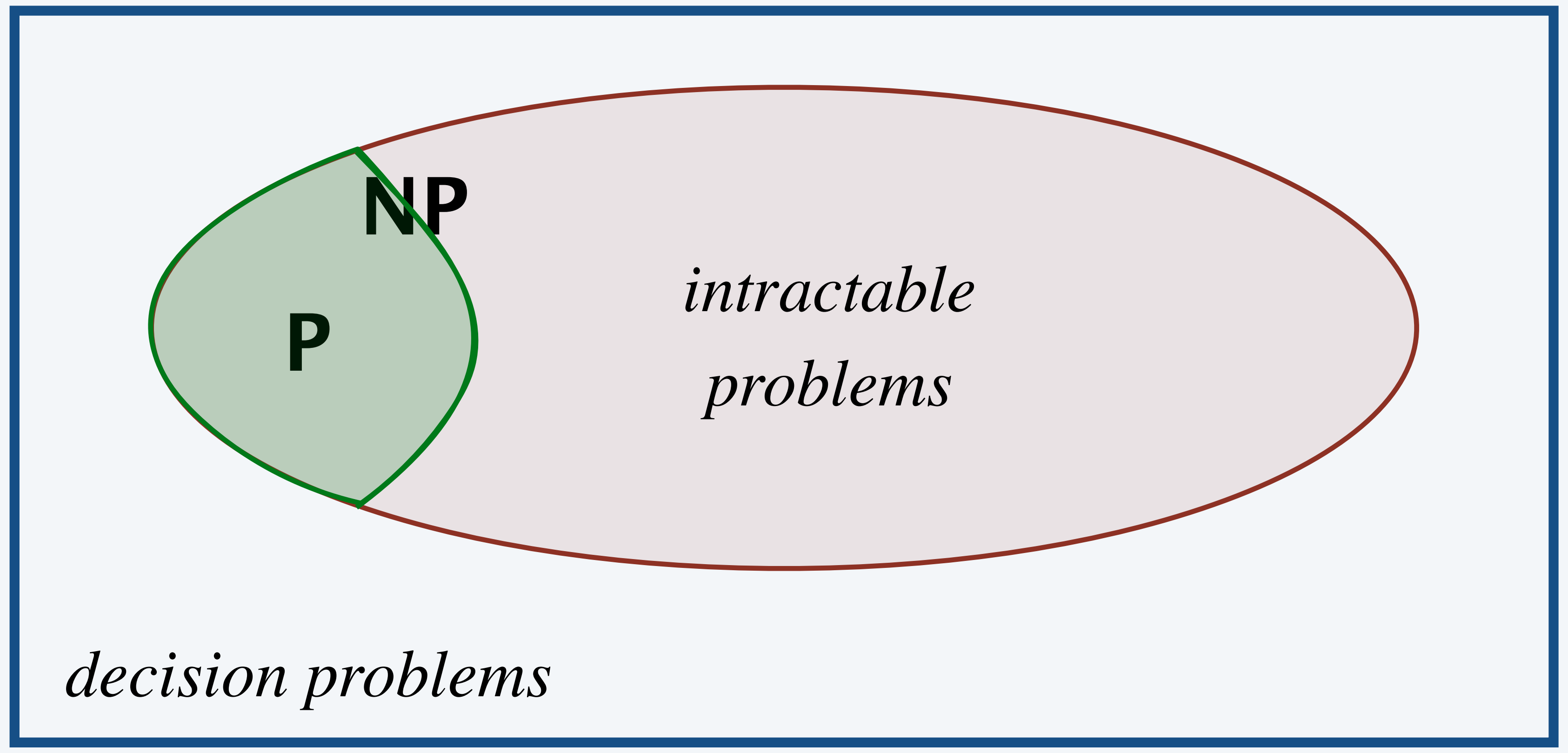
Observation. **NP** contains **P** ⟵ *any string serves as witness*

$ 1M

THE question. **P** = **NP** ?

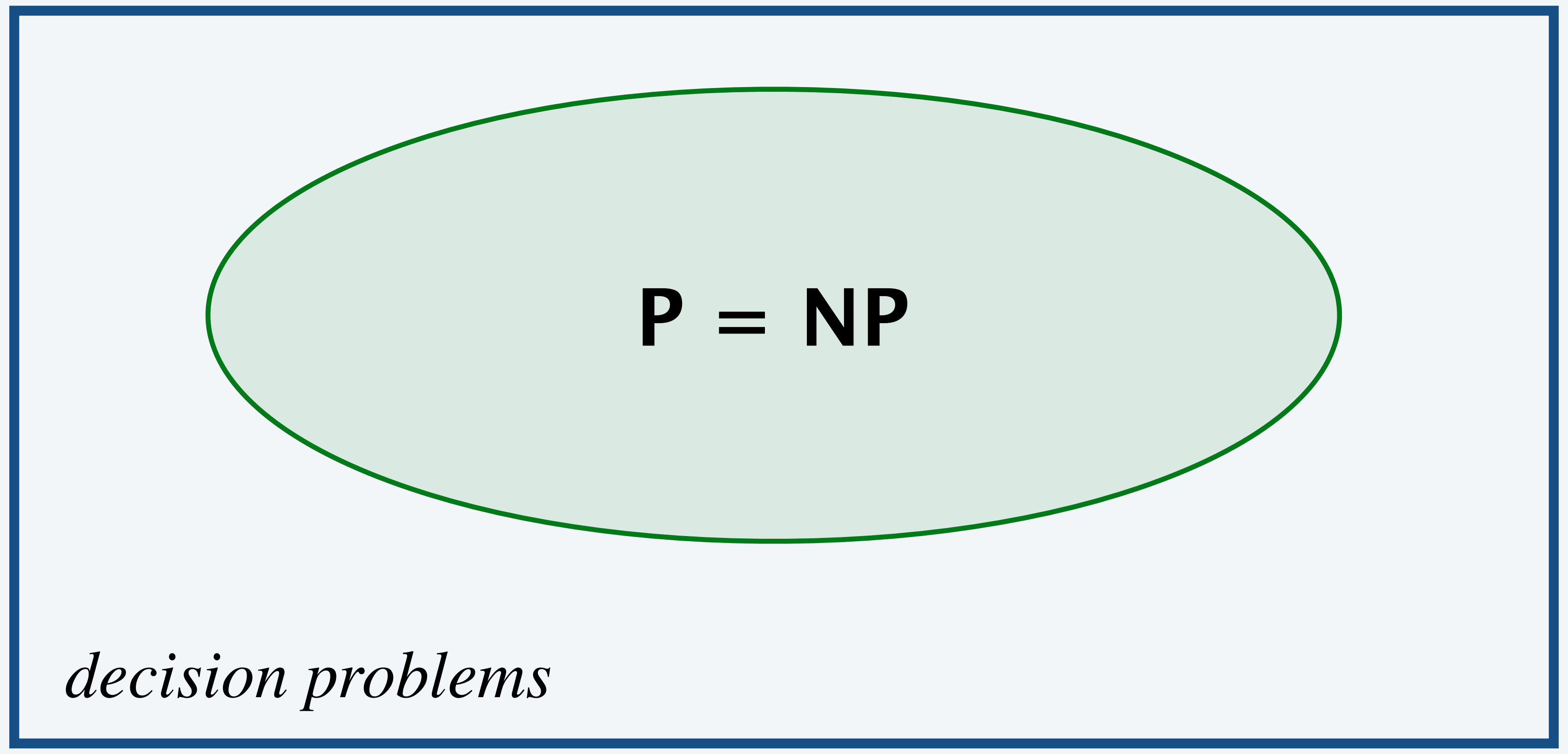
Is *solving* harder than *verifying*?

Two possible worlds.



P ≠ NP

brute-force search may be the best we can do



P = NP

poly-time algorithms for factorization, 3-SAT, maxcut, …

long futile search for poly-time algorithms

Conjecture. **P ≠ NP.**

Why is P vs NP so central?

P vs **NP** is central in math, science, technology and beyond.

NP models many intellectual challenges humanity faces: *Why try to solve a problem if you cannot even determine whether a solution is good?*

domain	problem	witness/solution
mathematics	is a conjecture correct?	mathematical proof
engineering	given constraints (size, weight, energy), find a design (bridge, medicine, computer)	blueprint
science	given data on a phenomenon, find a theory explaining it	a scientific theory
the arts	write a beautiful poem / novel / pop song, draw a beautiful picture	a poem, novel, pop song, drawing

creative genius

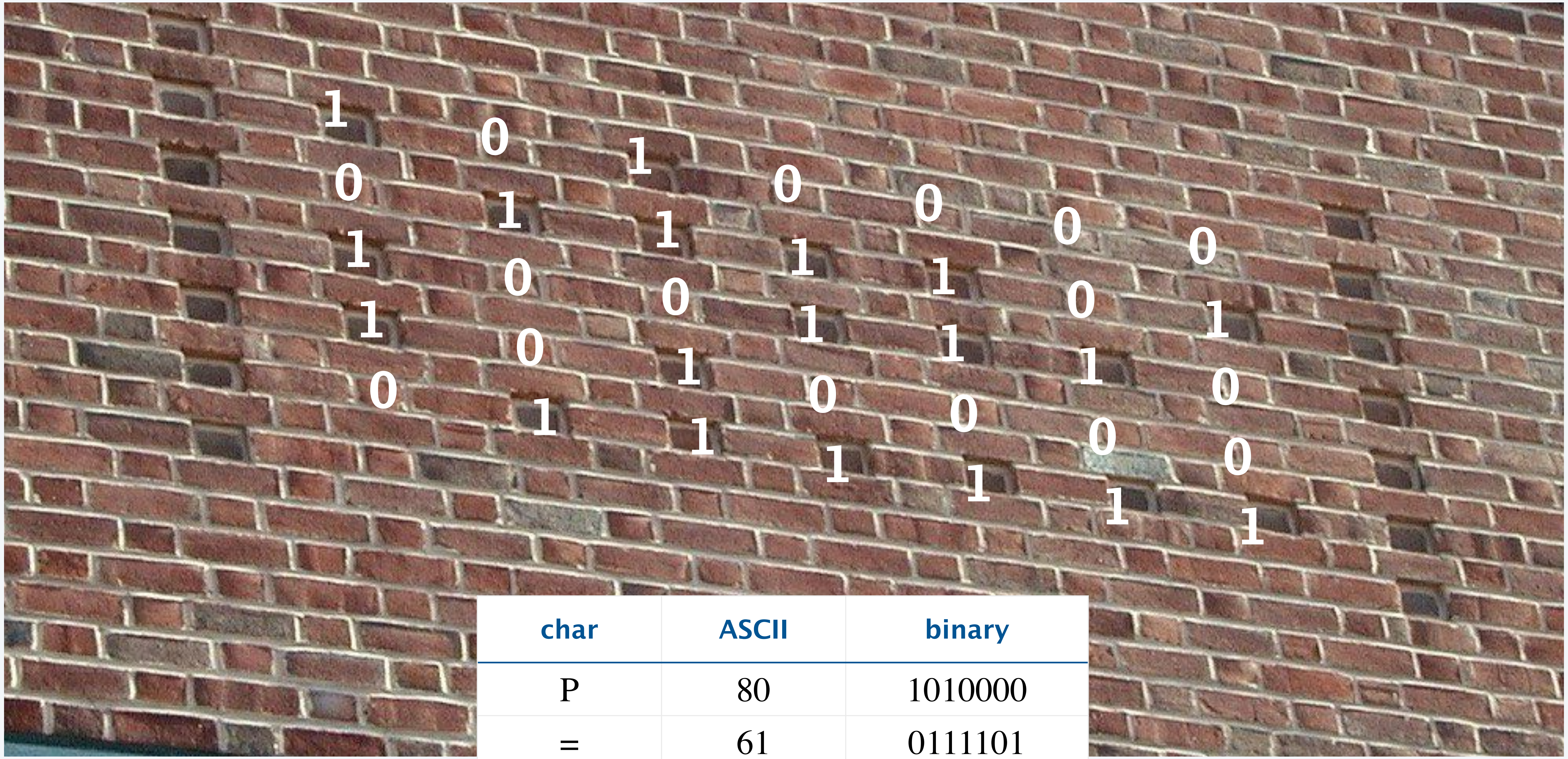
ordinary appreciation

Intuitively, verifying a solution should be way easier than finding it, supporting **P ≠ NP**.

Analogy for P vs NP. Creative genius vs. ordinary appreciation of creativity.

Princeton computer science building

Princeton computer science building (closeup)



char	ASCII	binary
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111

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Poly-time reduction

Goals.

- Classify problems according to computational requirements.
- If we can (or cannot) solve problem X efficiently, what other problems can (or cannot) be solved efficiently?

"solution to Y implies solution to X"
"Y is harder than X" (up to polys)
denoted $X \preceq Y$

Def. Problem X poly-time reduces to problem Y, if there exists a polynomial $p(n)$ such that any time-$T(n)$ algorithm for Y can be used to construct a time-$T(p(n))$ algorithm for X.

formal def in COS 240!

$T(n) \geq n$

ex. $T(n) = n^2$, $p(n) = n^3$ *implies* $T(p(n)) = n^6$.

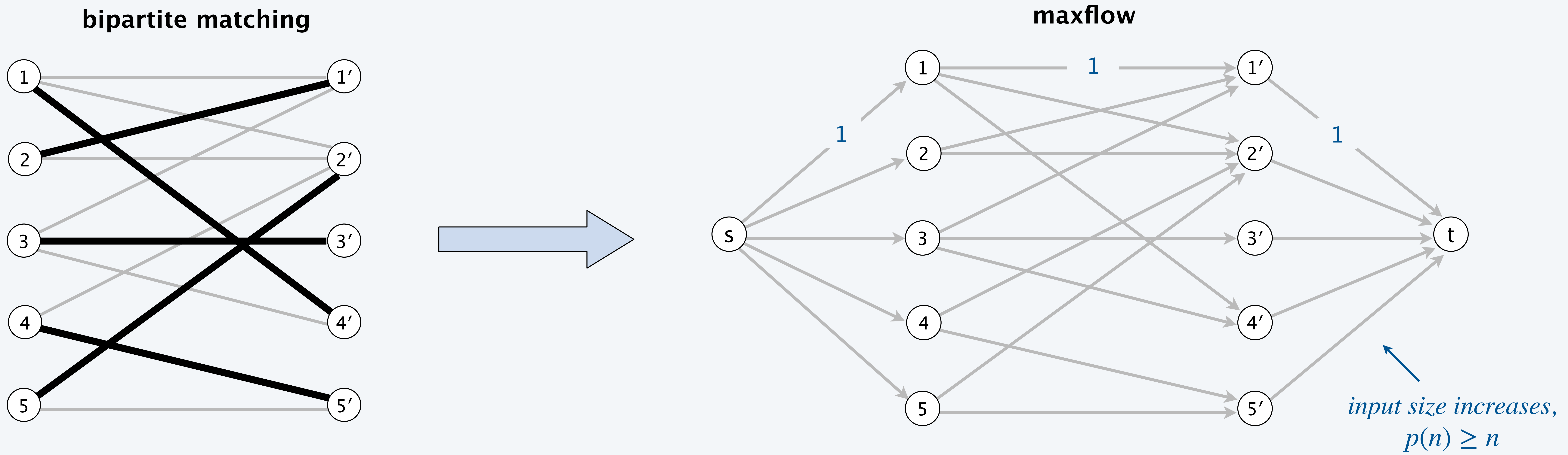
Algorithm design. If $X \preceq Y$ and Y can be solved efficiently, then X can also be solved efficiently.

Establishing intractability. If $X \preceq Y$ and X is intractable, then Y is also intractable.

Common mistake. Confusing *X poly-time reduces to Y* with *Y poly-time reduces to X*.

Poly-time reduction example 1

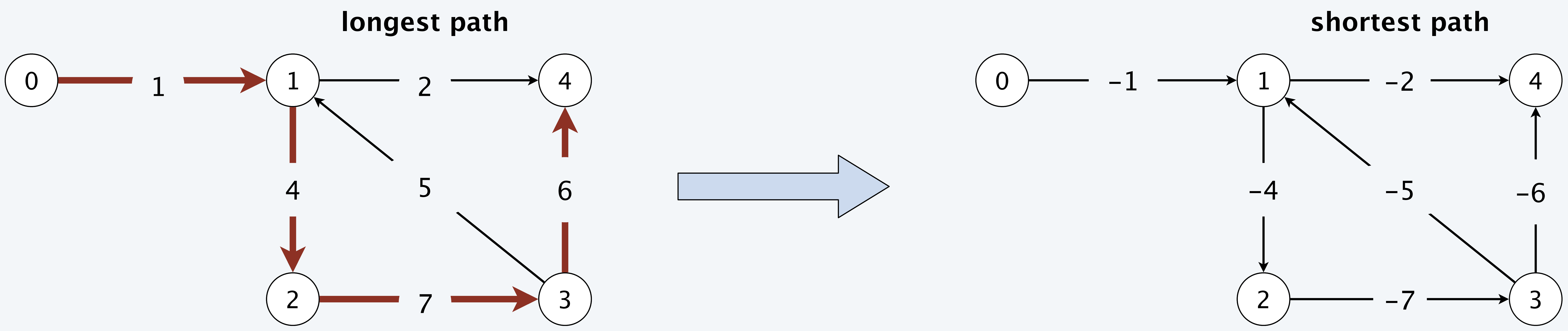
Bipartite matching ≤ maxflow:



Algorithm design. Since maxflow can be solved efficiently, so can bipartite matching.

Poly-time reduction example 2

Longest path ≤ shortest path with negative weights:



Establishing intractability. If longest path is intractable (as conjectured), so is shortest path with negative weights.

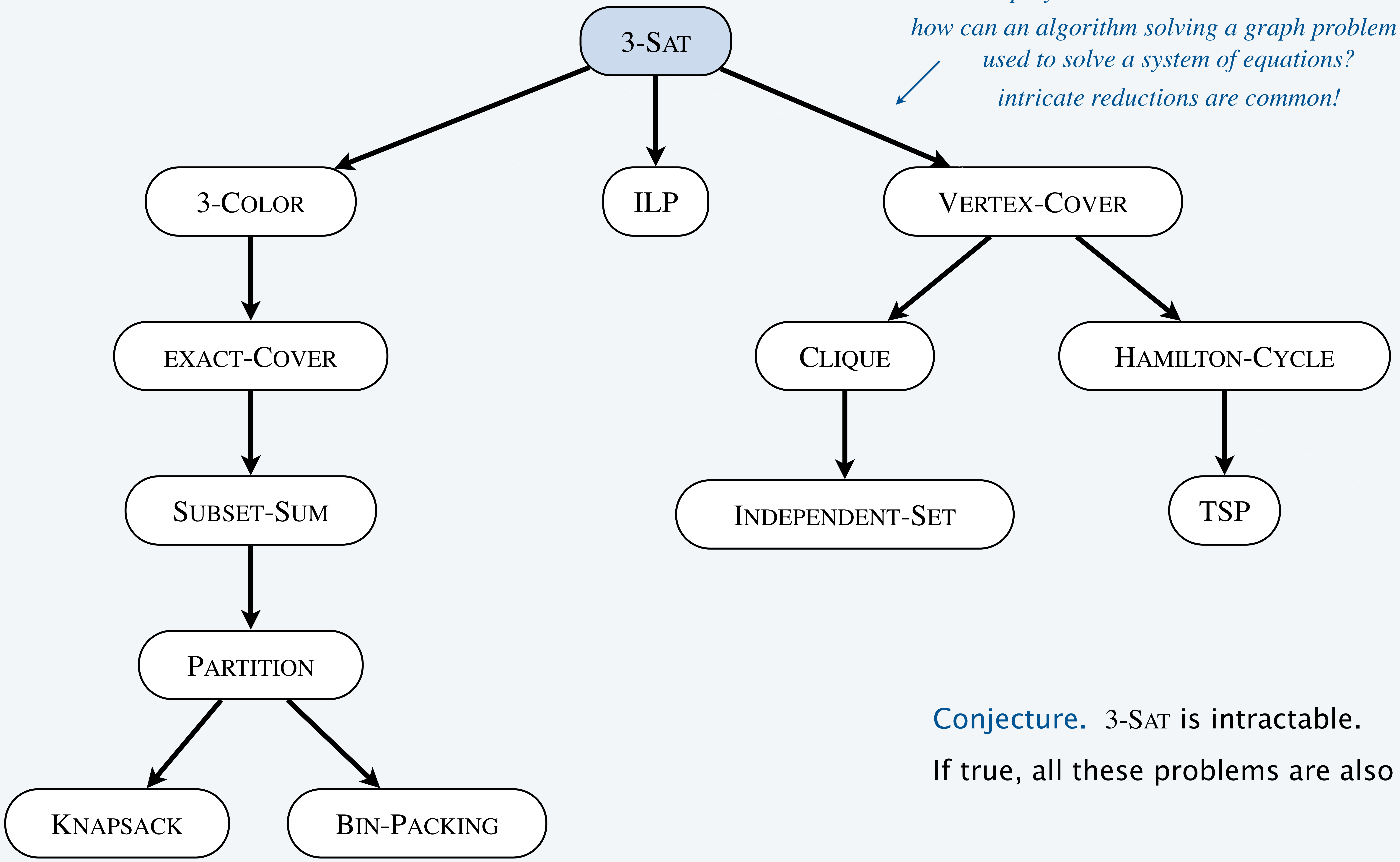
Intractability: quiz 3

Suppose that Problem X poly-time reduces to Problem Y.

Which of the following can we infer?

- **A.** If Y can be solved in $\Theta(n^3)$ time, then X can be solved in $\Theta(n^3)$ time.
- **B.** If Y can be solved in $\Theta(n^3)$ time, then X can be solved in poly-time.
- **C.** If X cannot be solved in $\Theta(n^3)$ time, then Y cannot be solved in poly-time.
- **D.** If Y cannot be solved in poly-time, then neither can X.

Some poly-time reductions from SAT



SAT poly-time reduces VERTEX-COVER.

how can an algorithm solving a graph problem be used to solve a system of equations?

intricate reductions are common!

Richard Karp
(1972)

Conjecture. 3-SAT is intractable.

If true, all these problems are also intractable!

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Intractability

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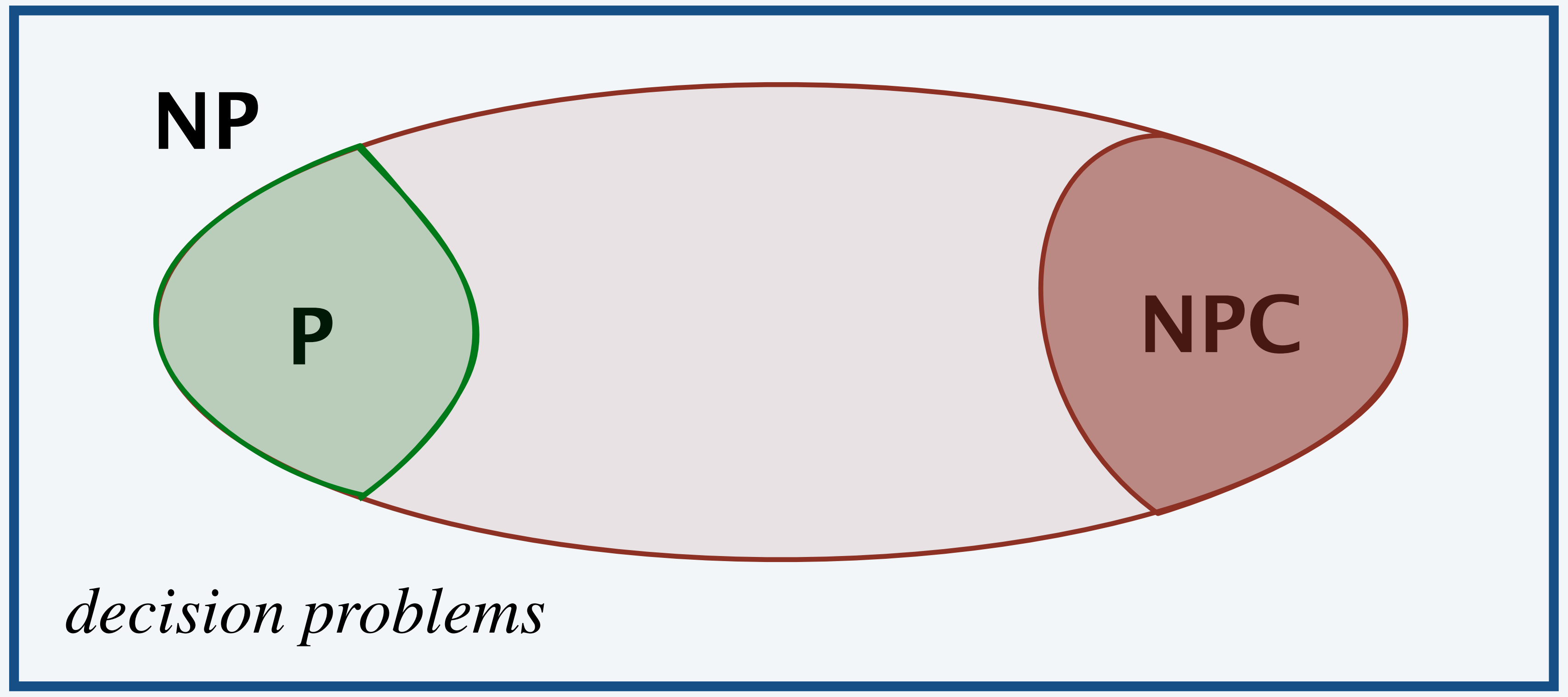
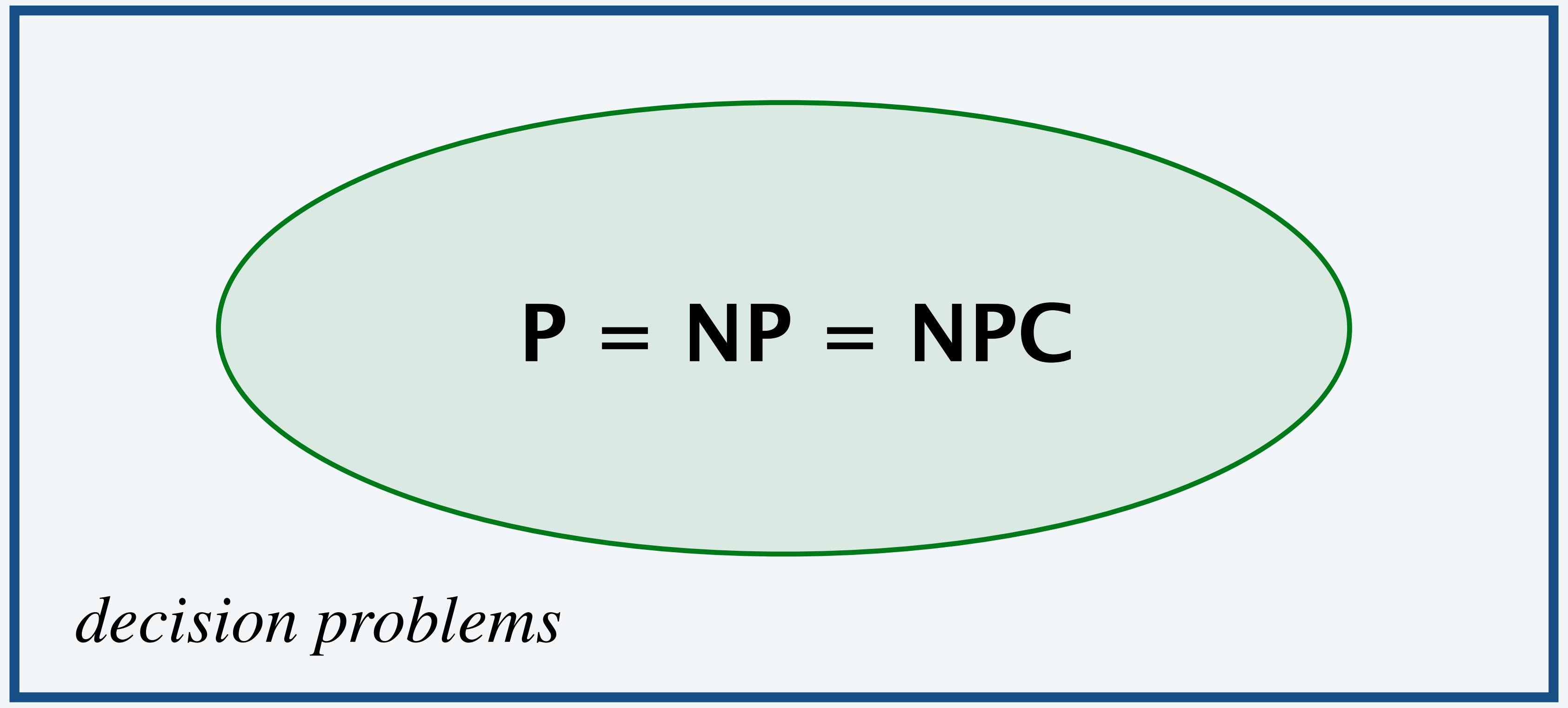
NP-completeness

Def. $Y \in \mathbf{NP}$ is NP-complete if for all $X \in \mathbf{NP}$, $X \leq Y$. ⟵ *X is maximally hard in* **NP**

Two worlds.



P ≠ NP



P = NP

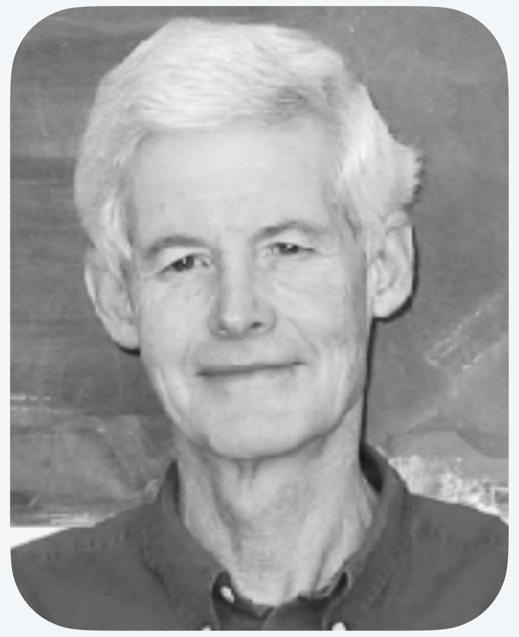
Cook–Levin theorem. 3-SAT is **NP**-complete. ⟵ *how can we prove $X \leq$ 3-SAT if we don't know X?*

Pioneering result in computer science!

Corollary 1. 3-SAT can be solved in poly-time if and only if **P** = **NP**.

Corollary 2. To show that $Y \in \mathbf{NP}$ is **NP**-complete, it suffices to show 3-SAT $\leq Y$.

Thousands of problems have been proven to be **NP**-complete!



Stephen Cook (1971)

Leonid Levin (1971)

NP-complete problems

field of study	NP-complete problem
Computer sience / Math	*maxcut, longest path, vertex cover, 3-SAT,…*
Aerospace engineering	*optimal mesh partitioning for finite elements*
Biology	*phylogeny reconstruction*
Chemical engineering	*heat exchanger network synthesis*
Chemistry	*protein folding*
Civil engineering	*equilibrium of urban traffic flow*
Economics	*computation of arbitrage in financial markets with friction*
Electrical engineering	*VLSI layout*
Environmental engineering	*optimal placement of contaminant sensors*
Financial engineering	*minimum risk portfolio of given return*
Game theory	*Nash equilibrium that maximizes social welfare*
Mechanical engineering	*structure of turbulence in sheared flows*
Medicine	*reconstructing 3d shape from biplane angiocardiogram*
Operations research	*traveling salesperson problem, integer programming*
Physics	*partition function of 3d Ising model*
Politics	*Shapley–Shubik voting power*
Pop culture	*versions of Sudoku, Checkers, Minesweeper, Tetris*
Statistics	*optimal experimental design*

6,000+ *scientific papers per year.*

NP-complete problems are different manifestations of the *same* fundamentally hard problem.

Solving any one of them in poly time solves all!

No field-specific math insights are required!

Suppose that X is NP-complete. What can you infer?

A. $X \in$ **NP**.

B. If X can be solved in poly-time, then **P** = **NP**.

C. If X cannot be solved in poly-time, then **P** ≠ **NP**.

D. If $Y \in$ **NP** and $X \le Y$ then Y is **NP**-complete.

E. All of the above.

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Dealing with intractability

Approaches to coping with intractability

… so your problem is **NP**-complete 😭

Safe to assume it is intractable: no worst-case poly-time algorithm solves all problem instances.

protein folding is **NP***-complete*

Do you need to solve *all* instances?

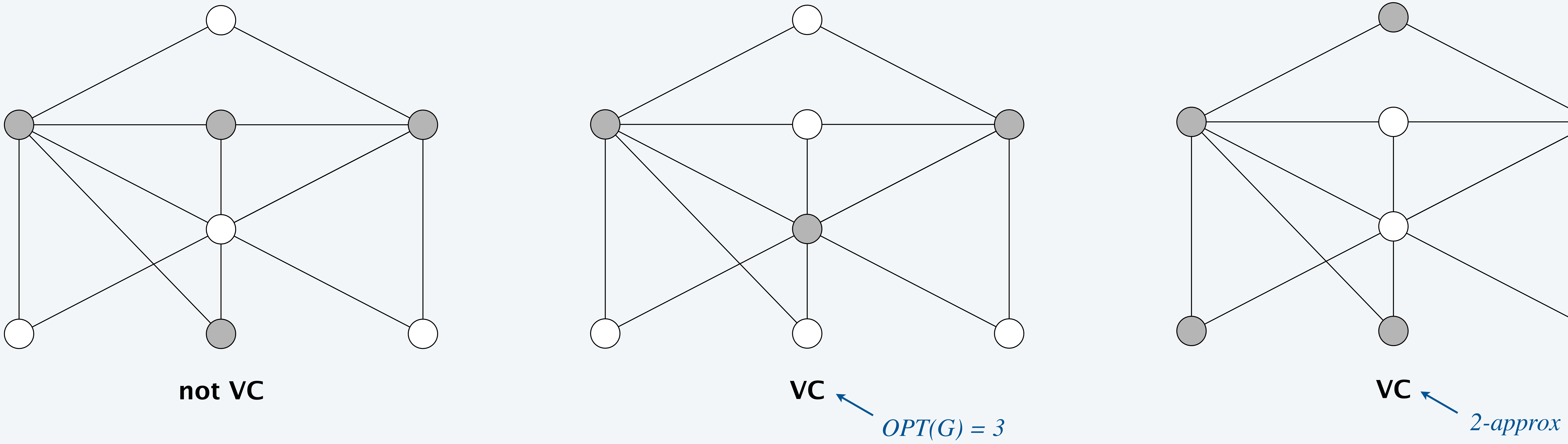
Model real-world instances. Worst-case inputs might not arise in practical applications.

Do you need the exact *optimal* solution?

Approximation algorithms. Look for good (though potentially suboptimal) solutions.

Vertex cover

A vertex cover of a graph G is a set of vertices such that every edge in G has at least one endpoint in the set.



$OPT(G)$ is the minimum size of a VC for G.

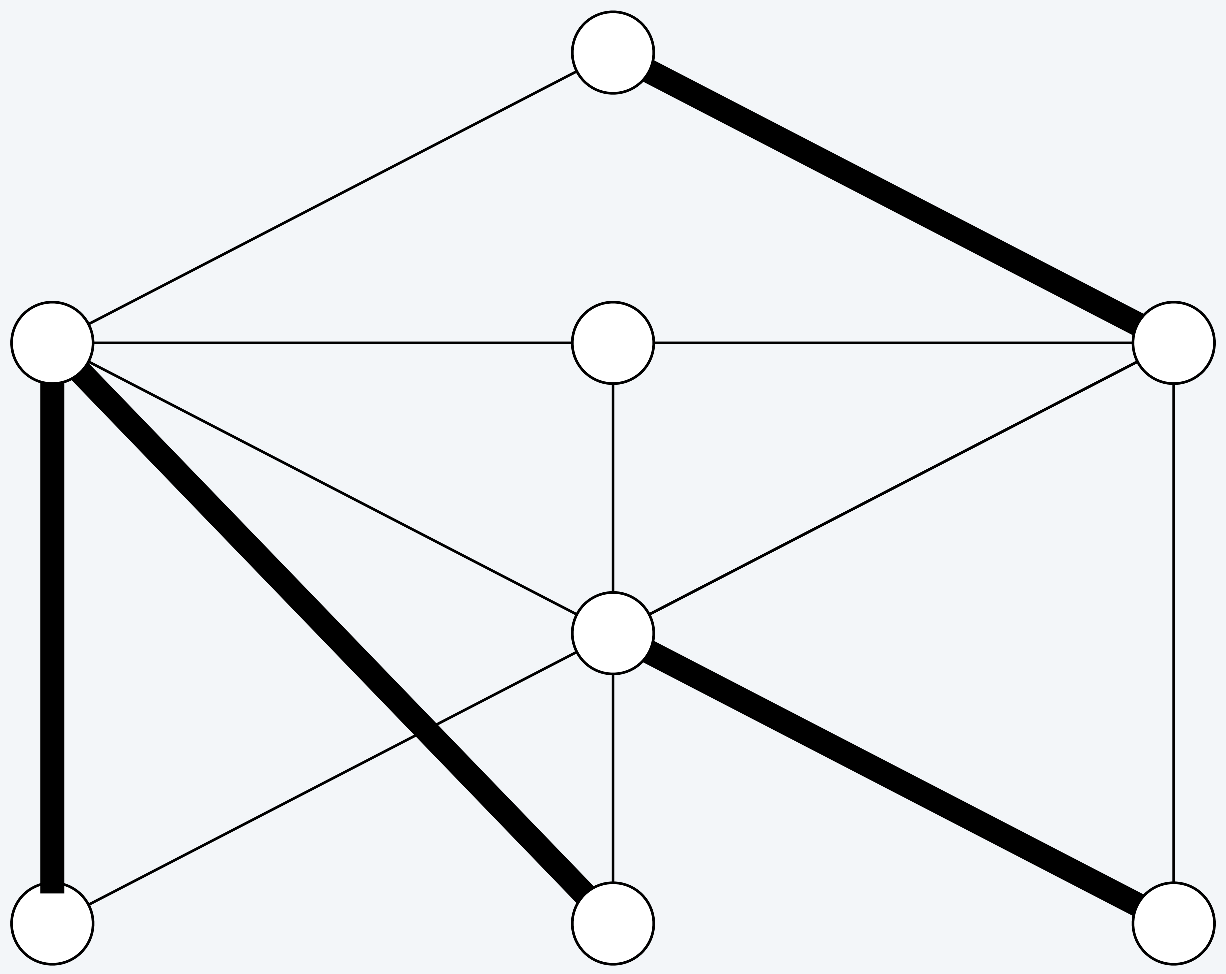
VC (decision): Given G, k, is $OPT(G) \leq k$? ← **NP**-complete 😭

VC (α–approx): Return a VC of size $\leq \alpha \cdot OPT(G)$. ⟵ *want $\alpha > 1$ as small as possible (min problem)*

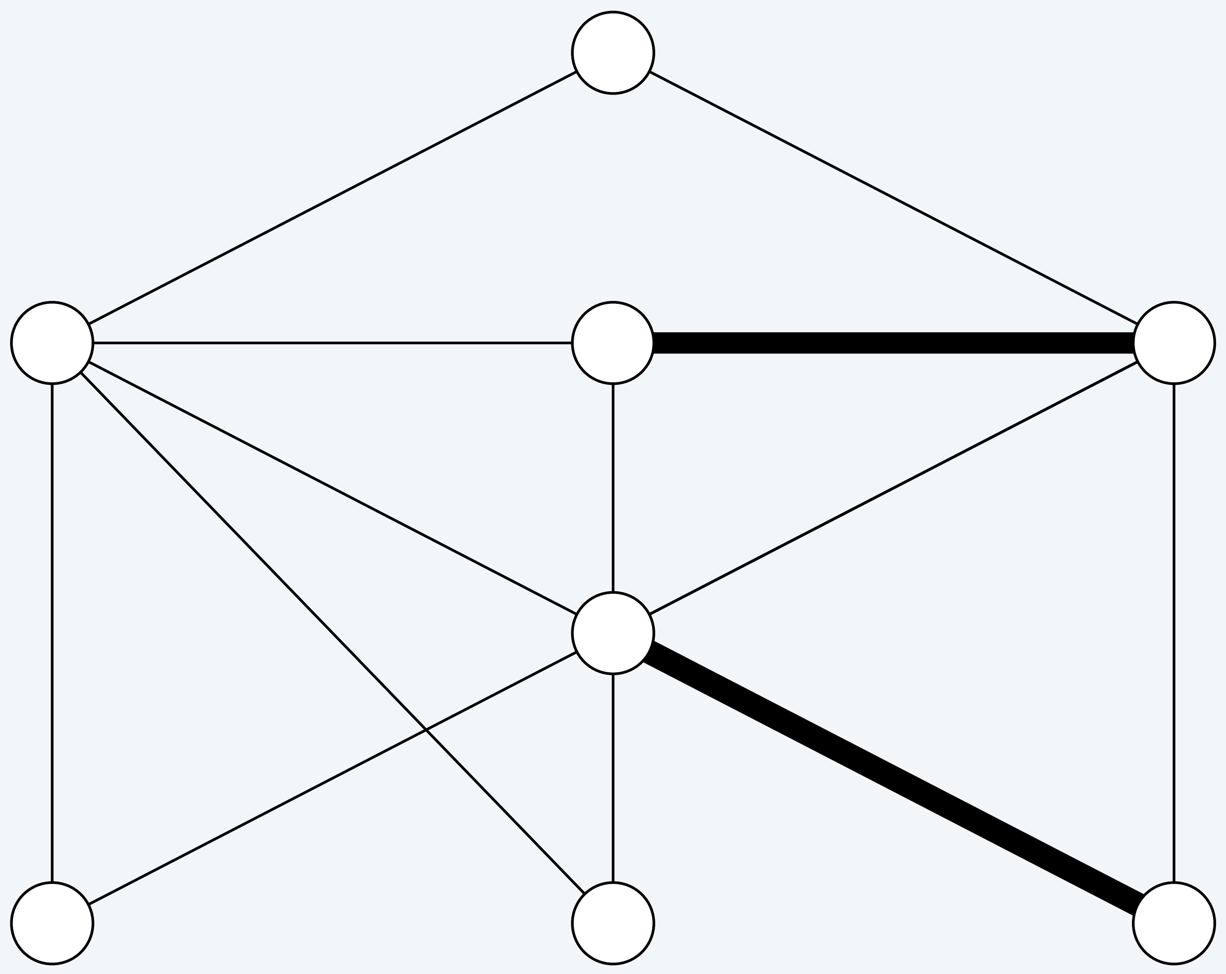
Detour: maximal matching

A matching in a graph G is a set of edges where no two edges share a common endpoint.

A maximal matching is a matching that cannot be extended by including an additional edge.

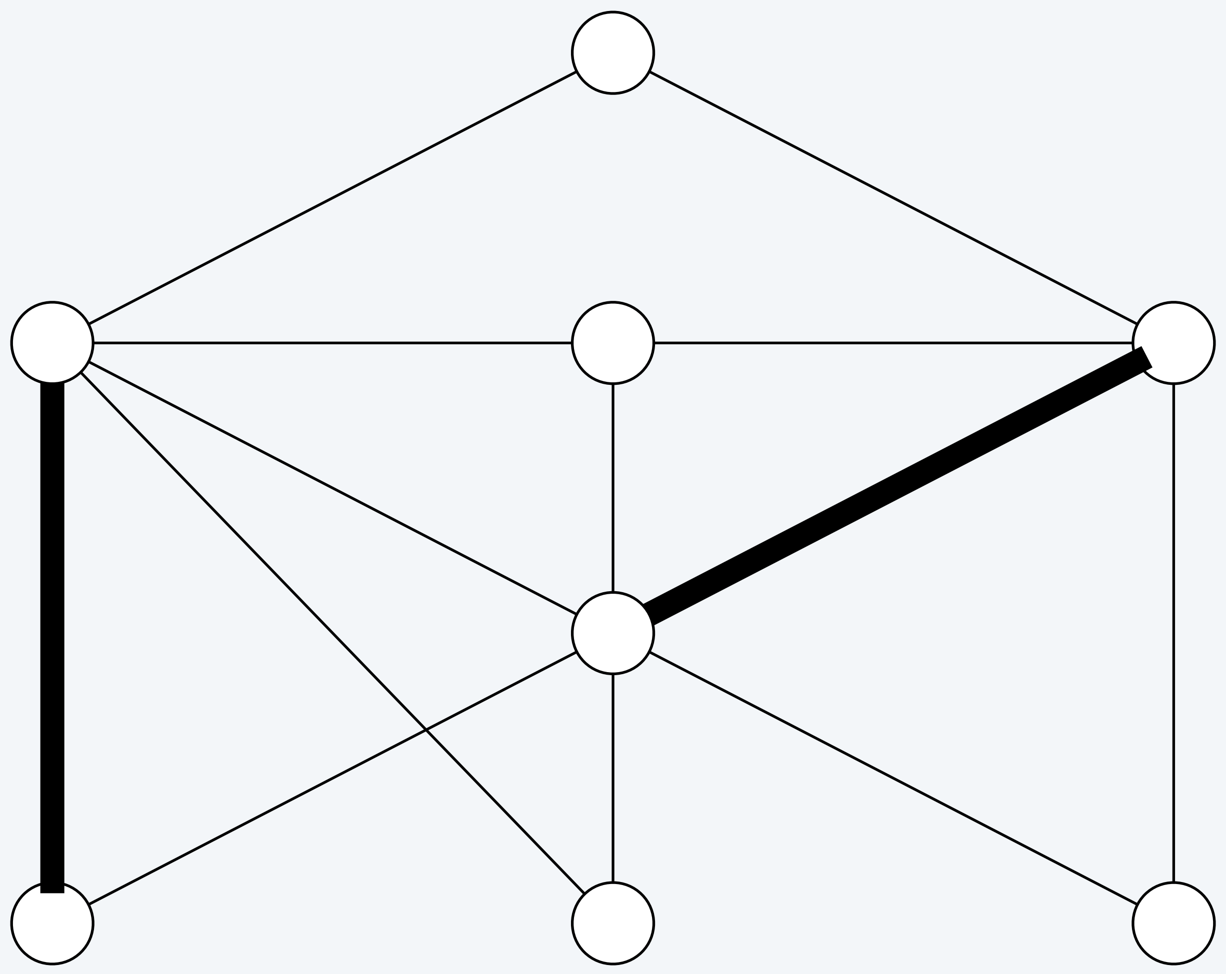


not matching



matching

not maximal



matching

maximal matching (but not perfect)

Greedy algorithm: $M = \varnothing$. Iterate through the edges, adding an edge e to M if neither endpoint of e is shared with any edge already in M. ← $\Theta(E + V)$

Vertex cover: 2-approximation algorithm

Algorithm.

Find a maximal matching M in G.

Return the set S of all endpoints of edges in M.

← $\Theta(E + V)$

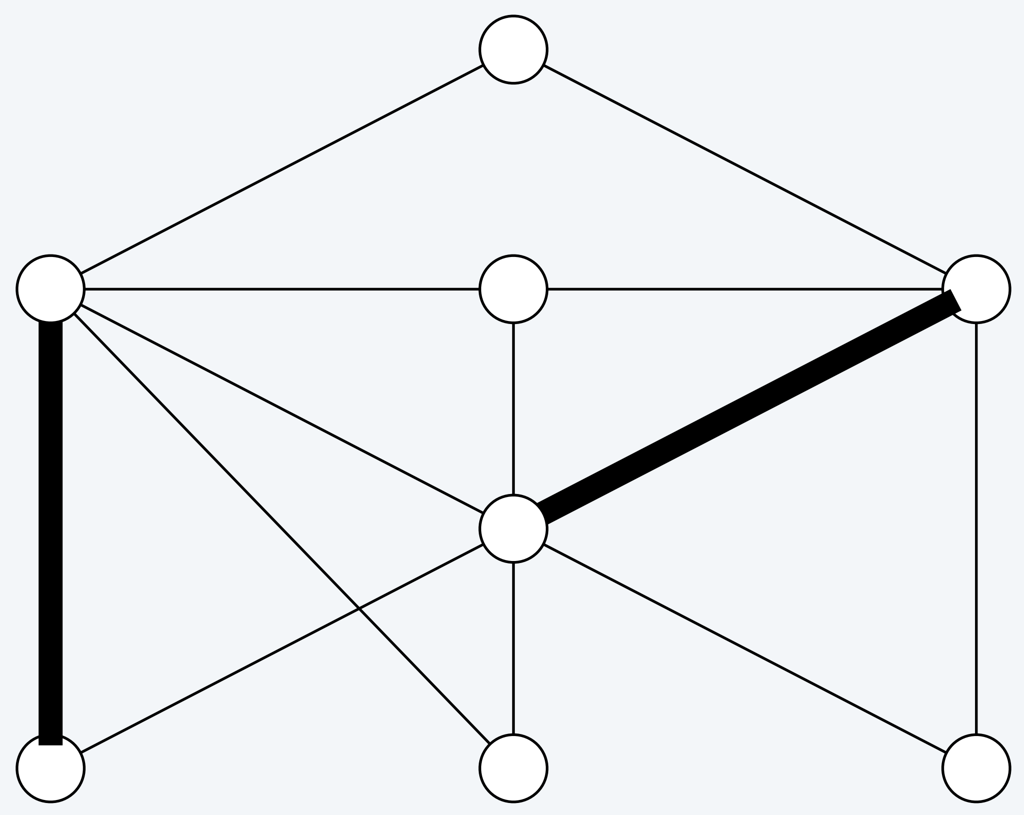
Claim. For every G, the algorithm returns a VC of size $\leq 2 \cdot OPT(G)$.

conjectured to be optimal

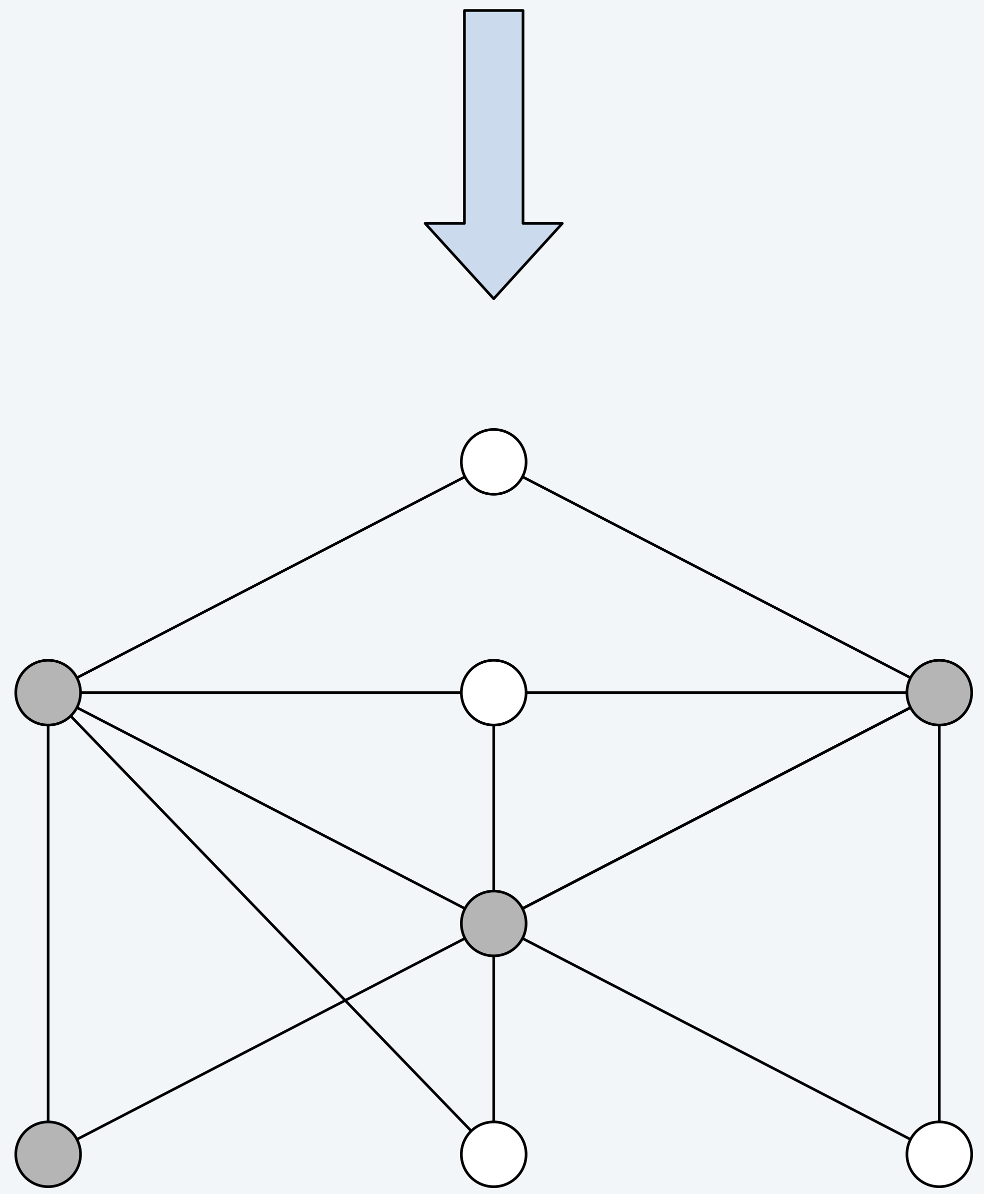
Proof.

S is a VC: Otherwise, there is an edge e with no endpoint in S.
e can be added to M, contradicting M's maximality.

$|S| \leq 2 \cdot OPT(G)$: Let S^* be a minimum VC.
Since M is a matching, the endpoints of all edges in M are distinct.
For every edge in M, at least one of its endpoints is in S^*.
So, $|S| = 2|M| \leq 2|S^*|$.



maximal matching



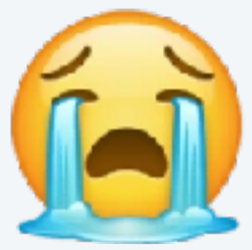
VC

3-SAT: randomized 7/8-approximation algorithm

$OPT(I)$ is the maximum fraction of equations in I that can be satisfied.

often $k = 1$

3-SAT (decision): Given I, k, is $OPT(I) \geq k$? ⟵ **NP**-complete



3-SAT (α–approx): Given I , return an assignment that satisfies $\geq \alpha \cdot OPT(I)$ fraction of the equations.

want $\alpha < 1$ as large as possible (max problem)

Algorithm.

m = # of equations

Generate $100m$ random assignments and return the one that satisfies the most equations. ⟵ *polynomial time*

optimal! (unless **P = NP***)*

Claim. For any I, with probability .99, the returned assignment satisfies $\geq 7/8$ fraction of the equations.

Proof idea. A core observation is that a random assignment satisfied each equation with probability $7/8$.

E.g., "$\neg\, x_5$ *or* x_8 *or* $\neg\, x_9 =$ *true*" is not satisfied only when $x_5 = T, x_8 = F, x_9 = T$, which happens with probability $(1/2)^3$.

assume 3 district variables in an equation

Approximation algorithm

α–approximation algorithm:

For a *minimization* problems: return a solution with value $\leq \alpha \cdot OPT$, $\alpha > 1$.

For a *maximization* problems: return a solution with value $\geq \alpha \cdot OPT$, $\alpha < 1$.

An **NP**–complete problem may admit a *polynomial-time* α–approximation algorithm:

- For no constant α. ⟵ *hard to solve with any precision*
- For some constant α (e.g., 2, 1/2 or 7/8). ⟵ *easy to solve with precision α, hard with better precision*
- For every $\alpha \neq 0, 1$ (PTAS/FPTAS). ⟵ *easy to solve with any precision, hard to solve exactly*

The field of hardness of approximation studies the optimal α achievable for different **NP**–complete problems.

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Leveraging intractability: guiding scientific inquiry

1926. Ising introduces a mathematical model for ferromagnetism.

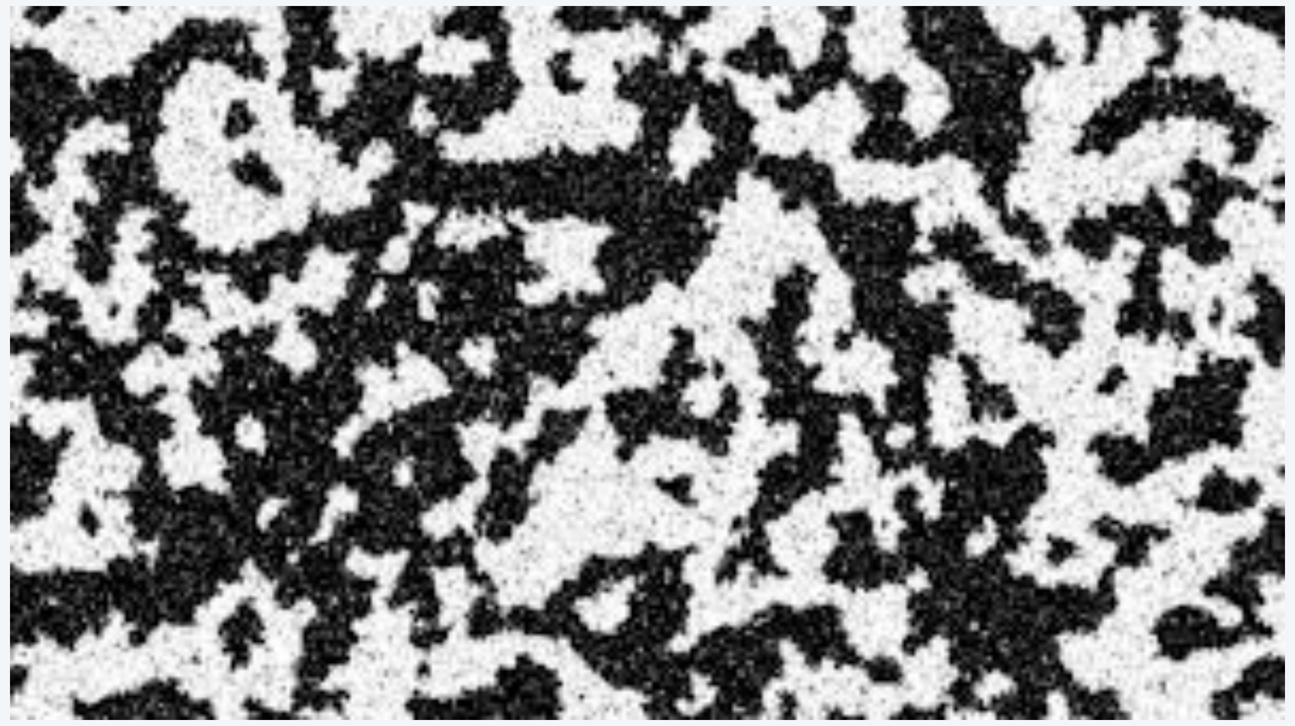
1930s. Closed form solution is a holy grail of statistical mechanics.

1944. Onsager finds closed form solution to 2D version in tour de force.

1950s. Feynman (and others) seek closed form solution to 3D version.

2000. Istrail shows that ISING-3D is **NP**-complete.

Bottom line. Search for a closed formula seems futile.



Leveraging intractability: cryptography

Secure password system. A user creates a password to enable login to their account.

How can the server store the password securely?

Solution. Convert password into two large primes p, q. Server stores only the product $N = pq$.

To log in, user provides p, q. The server computes the product and compares to N.

Server: Multiply two integers (efficient).

Malicious user: Solve factorization (conjectured to be intractable).

Cryptographic schemes (e.g., RSA encryption) require malicious parties to solve intractable (?) **NP** problems.

P = **NP** $\implies$ *no crypto!*

Ron Rivest

Adi Shamir

Len Adelman

MasterCard SecureCode

Verified by VISA

multiply (easy)

$761838257287 \times 193707721$

147573952589676412927

factor (difficult)

Leveraging intractability: derandomization

Fun game. I toss a coin; you guess how it will land. What's the probability you guess correctly? *50%*

Fun game 2. I toss a coin; you can use your computer to guess how it will land.
What's the probability you guess correctly? *still 50%…*

Fun game 3. I toss a coin; you are a Martian with complete knowledge of the physics of the universe and access to sophisticated equipment.
You guess how it will land—what's the probability you guess correctly? *100%?*

Randomness is in the ~~eye~~ of the beholder!
computational power

Hardness vs. Randomness. The outcome of intractable problems often appears random. We can feed such outcomes to randomized algorithms instead of real randomness, thereby making them deterministic.

A final thought

" *Now my general conjecture is as follows: for almost all sufficiently complex types of enciphering, […] the mean key computation length increases exponentially with the length of the key […].*

The nature of this conjecture is such that I cannot prove it […].
Nor do I expect it to be proven. "

— John Nash