https://algs4.cs.princeton.edu

2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [last lecture]

…

Quicksort. [this lecture]

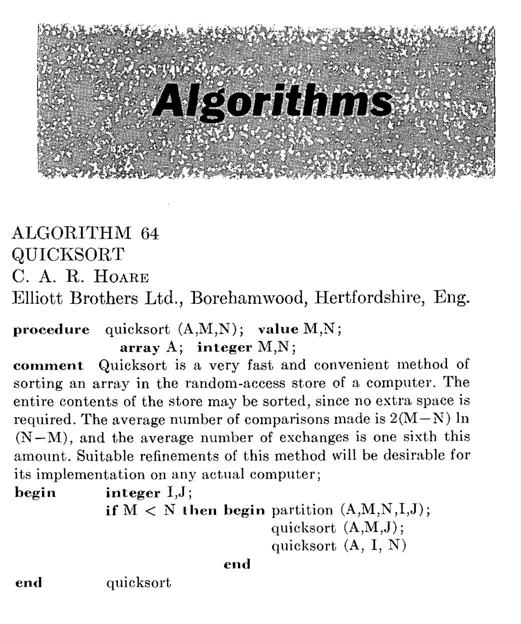
…

A brief history

Tony Hoare.

- Invented quicksort in 1960 to translate Russian into English.
- Later learned Algol 60 (and recursion) to implement it.

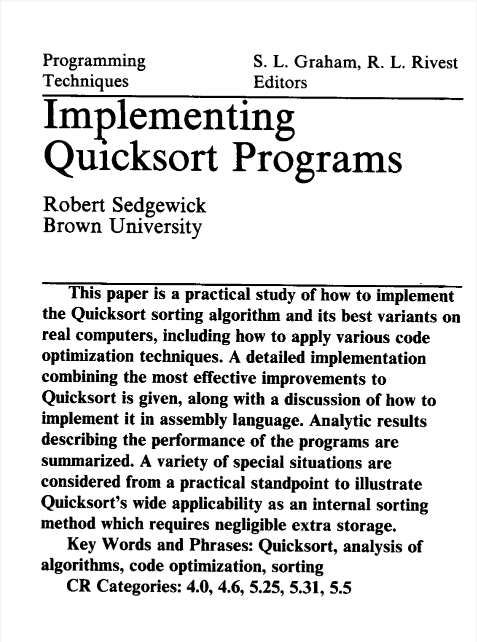
Tony Hoare
1980 Turing Award



Algorithms

ALGORITHM 64
QUICKSORT
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure   quicksort (A,M,N);  value M,N;
              array A;  integer M,N;
comment  Quicksort is a very fast and convenient method of
sorting an array in the random-access store of a computer. The
entire contents of the store may be sorted, since no extra space is
required. The average number of comparisons made is 2(M−N) ln
(N−M), and the average number of exchanges is one sixth this
amount. Suitable refinements of this method will be desirable for
its implementation on any actual computer;
begin       integer I,J;
            if M < N then begin partition (A,M,N,I,J);
                                quicksort (A,M,J);
                                quicksort (A, I, N)
                          end
end         quicksort
```



Programming Techniques — S. L. Graham, R. L. Rivest, Editors

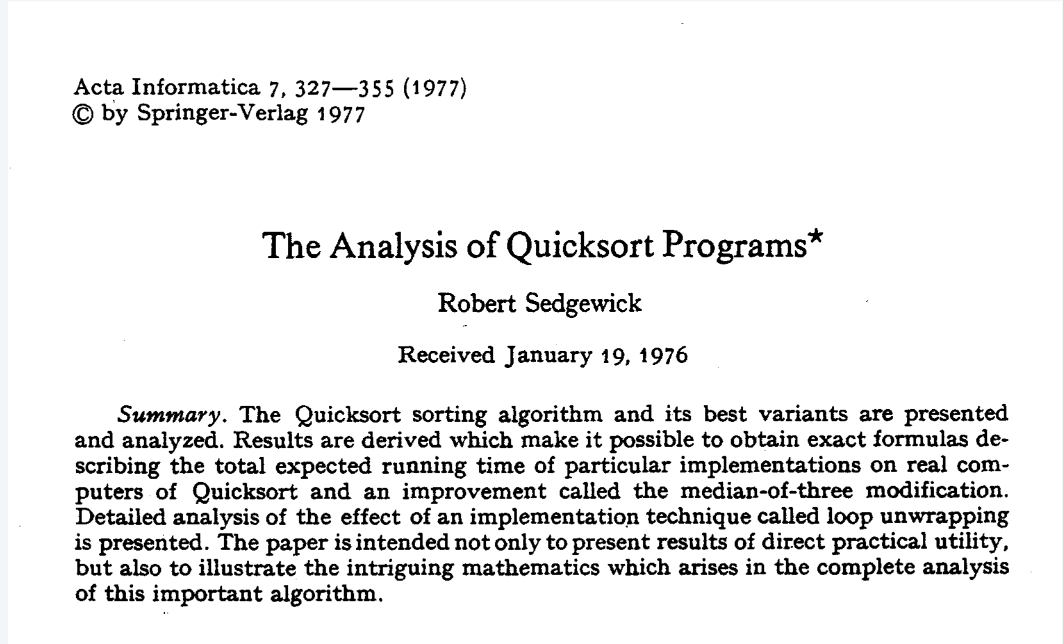
Implementing Quicksort Programs

Robert Sedgewick
Brown University

This paper is a practical study of how to implement the Quicksort sorting algorithm and its best variants on real computers, including how to apply various code optimization techniques. A detailed implementation combining the most effective improvements to Quicksort is given, along with a discussion of how to implement it in assembly language. Analytic results describing the performance of the programs are summarized. A variety of special situations are considered from a practical standpoint to illustrate Quicksort's wide applicability as an internal sorting method which requires negligible extra storage.

Key Words and Phrases: Quicksort, analysis of algorithms, code optimization, sorting

CR Categories: 4.0, 4.6, 5.25, 5.31, 5.5



Acta Informatica 7, 327—355 (1977)
© by Springer-Verlag 1977

The Analysis of Quicksort Programs*

Robert Sedgewick

Received January 19, 1976

Summary. The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.

Bob Sedgewick

Bob Sedgewick.

- Refined and popularized quicksort in 1970s.
- Analyzed many versions of quicksort.

https://algs4.cs.princeton.edu

2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Quicksort overview

Step 1. Shuffle the array.

Step 2. Partition the array so that, for some index `j` :

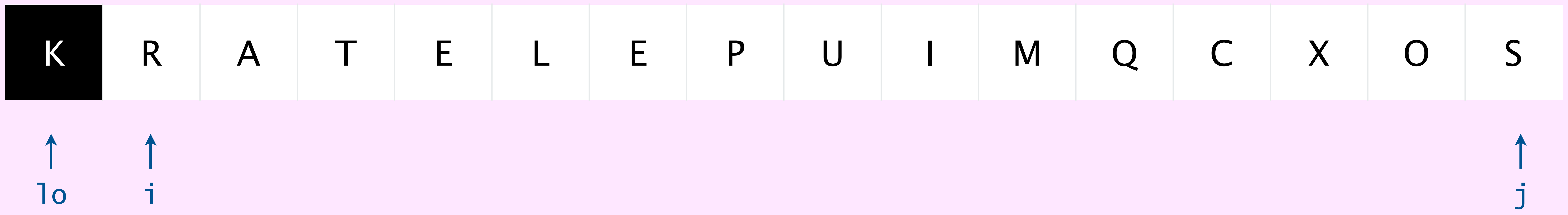
- Entry `a[j]` is in place. ⟵ *"pivot" or "partitioning item"*
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Step 3. Sort each subarray recursively.

Quicksort partitioning demo

Repeat until pointers cross:

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.



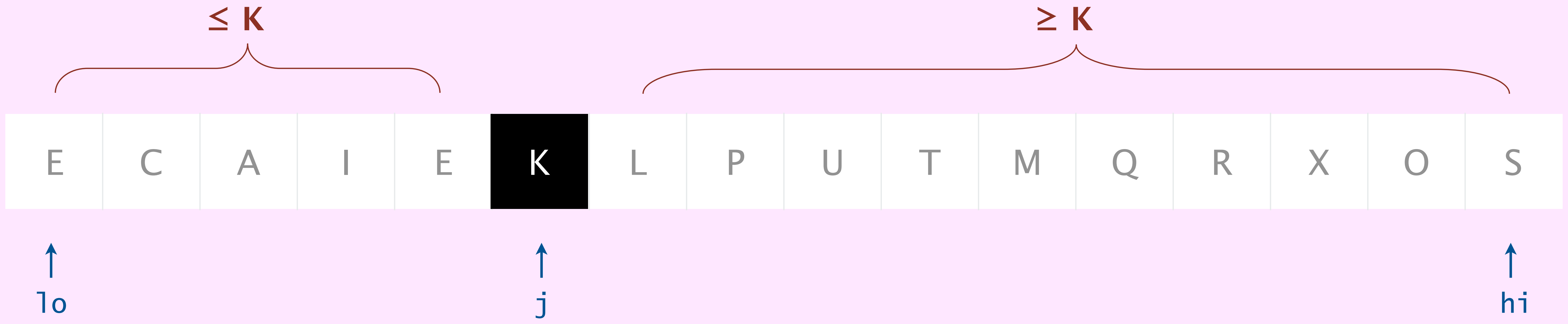
stop i scan because a[i] >= a[lo]

Quicksort partitioning demo

Repeat until pointers cross:

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross. Exchange `a[lo]` with `a[j]`.

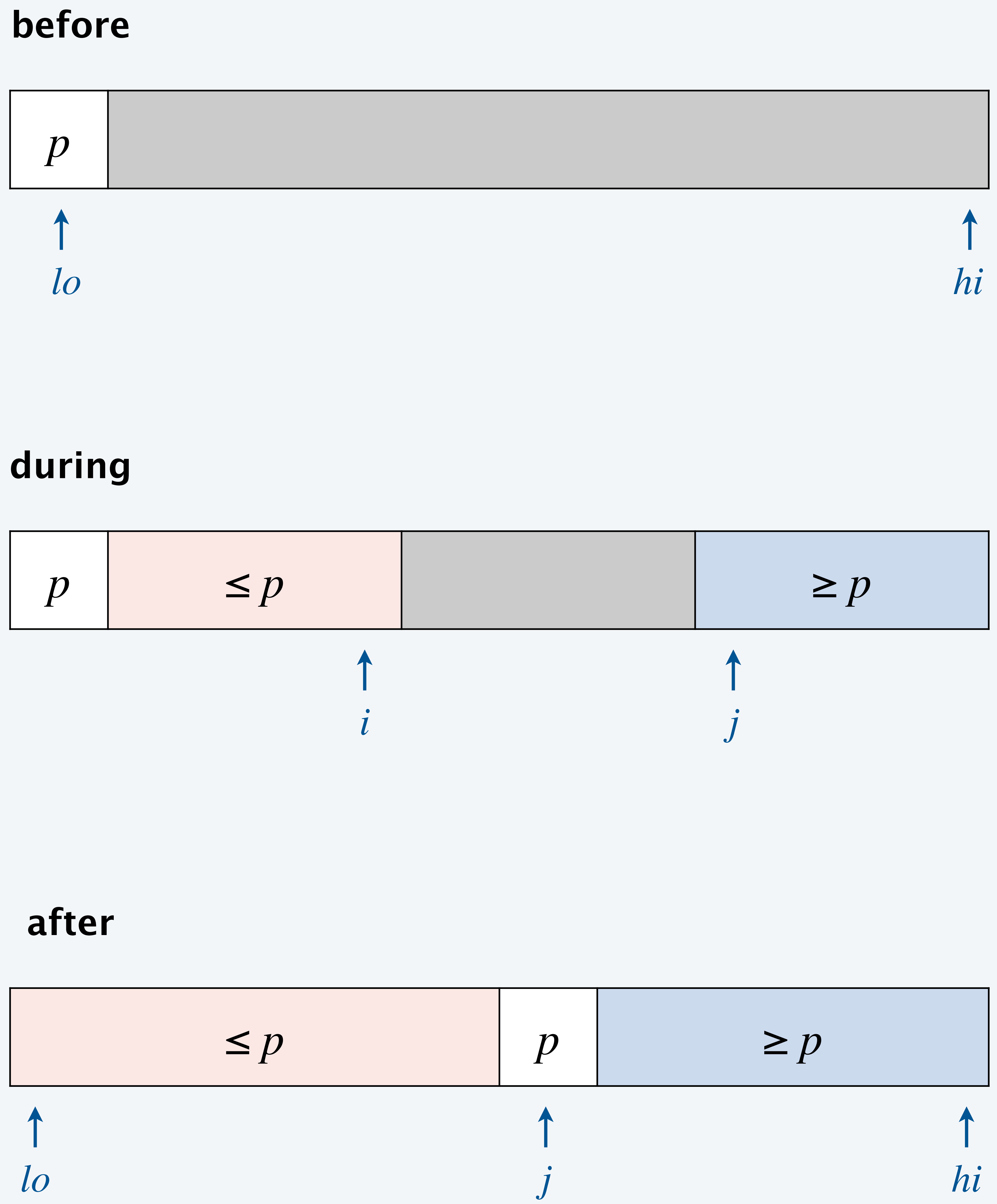


partitioned!

Quicksort partitioning: Java implementation

```java
private static int partition(Comparable[] a, int lo, int hi) {
   Comparable p = a[lo];
   int i = lo, j = hi+1;
   while (true) {
      while (less(a[++i], p))           // find item on left to swap
         if (i == hi) break;

      while (less(p, a[--j]))           // find item on right to swap
         if (j == lo) break;

      if (i >= j) break;                // check if pointers cross

      exch(a, i, j);                    // swap
   }

   exch(a, lo, j);                      // swap with pivot
   return j;                            // index of element known to be in place
}
```

https://algs4.cs.princeton.edu/23quick/Quick.java.html



Quicksort: quiz 2

In the worst case, how many compares and exchanges does `partition()` make to partition a subarray of length n?

A. $\sim \frac{1}{2} n$ and $\sim \frac{1}{2} n$

B. $\sim \frac{1}{2} n$ and $\sim n$

C. $\sim n$ and $\sim \frac{1}{2} n$

D. $\sim n$ and $\sim n$

M	A	B	C	D	E	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10

Quicksort: Java implementation

```
public class Quick {

   private static int partition(Comparable[] a, int lo, int hi) {
      /* see previous slide */
   }

   public static void sort(Comparable[] a) {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi) {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
   }

}
```

shuffle needed for performance guarantee (stay tuned)

https://algs4.cs.princeton.edu/23quick/Quick.java.html

Quicksort trace

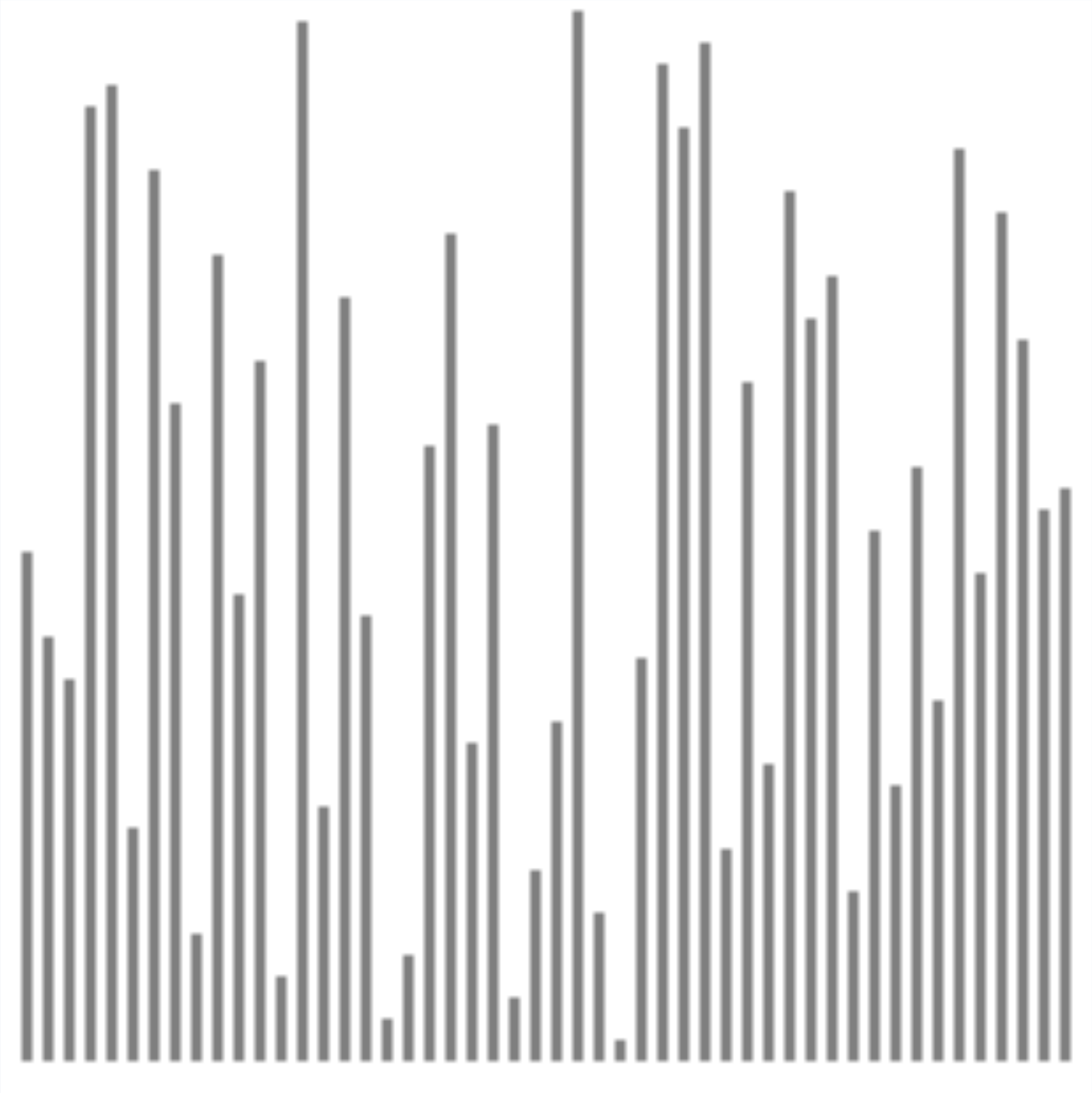
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



https://www.toptal.com/developers/sorting-algorithms/quick-sort

▲ algorithm position

in order

current subarray

not in order

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but it is not worth the cost.

Loop termination. Terminating the loop (when pointers cross) is more subtle than it appears.

Equal keys. Handling duplicate keys is trickier that it appears. [stay tuned]

Preserving randomness. Shuffling is needed for performance guarantee.
Equivalent alternative. Pick a random pivot in each subarray.

not stable!

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (n^2)			mergesort (n log n)			quicksort (n log n)		
computer	**thousand**	**million**	**billion**	**thousand**	**million**	**billion**	**thousand**	**million**	**billion**
home	*instant*	2.8 hours	317 years	*instant*	1 second	18 min	*instant*	0.6 sec	12 min
super	*instant*	1 second	1 week	*instant*	*instant*	*instant*	*instant*	*instant*	*instant*

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: quiz 3

Why is quicksort typically faster than mergesort in practice?

- **A.** Fewer compares.
- **B.** Fewer array acceses.
- **C.** Both A and B.
- **D.** Neither A nor B.

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} n^2$.

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

a[ ]

after random shuffle (second row)

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} n^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

after random shuffle

Good news. Worst case for randomized quicksort is mostly irrelevant in practice.

- Exponentially small chance of occurring.
  (unless bug in shuffling or no shuffling)
- More likely that computer is struck by lightning bolt during execution.

Quicksort: probabilistic analysis

Proposition. The expected number of compares C_n to quicksort an array of n distinct keys is $\sim 2n \ln n$ (and the number of exchanges is $\sim \frac{1}{3} n \ln n$).

Recall. Any algorithm with the following structure takes $\Theta(n \log n)$ time.

```
public static void f(int n) {
    if (n == 0) return;
    f(n/2);          ←  solve two problems
    f(n/2);             of half the size
    linear(n);       ←  do Θ(n) work
}
```

Intuition. Each partitioning step divides the problem into two subproblems, each of approximately one-half the size.

↑ *probabilistically "close enough"*

Quicksort properties

Quicksort analysis summary.

39% more than mergesort

- Expected number of compares is $\sim 1.39\, n \log_2 n$.

  [standard deviation is $\sim 0.65\, n$]
- Expected number of exchanges is $\sim 0.23\, n \log_2 n$. ⟵ *much less than mergesort*
- Min number of compares is $\sim n \log_2 n$. ⟵ *never less than mergesort*
- Max number of compares is $\sim \frac{1}{2} n^2$. ⟵ *but never happens*

Context. Quicksort is a (Las Vegas) randomized algorithm.

- Guaranteed to be correct.
- Running time depends on outcomes of random coin flips (shuffle).

Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

- Partitioning: $\Theta(1)$ extra space.
- Function-call stack: $\Theta(\log n)$ extra space (with high probability).

can guarantee $\Theta(\log n)$ depth by recurring on smaller subarray before larger subarray (but this involves using an explicit stack)

Proposition. Quicksort is not stable.

Pf. [by counterexample]

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, int lo, int hi) {

   if (hi <= lo + CUTOFF - 1) {
      Insertion.sort(a, lo, hi);
      return;
   }

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

$\sim$ 12 / 7 $n \ln n$ *compares* (14% *fewer*)
$\sim$ 12 / 35 $n \ln n$ *exchanges* (3% *more*)

```
private static void sort(Comparable[] a, int lo, int hi) {
   if (hi <= lo) return;

   int median = medianOf3(a, lo, mid + (hi - lo) / 2, hi);
   swap(a, lo, median);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

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2.3 Quicksort

- quicksort
- **selection**
- duplicate keys
- system sorts

Selection

Goal. Given an array of n items, find item of rank k.

Ex. Min $(k = 0)$, max $(k = n - 1)$, median $(k = n / 2)$.

Applications.

- Order statistics.
- Find the "top k."

Use complexity theory as a guide.

- Easy $O(n \log n)$ algorithm. How?
- Easy $O(n)$ algorithm for $k = 0$ or 1. How?
- Easy $\Omega(n)$ lower bound. Why?

Which is true?

- $O(n)$ algorithm? [is there a linear-time algorithm?]
- $\Omega(n \log n)$ lower bound? [is selection as hard as sorting?]

Quickselect demo

Partition array so that for some `j`:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; stop when `j` equals `k`.

select element of rank k = 5

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
50	21	28	65	39	59	56	22	95	12	90	53	32	77	33

k = 5

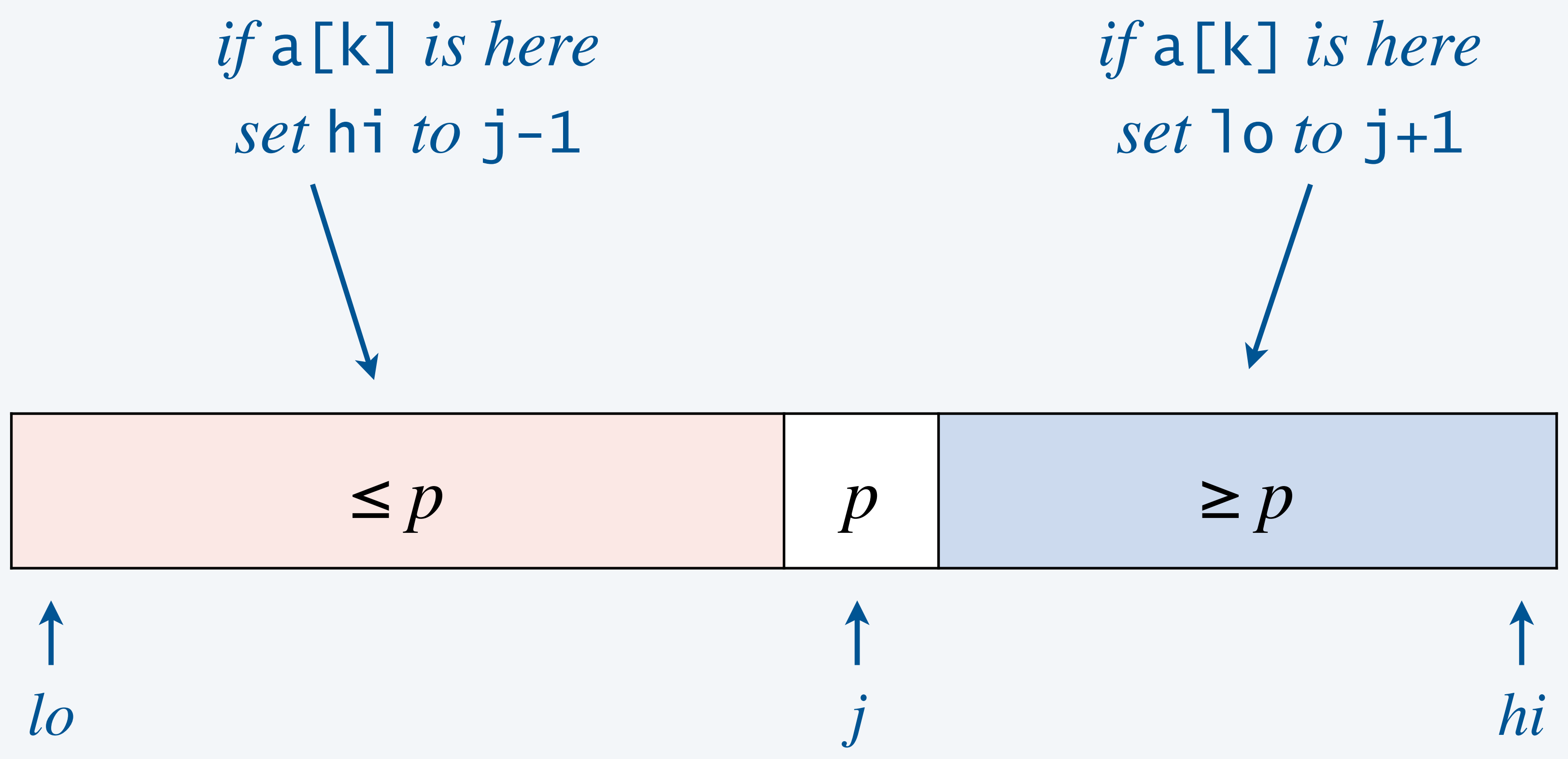
Quickselect

Partition array so that for some `j`:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; stop when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k) {
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo) {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



Quickselect: probabilistic analysis

Proposition. The expected number of compares C_n to quickselect the item of rank k in an array of length n is $\Theta(n)$.

probabilistically "close enough"

Intuition. Each partitioning step approximately halves the length of the array.

Recall. Any algorithm with the following structure takes $\Theta(n)$ time.

```
public static void f(int n) {
   if (n == 0) return;
   linear(n);        ← do Θ(n) work
   f(n/2);           ← solve one subproblem of half the size
}
```

$$n + n/2 + n/4 + \ldots + 1 \sim 2n$$

Careful analysis yields:

$$C_n \sim 2n + 2k \ln(n/k) + 2(n-k)\ln(n/(n-k))$$

$$\leq (2 + 2\ln 2)\, n$$

$$\approx 3.38\, n$$

max occurs for median $(k = n/2)$

Theoretical context for selection

Q. Compare-based selection algorithm that makes $\Theta(n)$ compares in the worst case?

A. Yes! [ingenious divide-and-conquer]

> Time Bounds for Selection*
>
> Manuel Blum, Robert W. Floyd, Vaughan Pratt, Ronald L. Rivest, and Robert E. Tarjan
>
> *Department of Computer Science, Stanford University, Stanford, California 94305*
>
> Received November 14, 1972
>
> The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm—PICK. Specifically, no more than $5.430\dot{5}\, n$ comparisons are ever required. This bound is improved for extreme values of i, and a new lower bound on the requisite number of comparisons is also proved.

$$T(n) = T(n/5) + T(7n/10) + \Theta(n)$$

$T(n/5)$: *find pivot*; $T(7n/10)$: *that eliminates 30% of items*

Caveat. Constants are high $\Rightarrow$ not used in practice.

Use theory as a guide.

- Open problem: practical algorithm that makes $\Theta(n)$ compares in the worst case.
- Until one is discovered, use quickselect (if you don't need a full sort).

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2.3 Quicksort

- quicksort
- selection
- *duplicate keys*
- system sorts

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

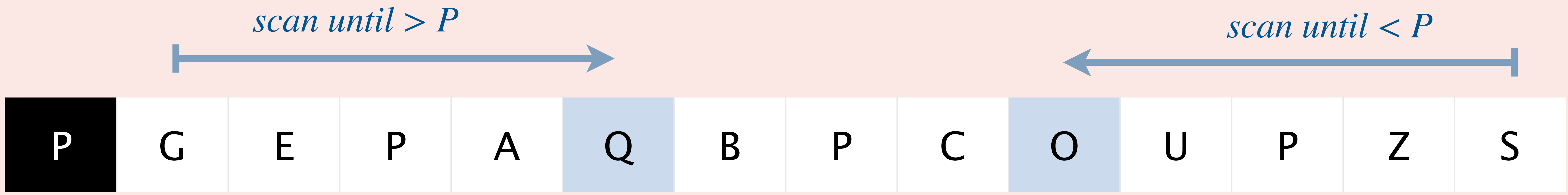
```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

↑ *key*

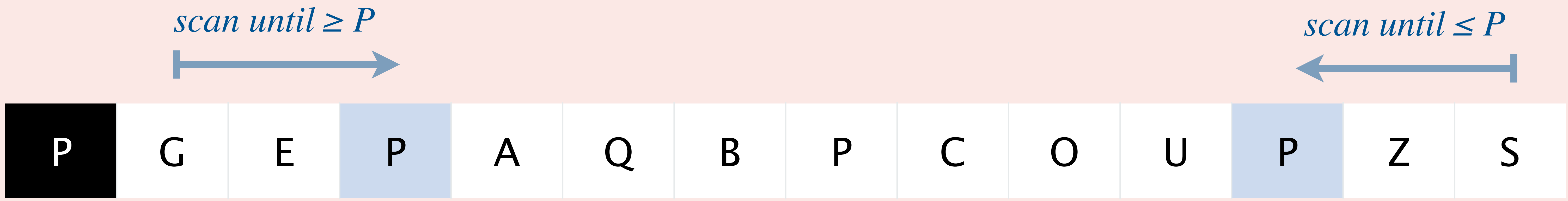
Quicksort: quiz 4

When partitioning, how to handle keys equal to pivot?

A.



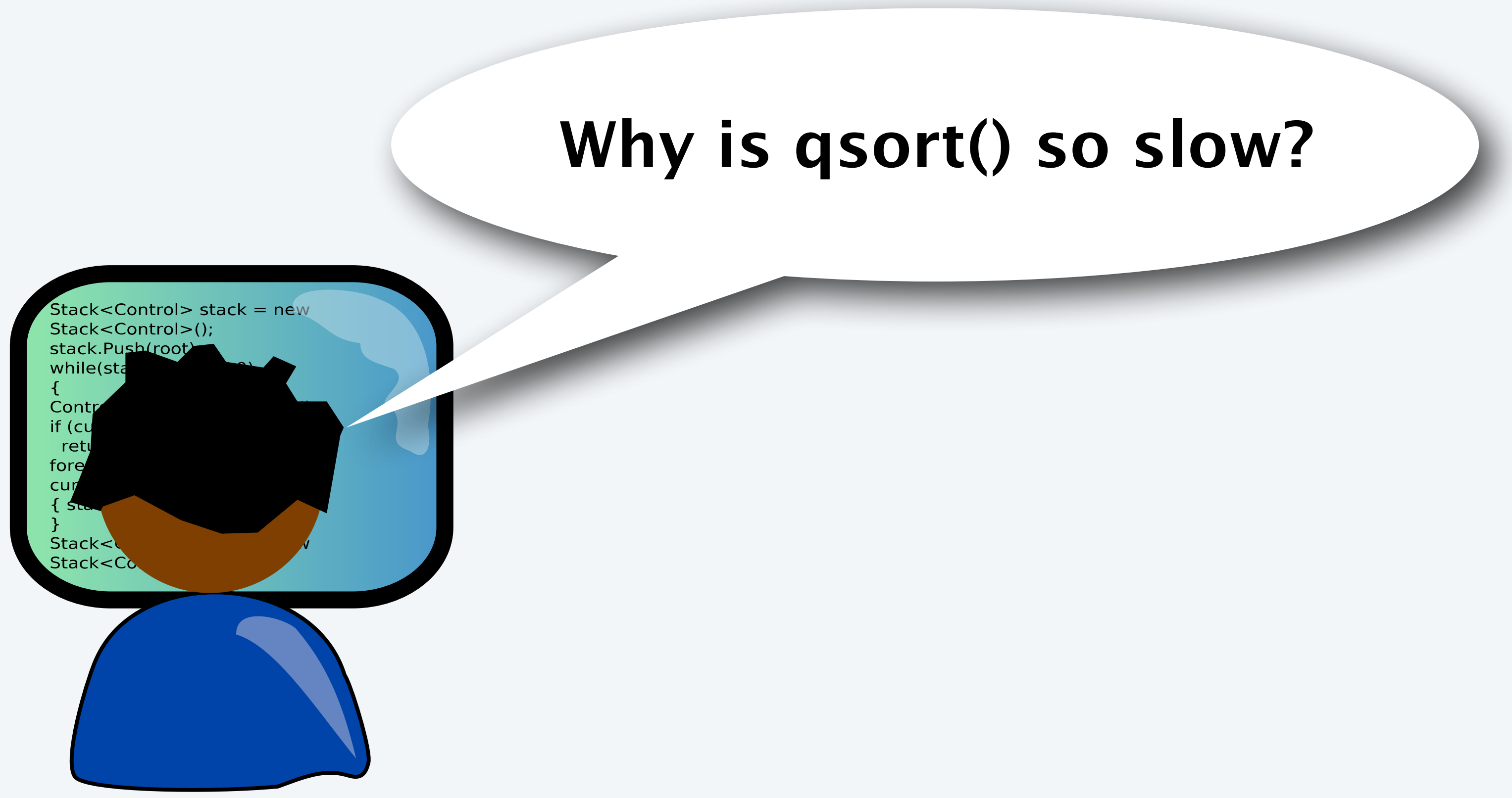
B.

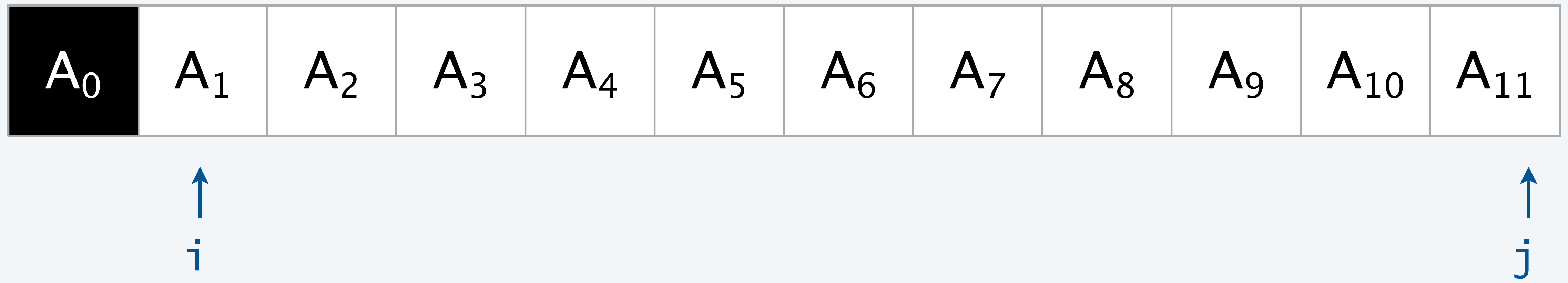


C. Either A or B.

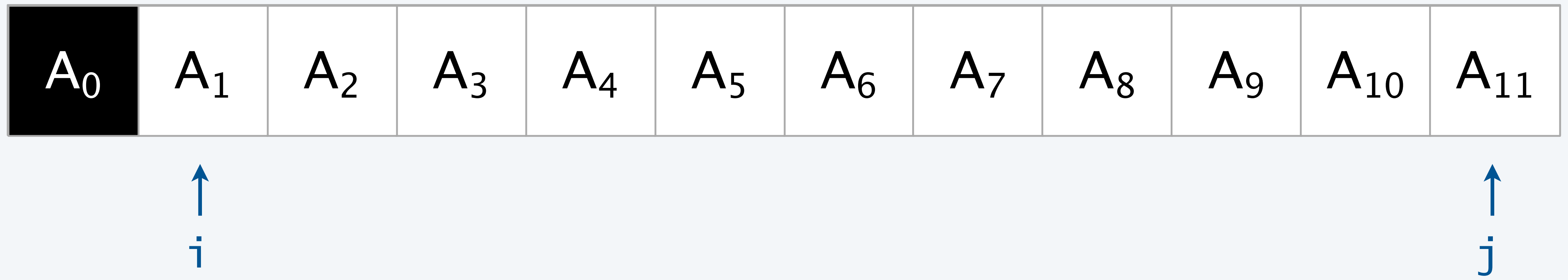
War story (system sort in C)

Bug. A `qsort()` call in C that should have taken seconds was taking minutes to sort a random array of 0s and 1s.





skip over equal keys



stop scan on equal keys

Duplicate keys: partitioning strategies

Bad. Don't stop scans on equal keys.

[$\Theta(n^2)$ compares when all keys equal]

B A A B A B B [B] C C C   A A A A A A A A A A [A]

Good. Stop scans on equal keys.

[$\sim n \log_2 n$ compares when all keys equal]

B A A B A [B] C C B C B   A A A A A [A] A A A A A

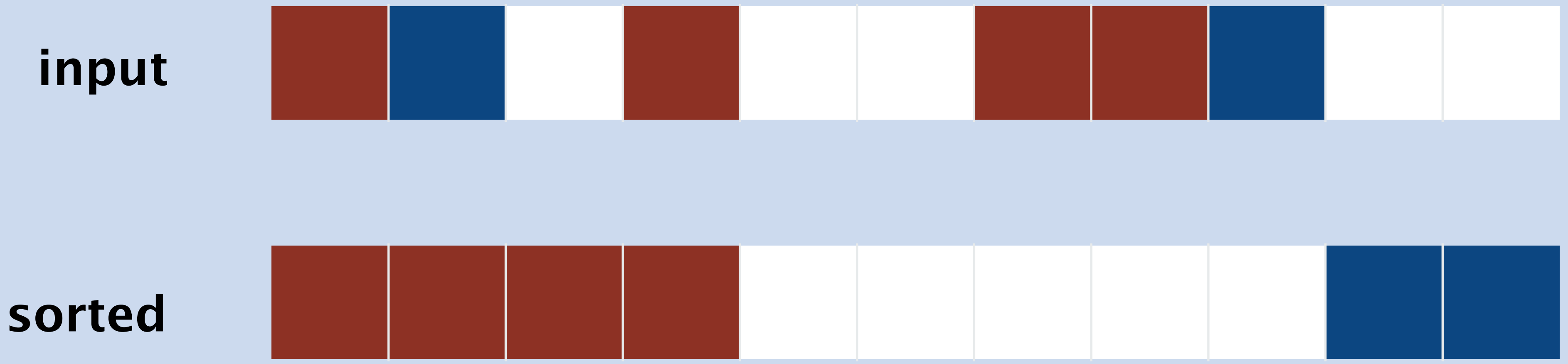
Better. Put all equal keys in place. How?

[$\sim n$ compares when all keys equal]

A A A [B B B B B] C C C   [A A A A A A A A A A A]

Dutch National Flag Problem

Problem. [Edsger Dijkstra] Given an array of n buckets, each containing a red, white, or blue pebble, sort them by color.



Operations allowed.

- $swap(i, j)$: swap the pebble in bucket i with the pebble in bucket j.
- $getColor(i)$: determine the color of the pebble in bucket i.

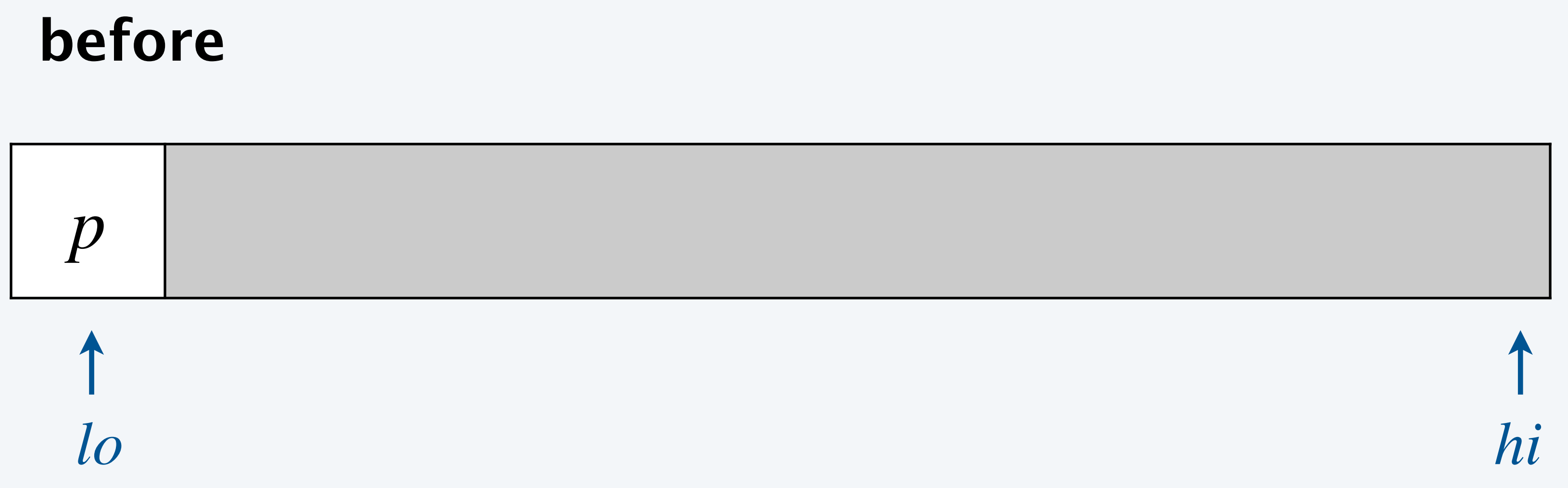
Performance requirements.

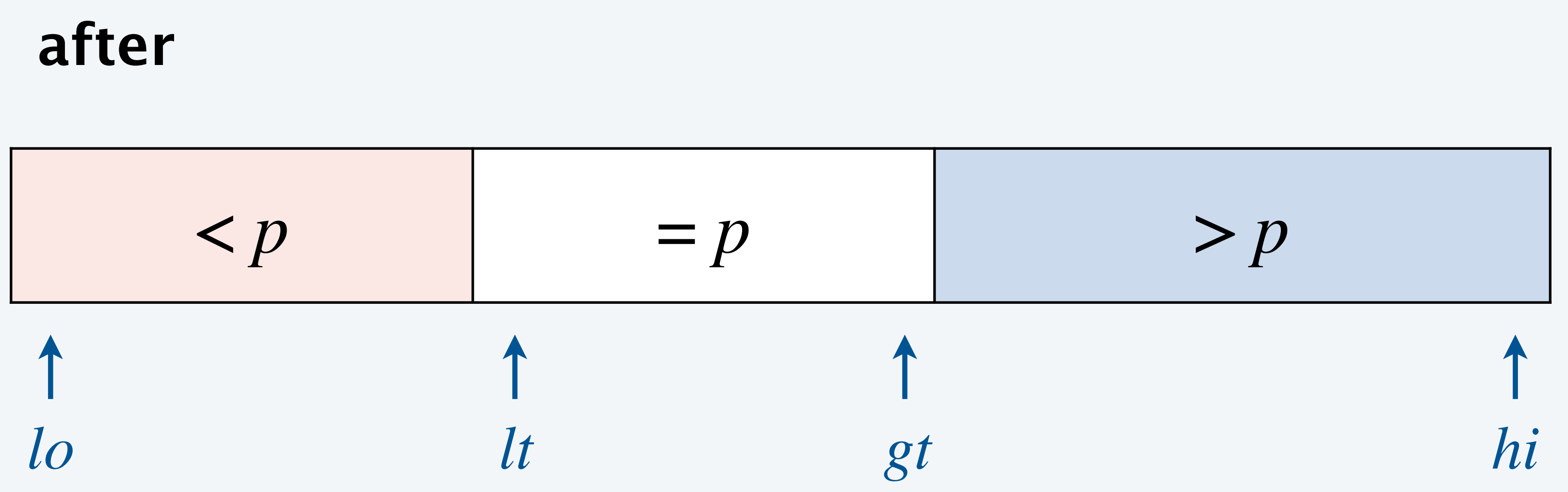
- Exactly n calls to $getColor()$.
- At most n calls to $swap()$.
- $\Theta(1)$ extra space.

3-way partitioning

Goal. Use pivot `p = a[lo]` to partition array into three parts so that:

- Red: smaller entries to the left of `lt`.
- White: equal entries between `lt` and `gt`.
- Blue: larger entries to the right of `gt`.





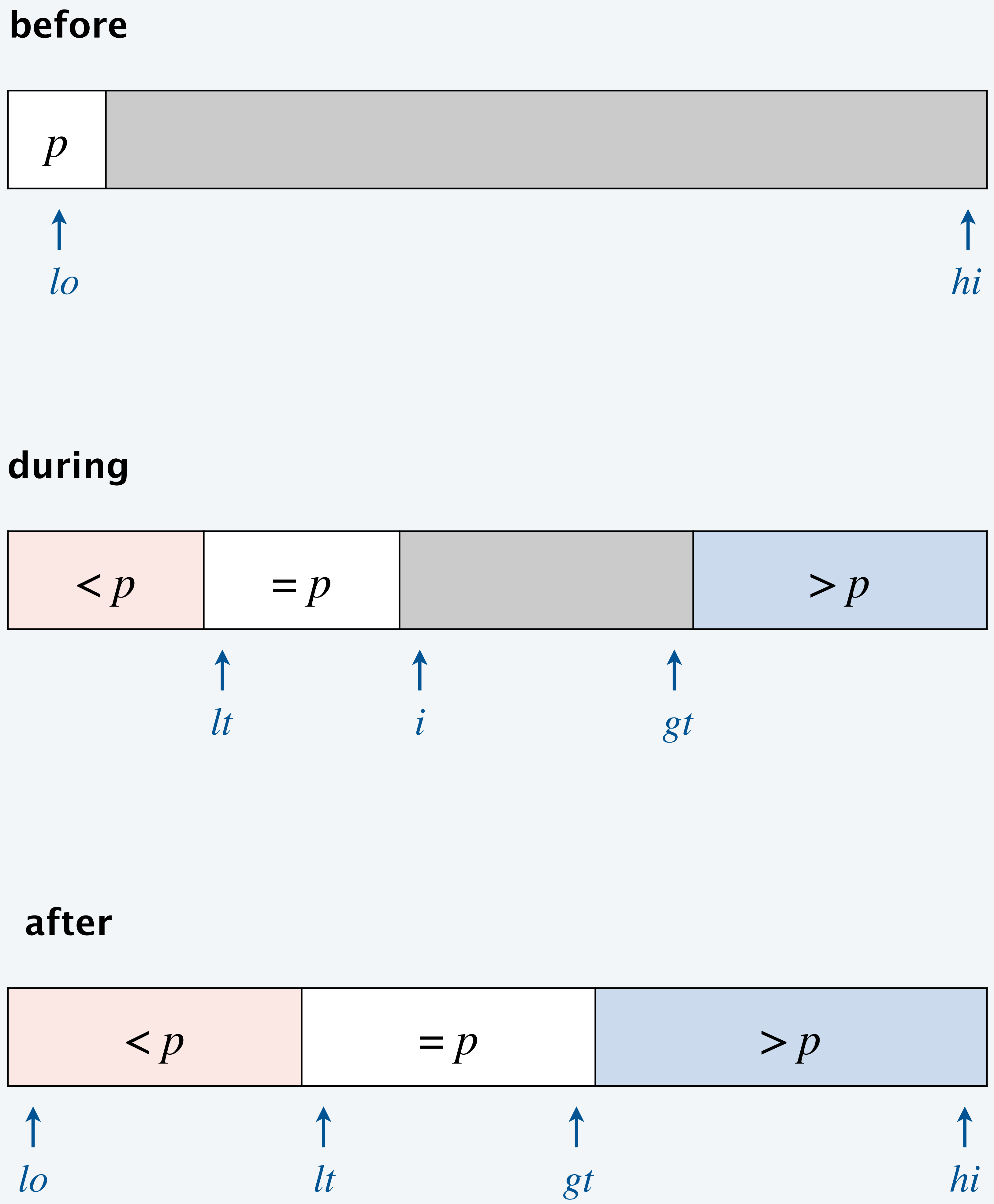
Dijkstra's 3-way partitioning algorithm: demo

- Let `p = a[lo]` be pivot.
- Scan `i` from left to right and compare `a[i]` to `p`.
  - less: exchange `a[i]` with `a[lt]`; increment both `lt` and `i`
  - greater: exchange `a[i]` with `a[gt]`; decrement `gt`
  - equal: increment `i`

lo lt	i											gt	hi
P_1	D	B	X	W	P_2	P_3	V	P_4	A	P_5	C	Y	Z

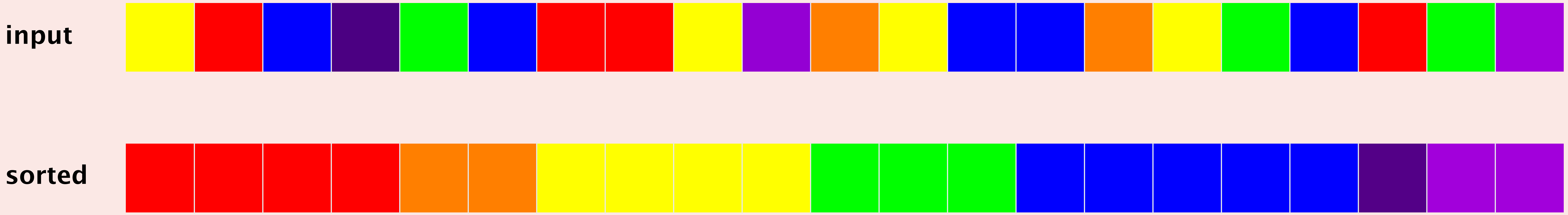
3-way quicksort: Java implementation

```java
private static void sort(Comparable[] a, int lo, int hi) {
   if (hi <= lo) return;
   Comparable p = a[lo];

   int lt = lo, gt = hi;
   int i = lo + 1;
   while (i <= gt) {
      int cmp = a[i].compareTo(p);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



What is the worst-case number of compares to 3-way quicksort an array of length n containing only 7 distinct values?

- **A.** $\Theta(n)$
- **B.** $\Theta(n \log n)$
- **C.** $\Theta(n^2)$
- **D.** $\Theta(n^7)$



Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2}n^2$	$\frac{1}{2}n^2$	$\frac{1}{2}n^2$	*n exchanges*
insertion	✔	✔	n	$\frac{1}{4}n^2$	$\frac{1}{2}n^2$	*use for small n or partially sorted arrays*
merge		✔	$\frac{1}{2}n\log_2 n$	$n\log_2 n$	$n\log_2 n$	$\Theta(n \log n)$ *guarantee; stable*
quick	✔		$n\log_2 n$	$2n\ln n$	$\frac{1}{2}n^2$	$\Theta(n \log n)$ *probabilistic guarantee; fastest in practice*
3-way quick	✔		n	$2n\ln n$	$\frac{1}{2}n^2$	*improves quicksort when duplicate keys*
?	✔	✔	n	$n\log_2 n$	$n\log_2 n$	*holy sorting grail*

number of compares to sort an array of n elements

https://algs4.cs.princeton.edu

2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

← *obvious applications*

- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.

← *problems become easy once items are in sorted order*

- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.

. . .

← *non-obvious applications*

System sort

Premise. Suppose you are the lead architect of a new programming language.

Q. Which sorting algorithm(s) would you use for the system sort? Defend your answer.

System sorts in Java 8 and Java 11

`Arrays.sort()` and `Arrays.parallelSort()`.

- Has one method for `Comparable` objects.
- Has an overloaded method for each primitive type.
- Has an overloaded method for use with a `Comparator`.
- Has overloaded methods for sorting subarrays.

Java

Algorithms.

- Version of mergesort (Timsort) for reference types.
- Version of quicksort (Dual-pivot quicksort) for primitive types.
- Parallel mergesort for `Arrays.parallelSort()`.

Q. Why use different algorithms for primitive and reference types?

Bottom line. Use the system sort!

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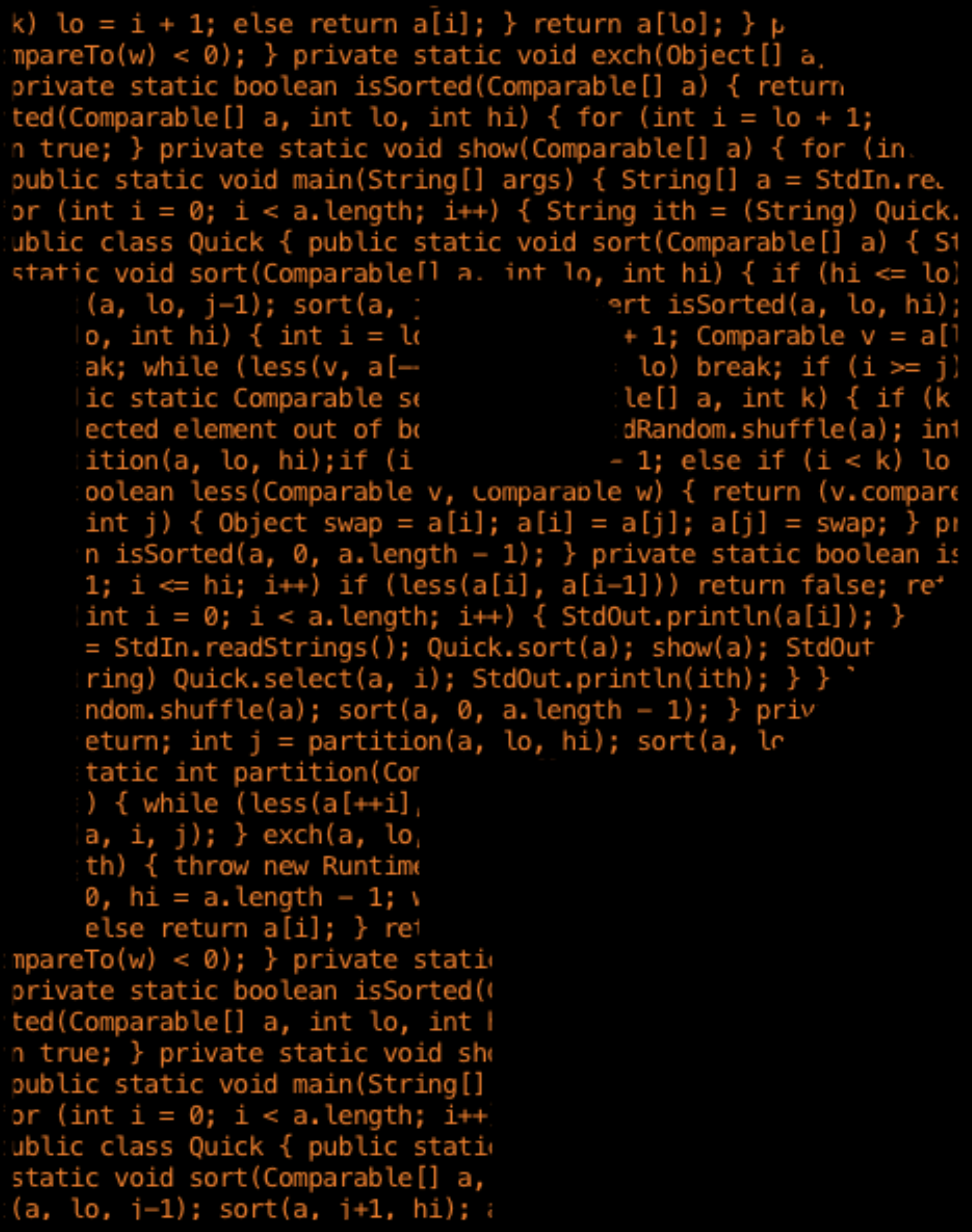
A final thought



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