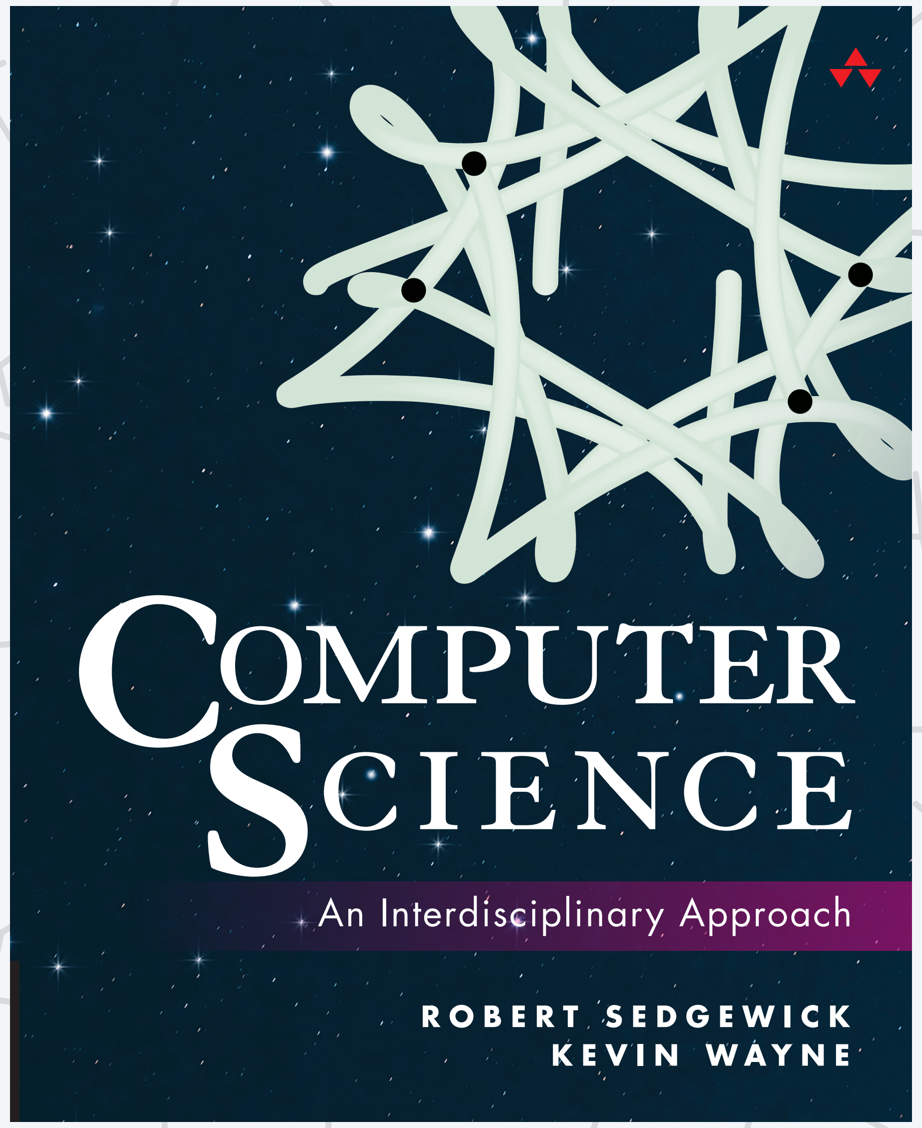


https://introcs.cs.princeton.edu

7. Digital Circuits

- *boolean algebra*
- *logic gates*
- *adder circuit*

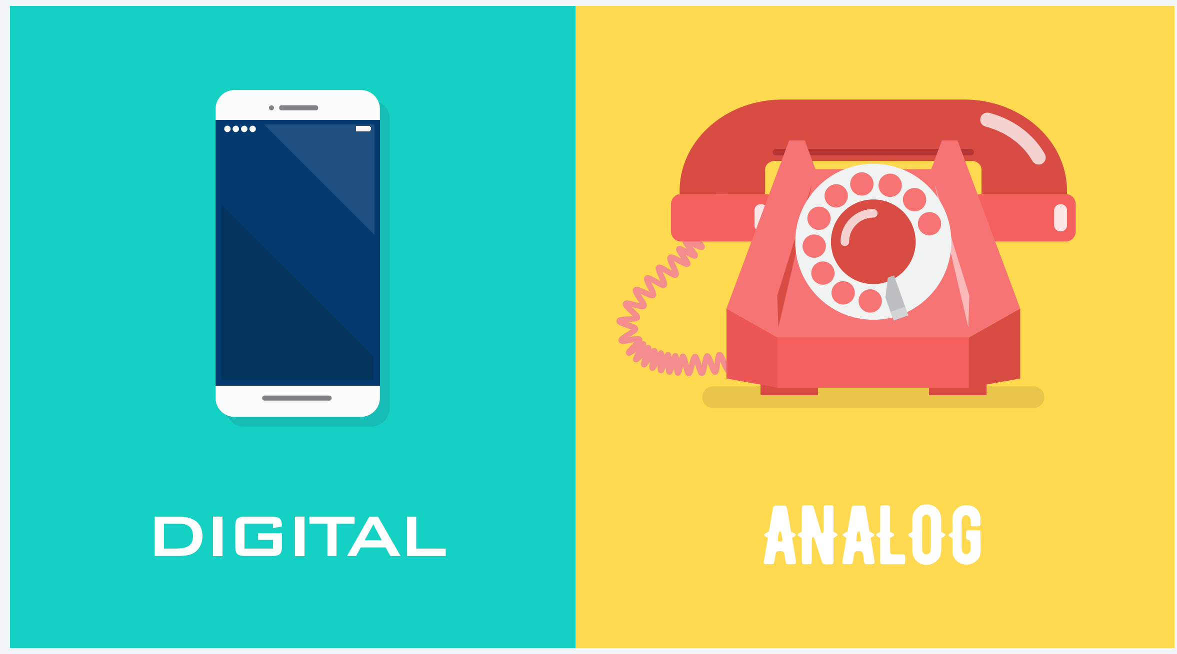
Context

Q. How are computers built?

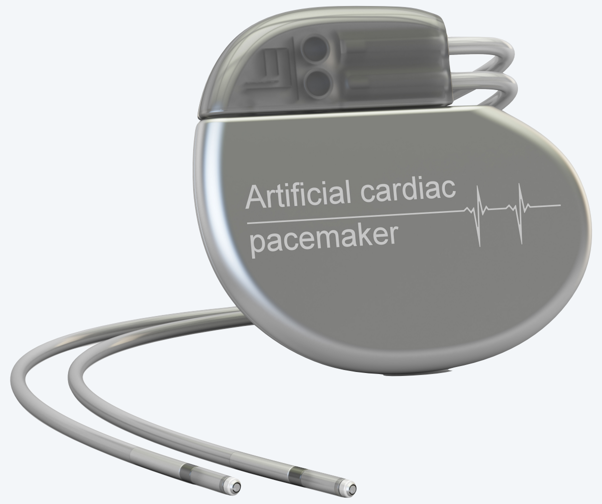
A. Not nearly as complicated as you might think.

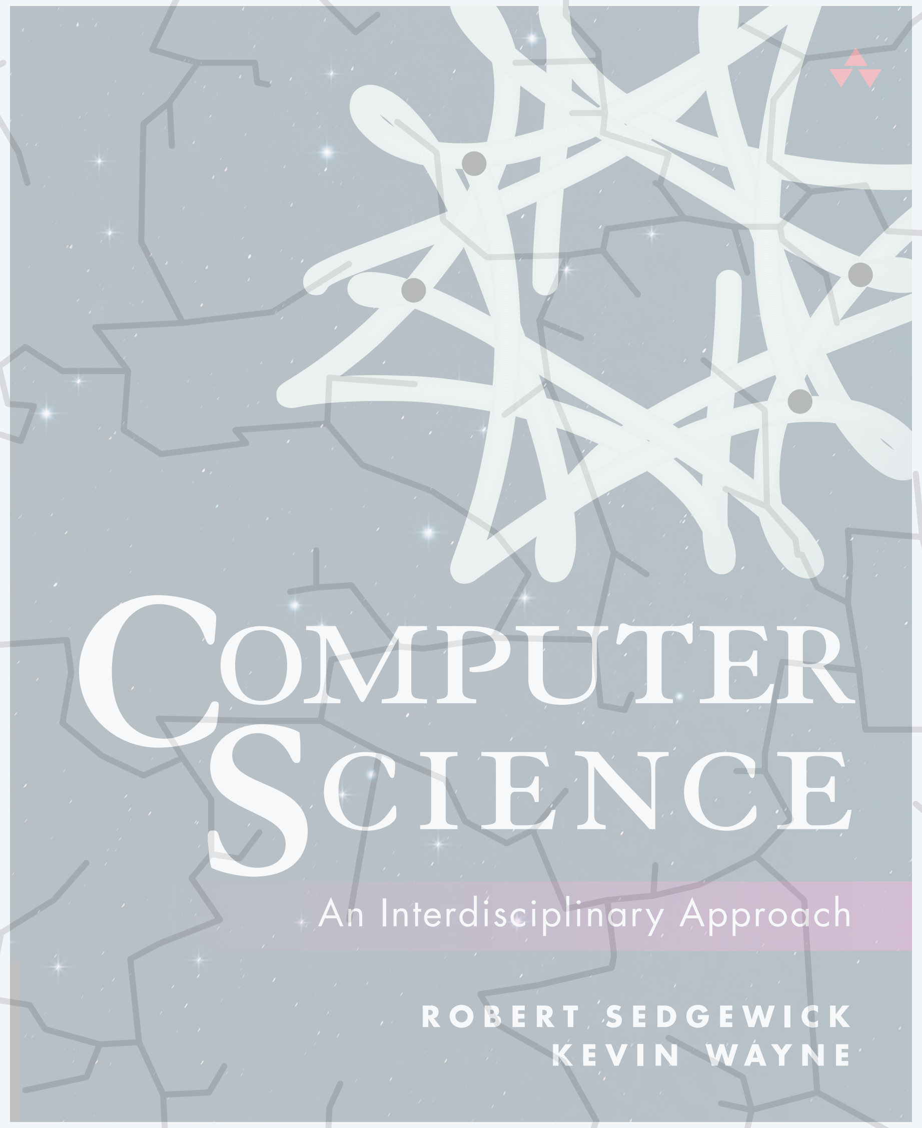
This lecture. Introduction to digital circuits.

- Digital = all signals are either 0 or 1.
- Analog = signals vary continuously.
- Advantages of digital: accurate, reliable, fast, cheap, scalable, …



Applications. Laptop, smartphone, gaming console, pacemaker, microprocessor, …





https://introcs.cs.princeton.edu

7. Digital Circuits

- *boolean algebra*
- *logic gates*
- *adder circuit*

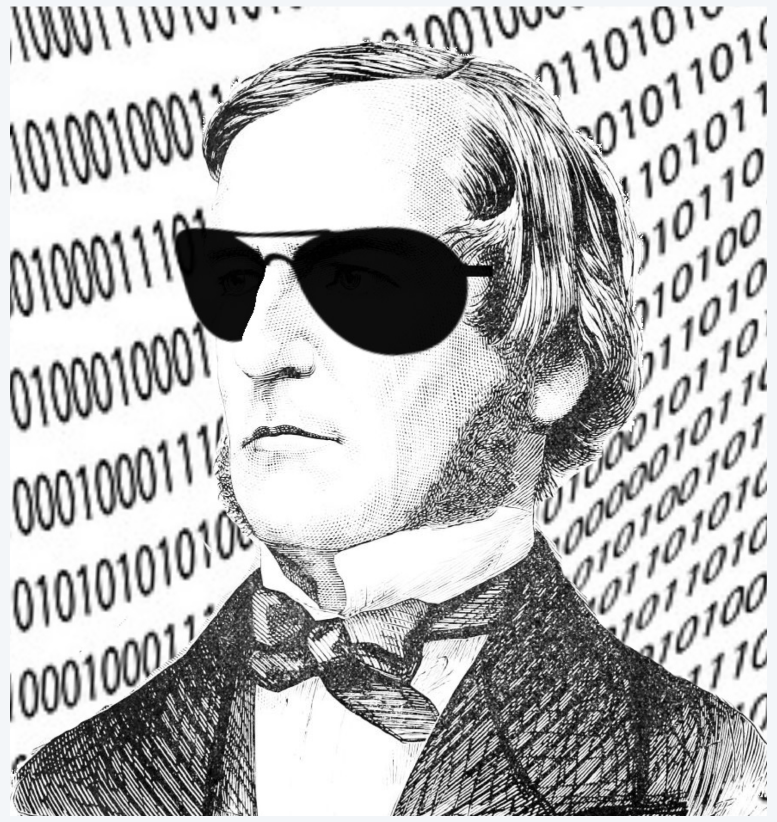
Boolean algebra

Boolean algebra. Developed by George Boole in 1840s to study logic problems.

- Values of variables are *true* (1) or *false* (0).
- Primitive operations are *NOT*, *AND*, and *OR*.
- Widely used in mathematics, logic, computer science, …

operation	logic notation	Java notation	circuit notation	precedence
NOT	$\neg$ x	`!x`	x'	*highest*
AND	x $\wedge$ y	`x && y`	$x \cdot y$	*middle*
OR	x $\vee$ y	`x || y`	$x + y$	*lowest*

this lecture



George Boole is Coole

Relevance to circuits. Provides the mathematical foundation.

Truth tables

Boolean function. A function whose arguments and result assume the values 0 and 1.

Truth table. A systematic way to define a boolean function.

- One row for each possible assignment of arguments.
- Each row gives the function value for the specified arguments.
- The truth table of a boolean function of n variables has 2^n rows.

x	x'
0	1
1	0

NOT

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

AND

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

OR

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

count in binary from 0 to $2^n - 1$

Boolean algebra properties

Boolean algebra shares many properties with elementary algebra. ← *justifies use of $\cdot$ and + for AND and OR*

	property	AND	OR	
axioms	*commutative*	$x \cdot y \;=\; y \cdot x$	$x + y \;=\; y + x$	*same as elementary algebra*
	associative	$x \cdot (y \cdot z) \;=\; (x \cdot y) \cdot z$	$x + (y + z) \;=\; (x + y) + z$	
	identity	$x \cdot 1 \;=\; x$	$x + 0 \;=\; x$	
	distributive	$x \cdot (y + z) \;=\; (x \cdot y) + (x \cdot z)$	$x + (y \cdot z) \;=\; (x + y) \cdot (x + z)$	*different from elementary algebra*
	complementary	$x \cdot x' \;=\; 0$	$x + x' \;=\; 1$	
theorems	*idempotent*	$x \cdot x \;=\; x$	$x + x \;=\; x$	
	De Morgan	$(x \cdot y)' \;=\; x' + y'$	$(x + y)' \;=\; x' \cdot y'$	
	duality	*in any law, can interchange* + *and* $\cdot$, *along with* 0 *and* 1		
	⋮	⋮		

Proving a theorem in Boolean algebra

Q. How to prove a theorem, such as De Morgan's law?

A1. Apply sequence of axioms or known theorems.

A2. For each possible assignment of truth values to variables, evaluate the purported theorem; confirm that it is *true*. ← *"method of perfect induction"*

Ex. De Morgan's law: $(x \cdot y)' = (x' + y')$.

x	y	$x \cdot y$	$(x \cdot y)'$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

truth table for LHS

x	y	x'	y'	$x' + y'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

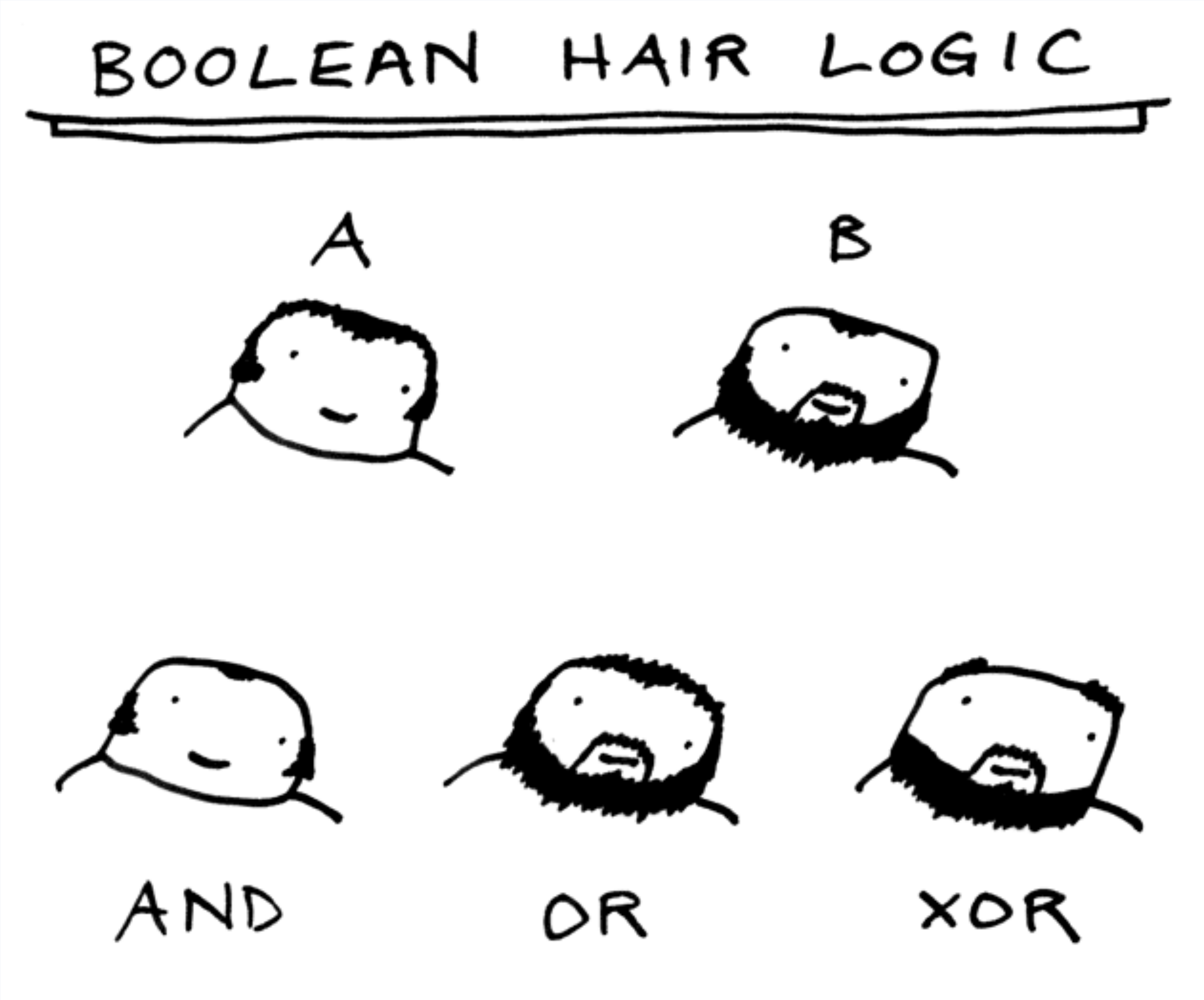
truth table for RHS

Boolean functions of two variables

Boolean function. A function whose arguments and result assume the values 0 and 1.

x	y	*AND*	*OR*	*NAND*	*NOR*	*XOR*
0	0	0	0	1	1	0
0	1	0	1	1	0	1
1	0	0	1	1	0	1
1	1	1	1	0	0	0

commonly used boolean functions of 2 variables



Boolean functions of three (and more) variables

Boolean function. A function whose arguments and result assume the values 0 and 1.

x	y	z	*AND*	*OR*	*MAJ*	*ODD*
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	1	1

some boolean functions of 3 variables

function	shorthand	description
logical AND	*AND*	all inputs are 1
logical OR	*OR*	any input is 1
majority	*MAJ*	more inputs are 1 than 0
odd parity	*ODD*	odd number of inputs are 1

these functions all extends to n variables

Sum-of-products

Sum-of-products. Every boolean function can be represented as a sum of products.

- Products: form an *AND* term for each 1 in truth table.
- Sum: combine the terms with the *OR* function.

also known as "disjunctive normal form"

$(x' \cdot y \cdot z) + (x \cdot y' \cdot z) + (x \cdot y \cdot z') + (x \cdot y \cdot z) \; = \; MAJ$

x	y	z	MAJ	$x' \cdot y \cdot z$	$x \cdot y' \cdot z$	$x \cdot y \cdot z'$	$x \cdot y \cdot z$	
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Expressing MAJ(x, y, z) as a sum of products

Digital circuits: quiz 1

Which of the following does NOT represent majority function?

A. $(x \cdot y) + (y \cdot z) + (x \cdot z)$

B. $z(x'y \ + \ xy') + xy$

C. $(x \cdot y) + (y \cdot z)$

D.

```
public static boolean majority(boolean x, boolean y, boolean z) {
   int count = 0;
   if (x) count++;
   if (y) count++;
   if (z) count++;
   return count >= 2;
}
```

Universality

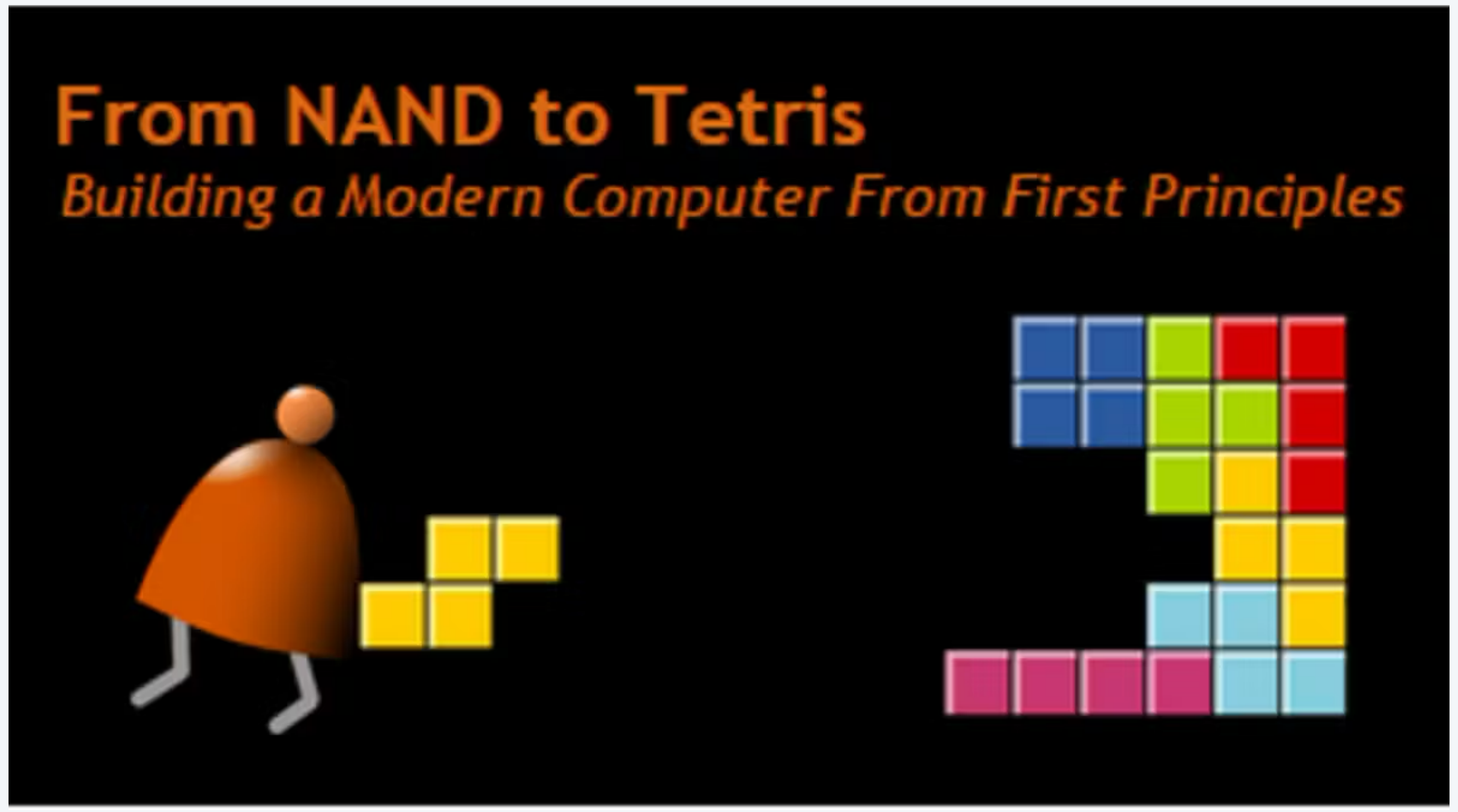
Def. A set of operations is universal if every boolean function can be expressed using just those operations.

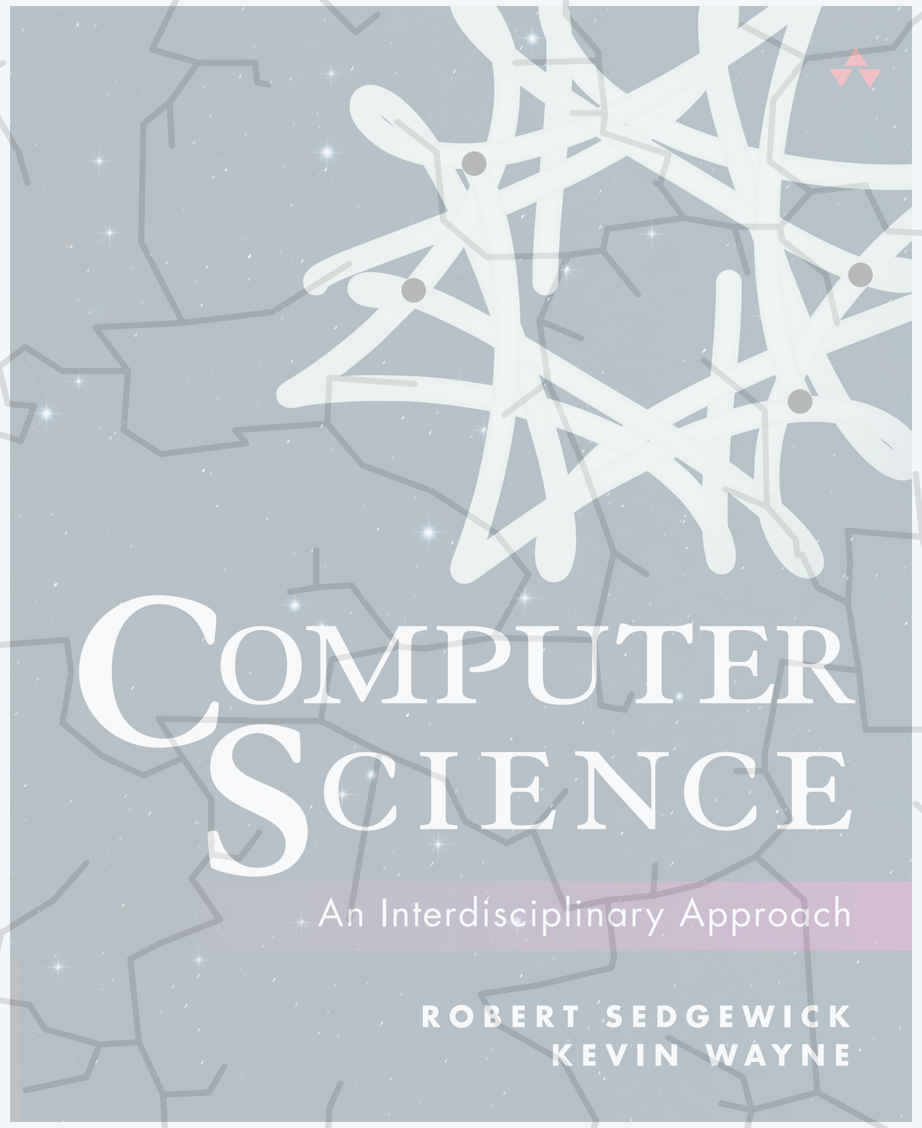
Proposition. $\{ AND, OR, NOT \}$ is a universal set of operations.
Pf. Sum-of-products construction on previous slide.

Proposition. $\{ NAND \}$ is a universal set of operations.
Pf. $\{ AND, OR, NOT \}$ can be constructed from $NAND$.

x	y	$NAND$
0	0	1
0	1	1
1	0	1
1	1	0

NAND





https://introcs.cs.princeton.edu

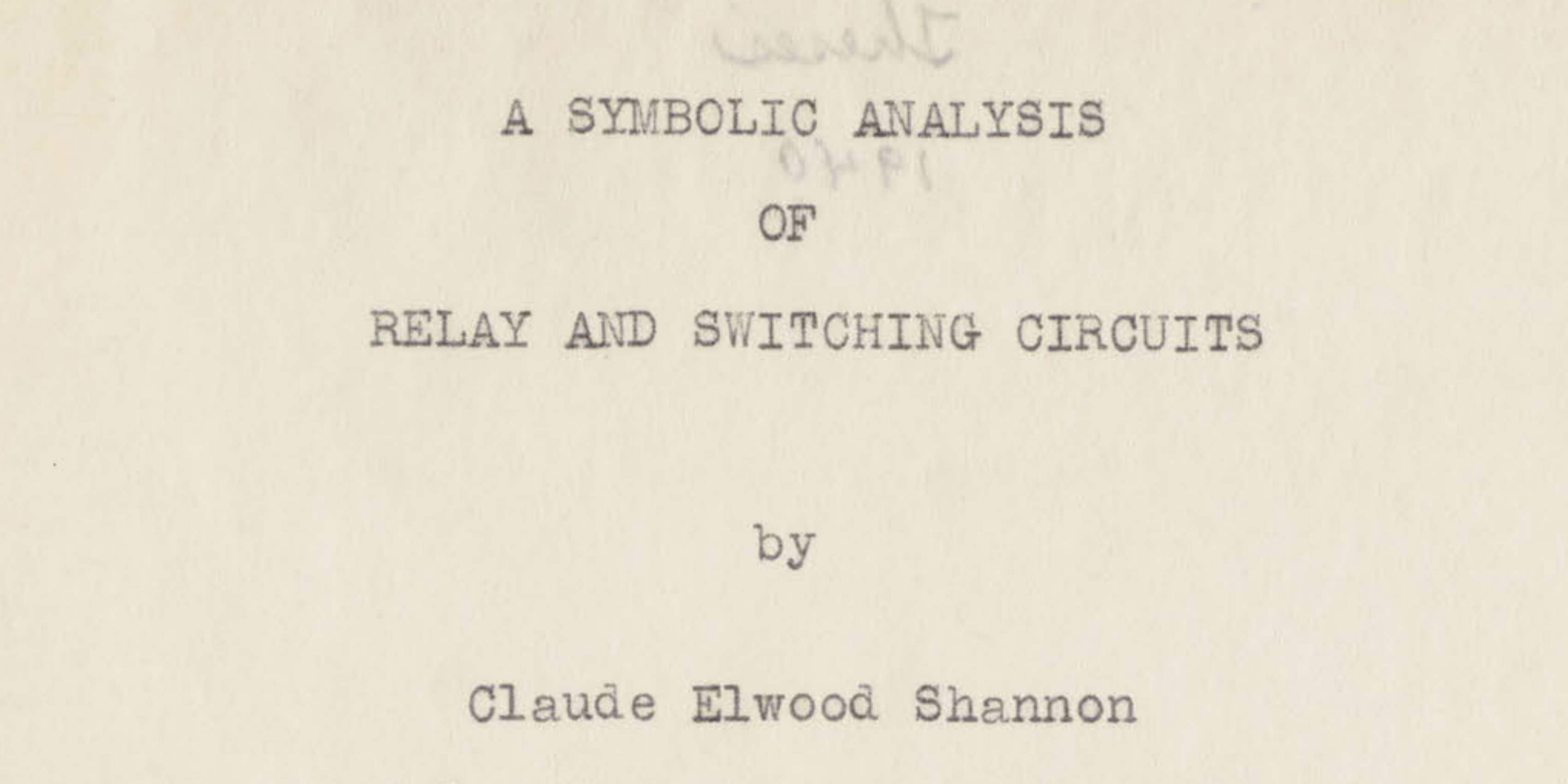
7. DIGITAL CIRCUITS

- ▸ *boolean algebra*
- ▸ *logic gates*
- ▸ *adder circuit*

A basis for digital devices

Claude Shannon. Identified the deep connection between Boolean algebra and circuits.

- Demonstrated how circuits could be analyzed using Boolean algebra.
- Designed circuits to perform mathematical operations on binary numbers. ← *add, subtract, multiply, factor, …*

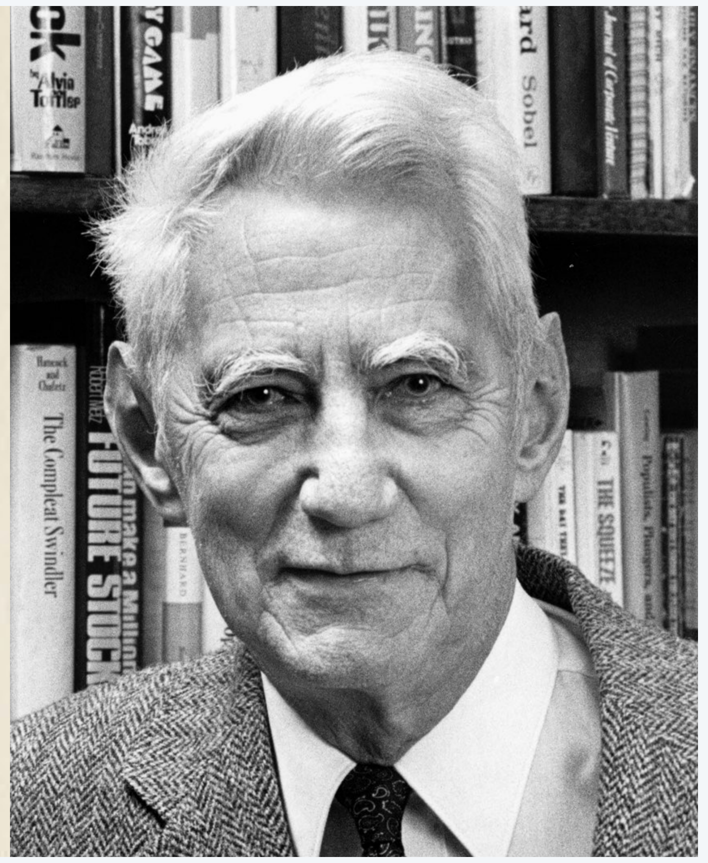

A SYMBOLIC ANALYSIS

OF

RELAY AND SWITCHING CIRCUITS

by

Claude Elwood Shannon



Claude Shannon's master's thesis at MIT (1937)

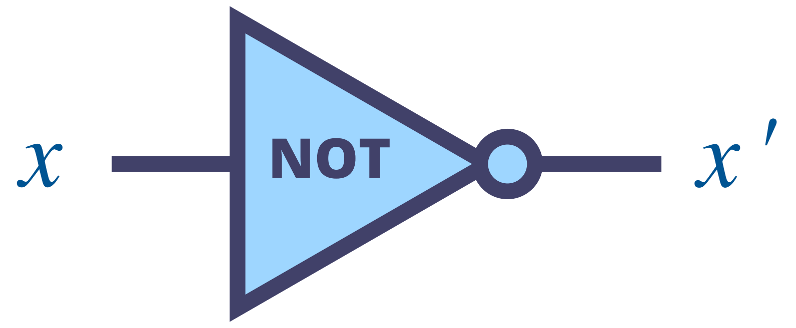
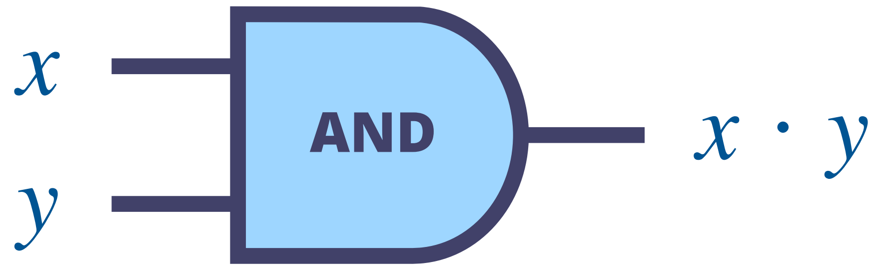
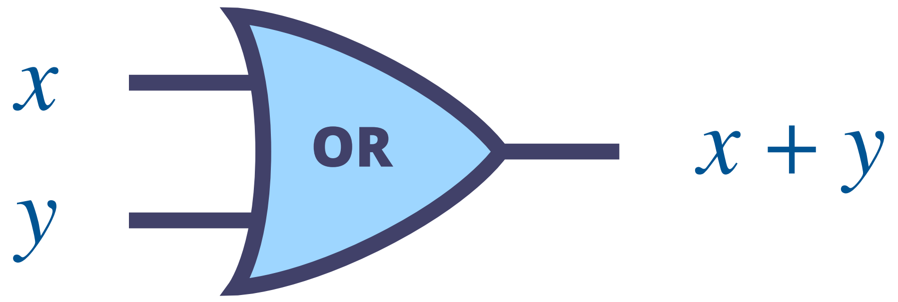
Impact. Every electronic device we use today is based upon Shannon's foundational work.

The Bit Player

available on COS 126 *Canvas*
(movie features Dean Andrea Goldsmith)

Primitive logic gates: AND, OR, and NOT

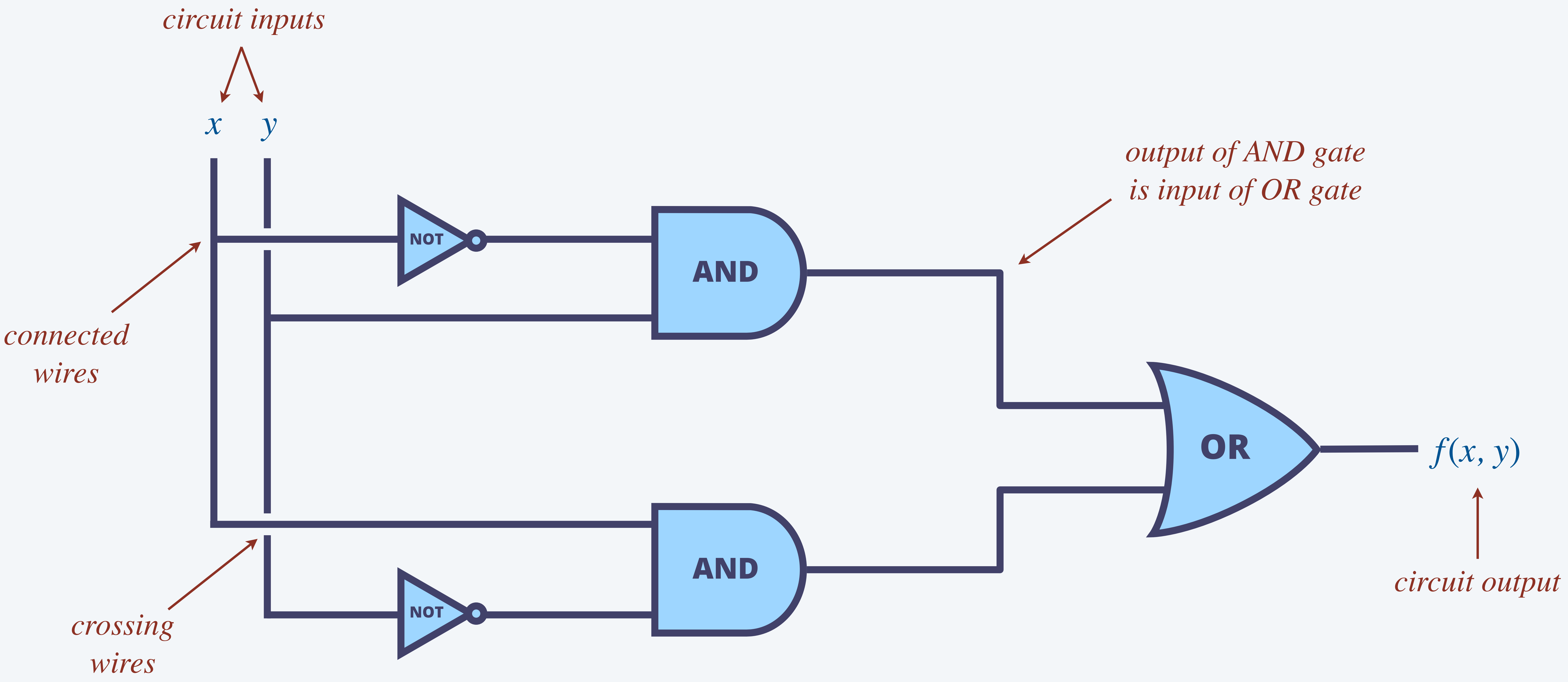
Logic gate. Physical device that implement a boolean function with one output.

<table>
<tr><th>gate</th><th>truth table</th><th>notation</th><th>symbol</th></tr>
<tr><td>NOT
(inverter)</td><td><table><tr><th>x</th><th>NOT</th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>0</td></tr></table></td><td>x'</td><td>
</td></tr>
<tr><td>AND</td><td><table><tr><th>x</th><th>y</th><th>AND</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td><td>$x \cdot y$</td><td>
</td></tr>
<tr><td>OR</td><td><table><tr><th>x</th><th>y</th><th>OR</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td><td>$x + y$</td><td>
</td></tr>
</table>

Digital circuits

Digital circuit. A network of logic gates connected by wires.

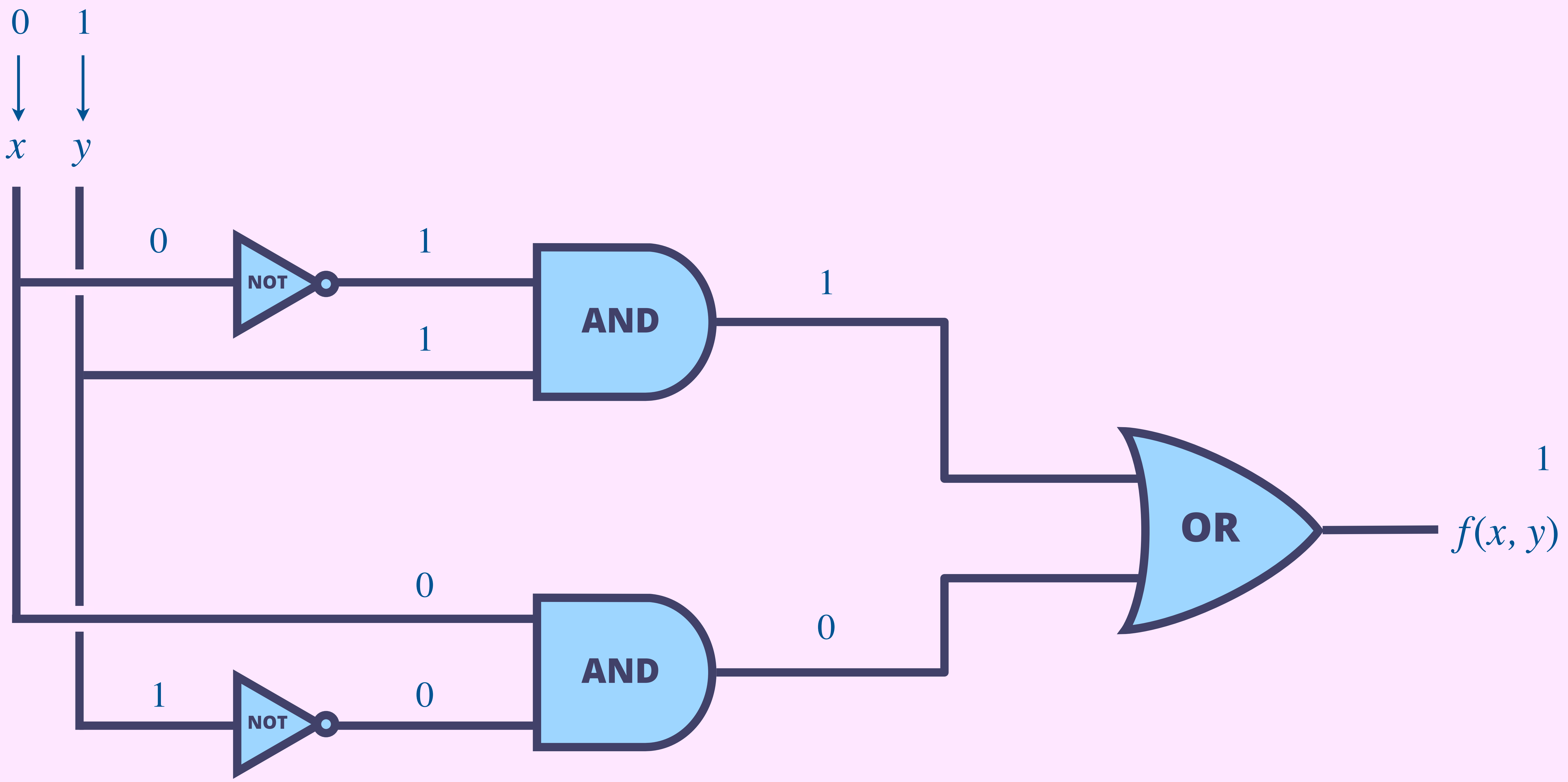
- Every wire is either *on* (1) or *off* (0).
- Can connect output of one gate to input of another gate.
- Any wire connected to a wire that is *on* is also *on* (and same for *off*).



x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Digital circuit. A network of logic gates connected by wires.

- Every wire is either *on* (1) or *off* (0).
- Can connect output of one gate to input of another gate.
- Any wire connected to a wire that is *on* is also *on* (and same for *off*).



x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

For which values of x and y does the following circuit output 1 ?

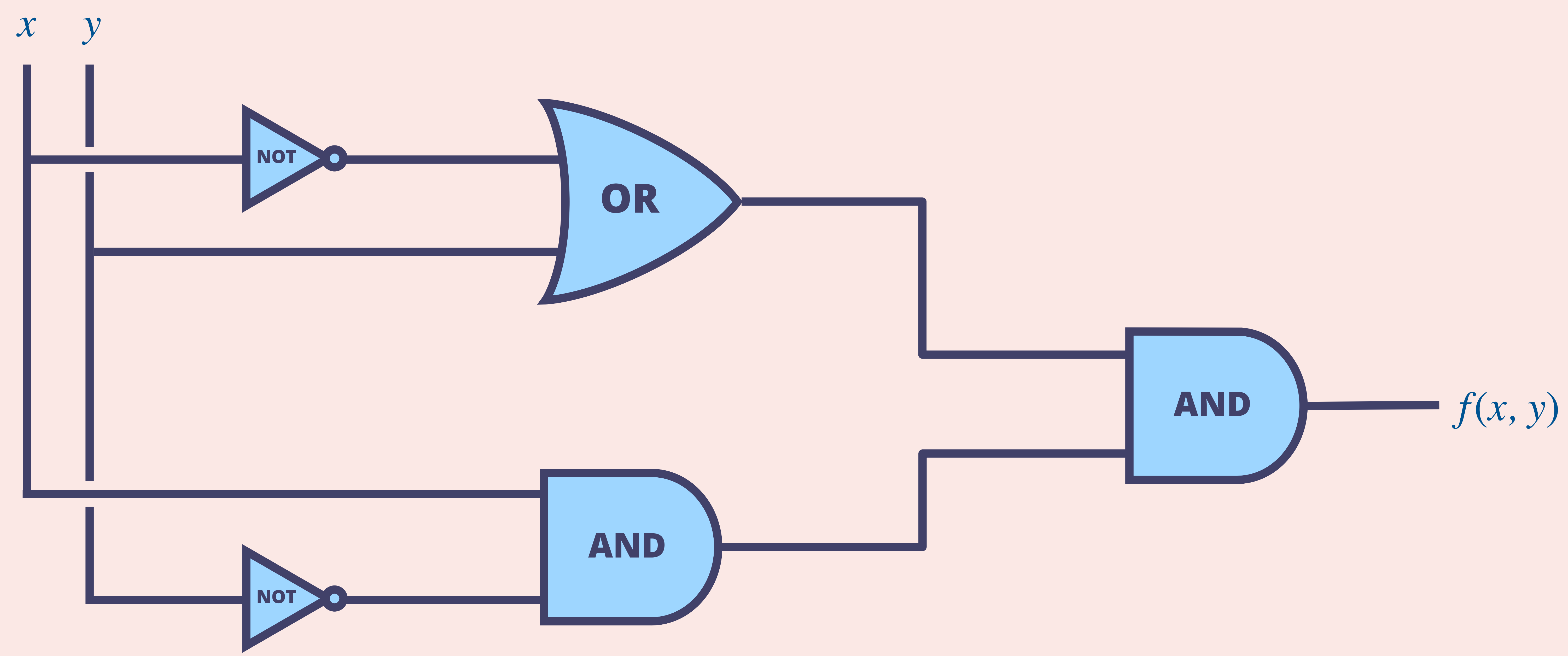
A. $x = 0,\ y = 0$

B. $x = 0,\ y = 1$

C. $x = 1,\ y = 0$

D. $x = 1,\ y = 1$

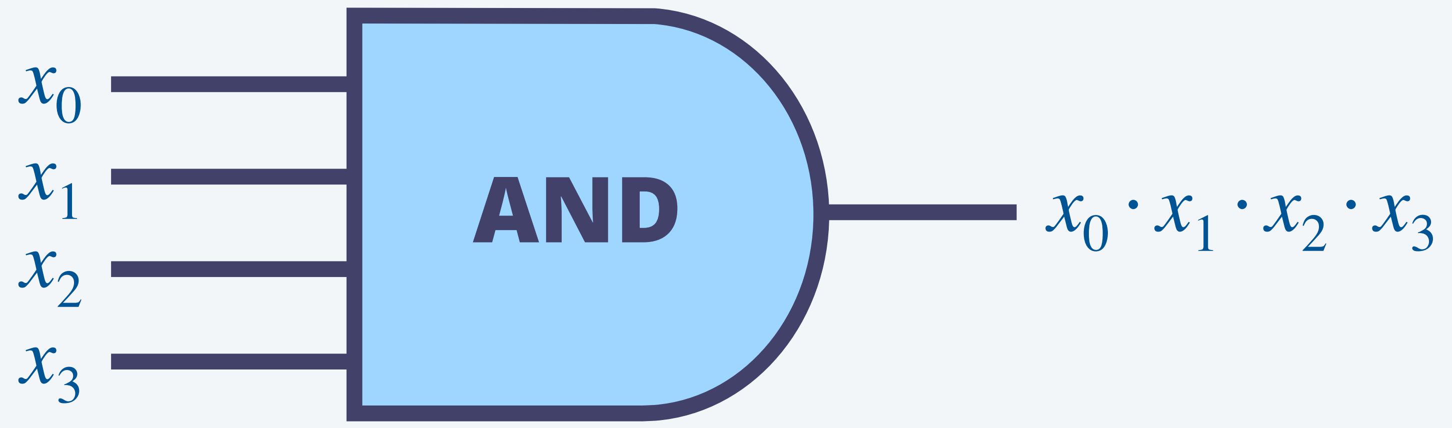
E. None of the above.



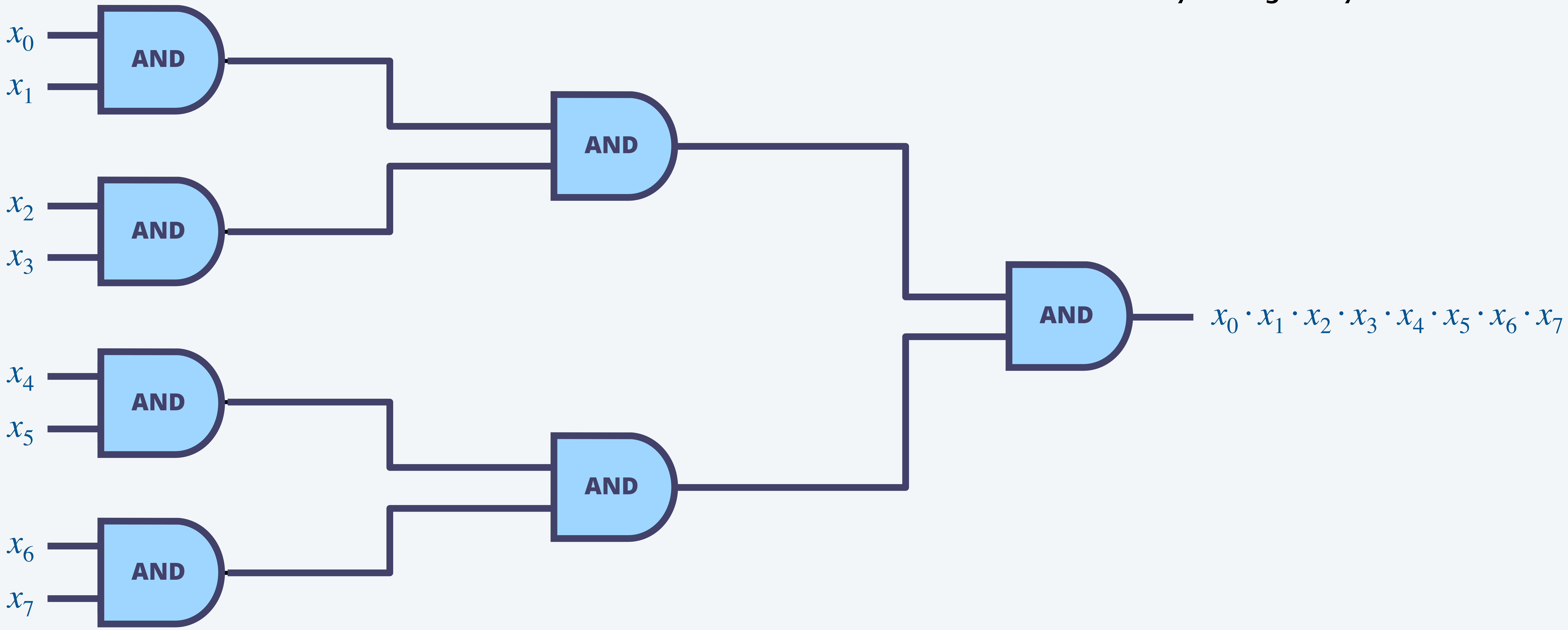
Multiway *AND* gates

Multiway *AND* gate.

- 1 if all inputs are 1.
- 0 if any input is 0.



4-way AND gate symbol

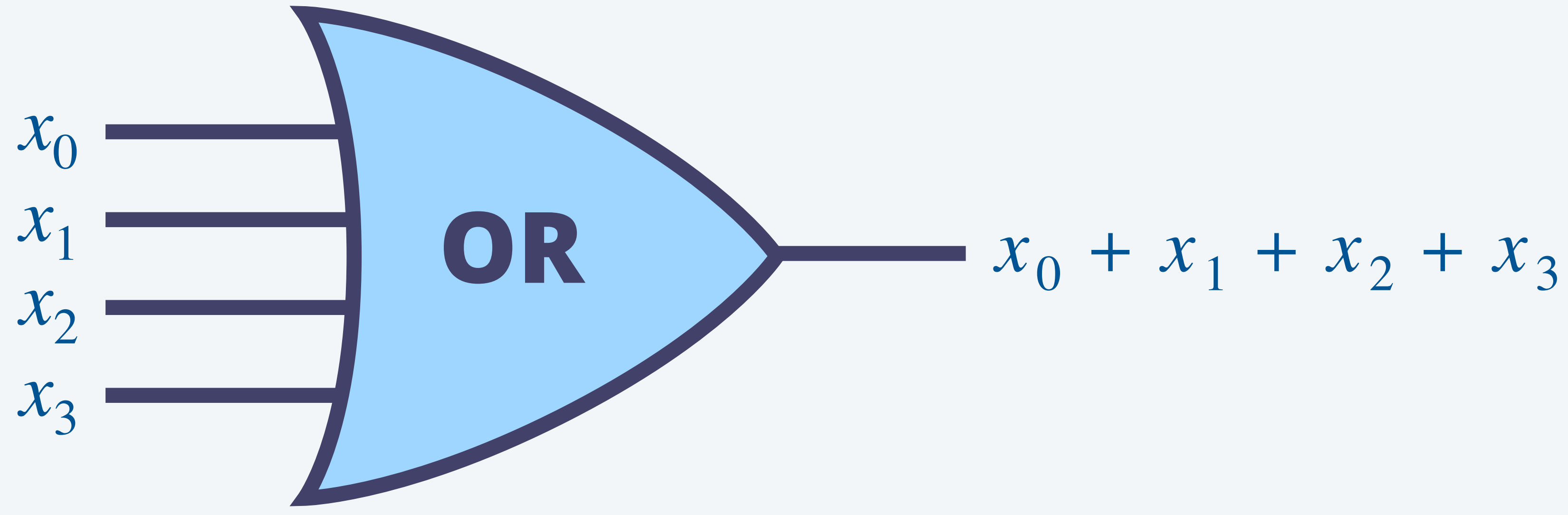


8-way AND gate implementation
(tree of 2-way AND gates)

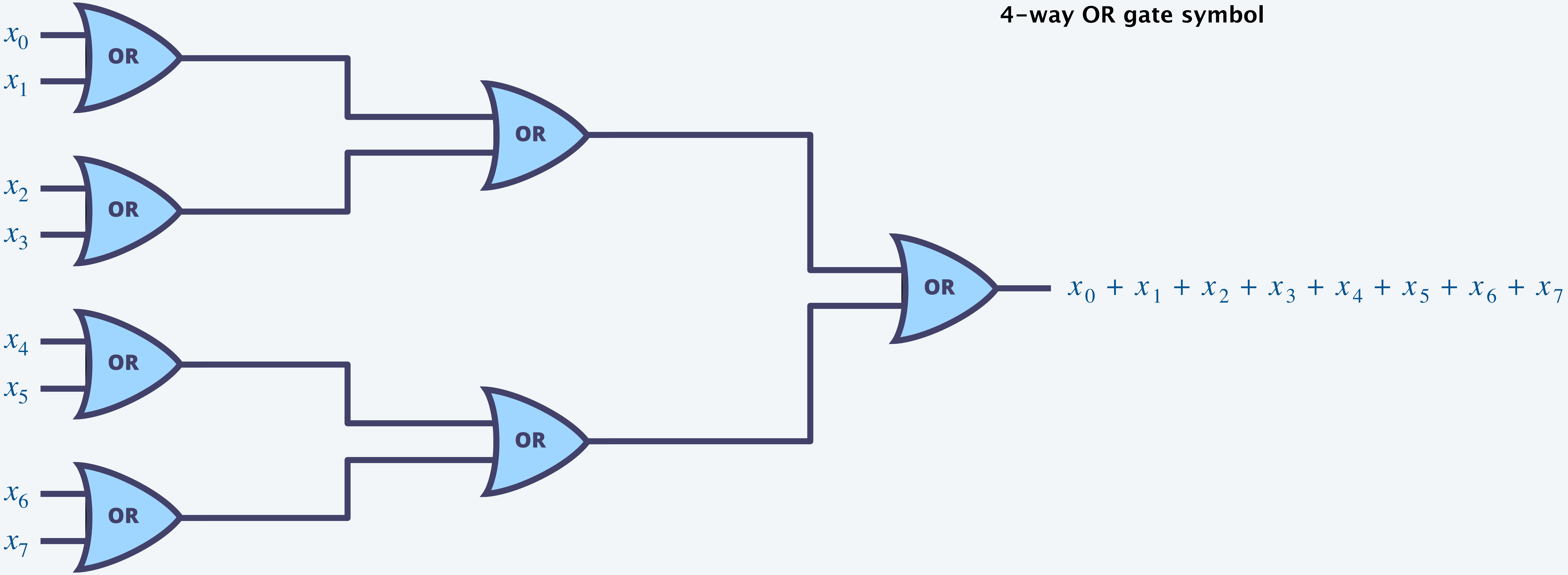
Multiway *OR* gates

Multiway *OR* gate.

- 1 if any input is 1.
- 0 if all inputs are 0.



4-way OR gate symbol

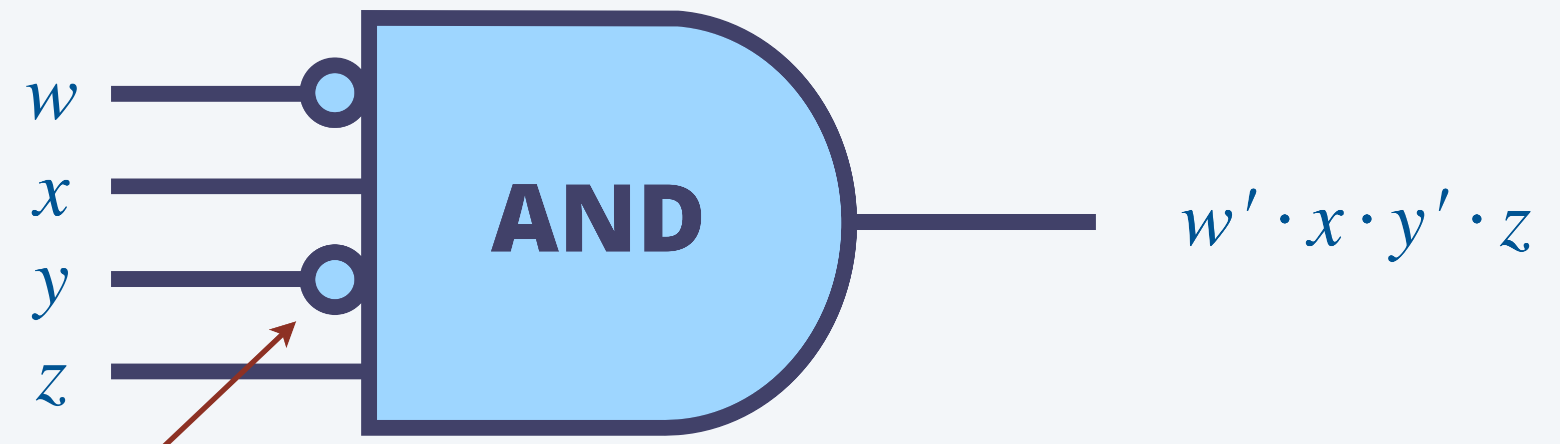


8-way OR gate implementation
(tree of 2-way OR gates)

Generalized *AND* gates

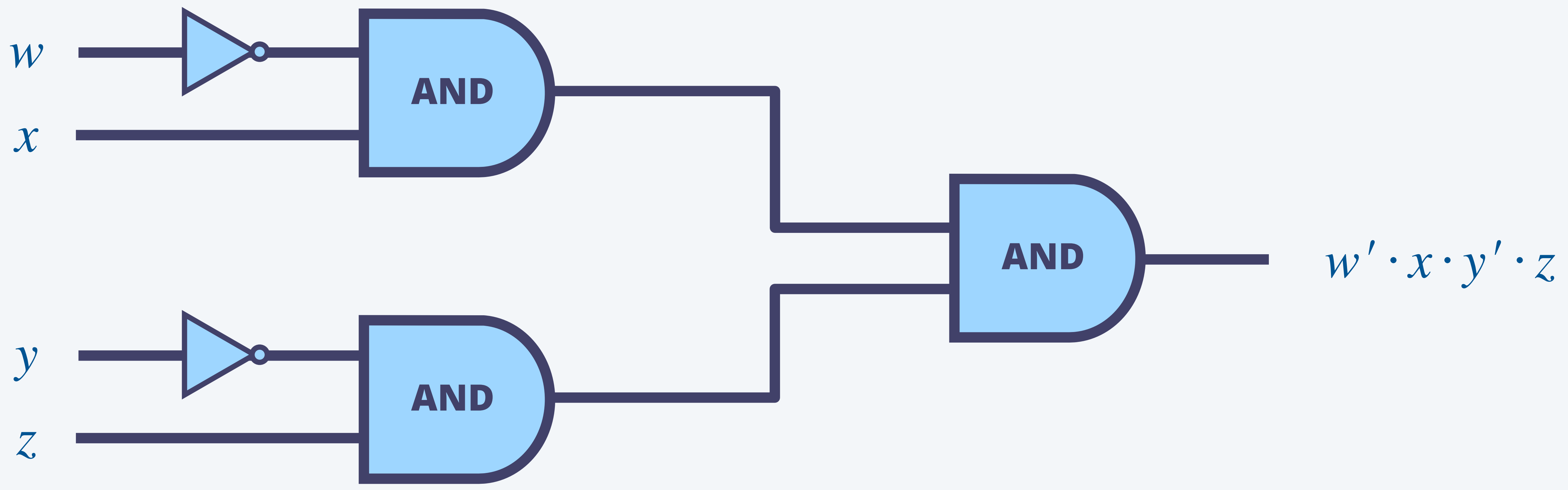
Generalized *AND* gate.

- 1 for exactly one set of input values.
- 0 for all other sets of input values.



4-way generalized
AND gate symbol

each "inversion bubble" denotes a NOT gate

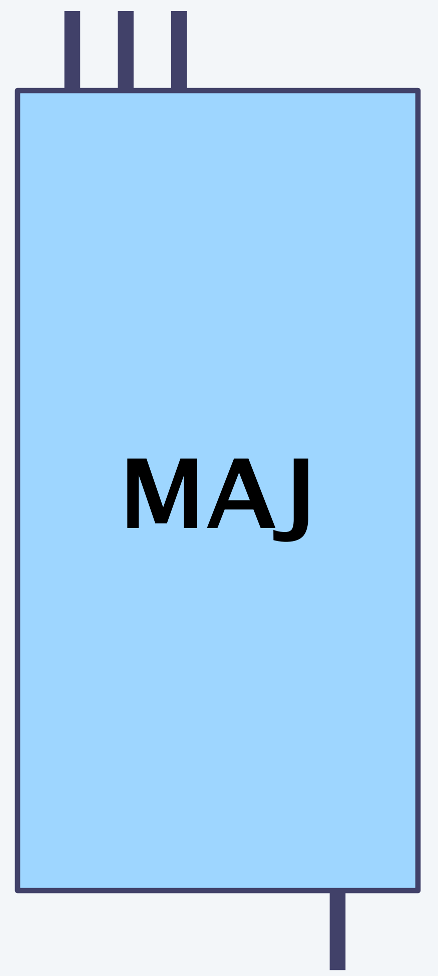


4-way generalized AND gate implementation
(tree of 2-way AND gates, plus NOT gates)

Majority function

Sum-of-products construction.

- Identify rows of truth table where the function is 1.
- Use a generalized *AND* gate for each term.
- Combine the terms using an *OR* gate.

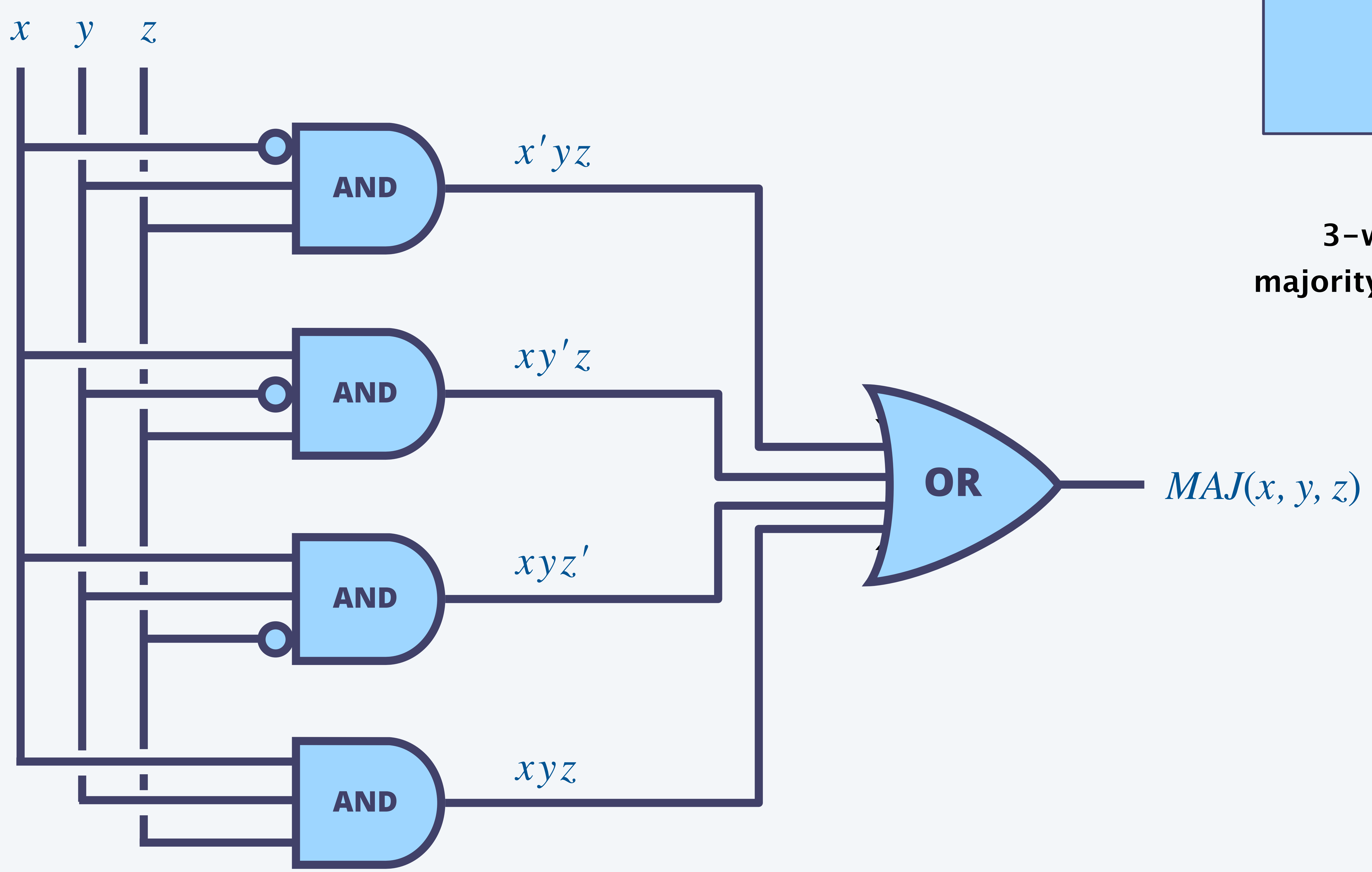


3-way majority circuit

Ex 1. Majority function.

x	y	z	MAJ	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	⟵ $x'yz$
1	0	0	0	
1	0	1	1	⟵ $xy'z$
1	1	0	1	⟵ xyz'
1	1	1	1	⟵ xyz

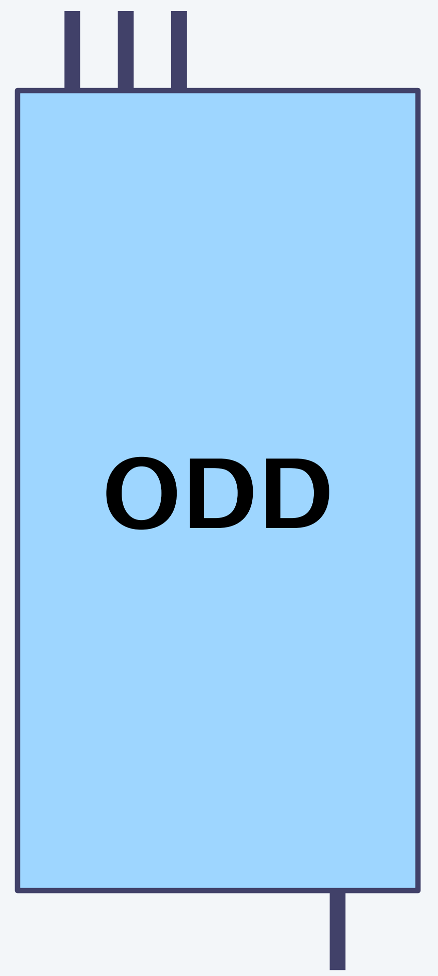
$$MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz$$



Odd-parity function

Sum-of-products construction.

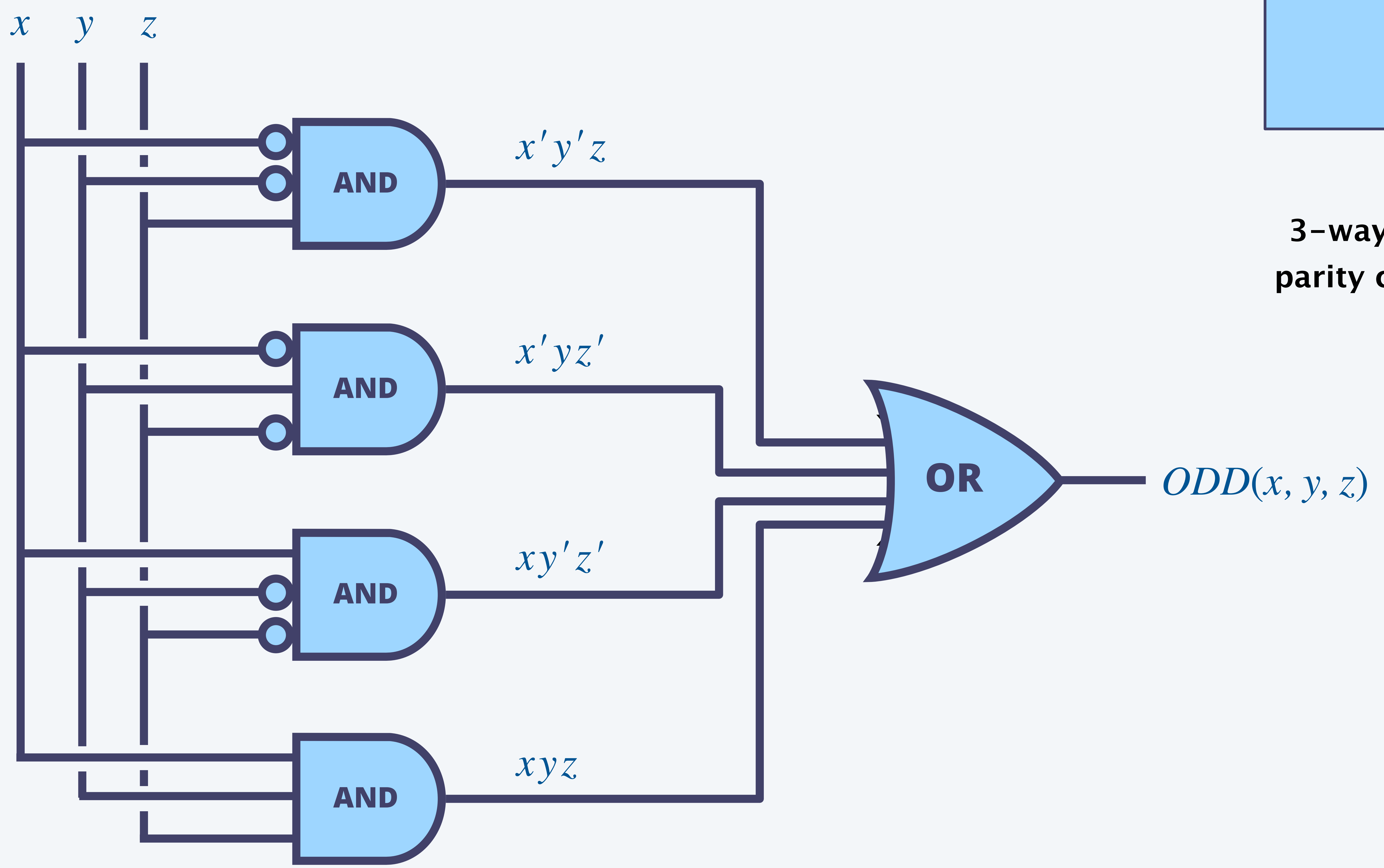
- Identify rows of truth table where the function is 1.
- Use a generalized *AND* gate for each term.
- Combine the terms using an *OR* gate.



Ex 2. Odd-parity function.

x	y	z	ODD	
0	0	0	0	
0	0	1	1	⟵ $x'y'z$
0	1	0	1	⟵ $x'yz'$
0	1	1	0	
1	0	0	1	⟵ $xy'z'$
1	0	1	0	
1	1	0	0	
1	1	1	1	⟵ xyz

$$ODD(x, y, z) = x'y'z + x'yz' + xy'z' + xyz$$



Sum-of-products construction (summary)

Goal. Design a digital circuit that computes a given boolean function.

Recipe.

- Step 1: Represent input and output with boolean variables.
- Step 2: Construct truth table to define the function.
- Step 3: Identify rows where the function is 1.
- Step 4: Use a generalized *AND* gate for each row, and *OR* the results. ← *sum-of-products construction*

Profound consequence. Can design a digital circuit for ANY boolean function.

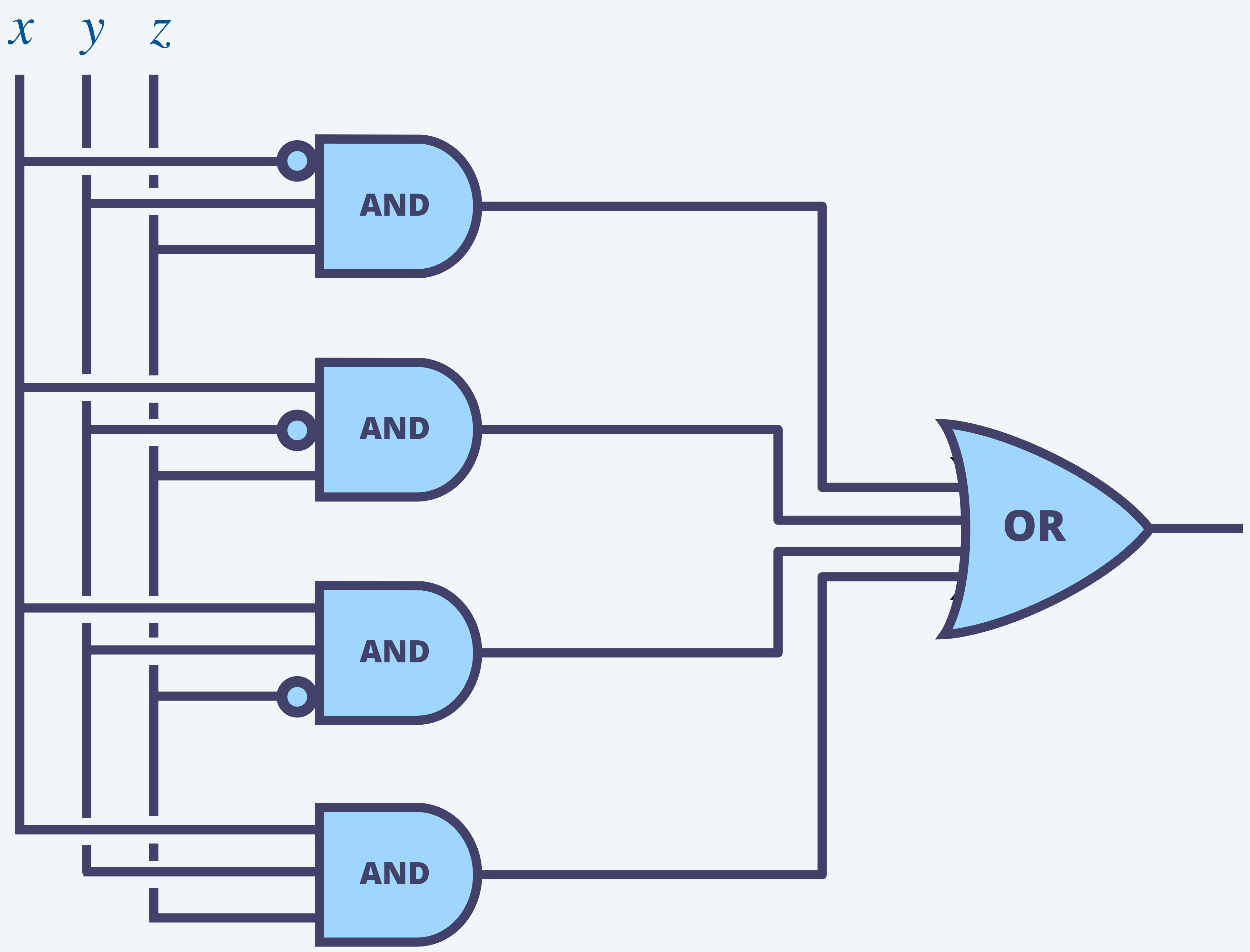
Optimized digital circuits

Caveat. Sum-of-products construction is not optimal in terms of:

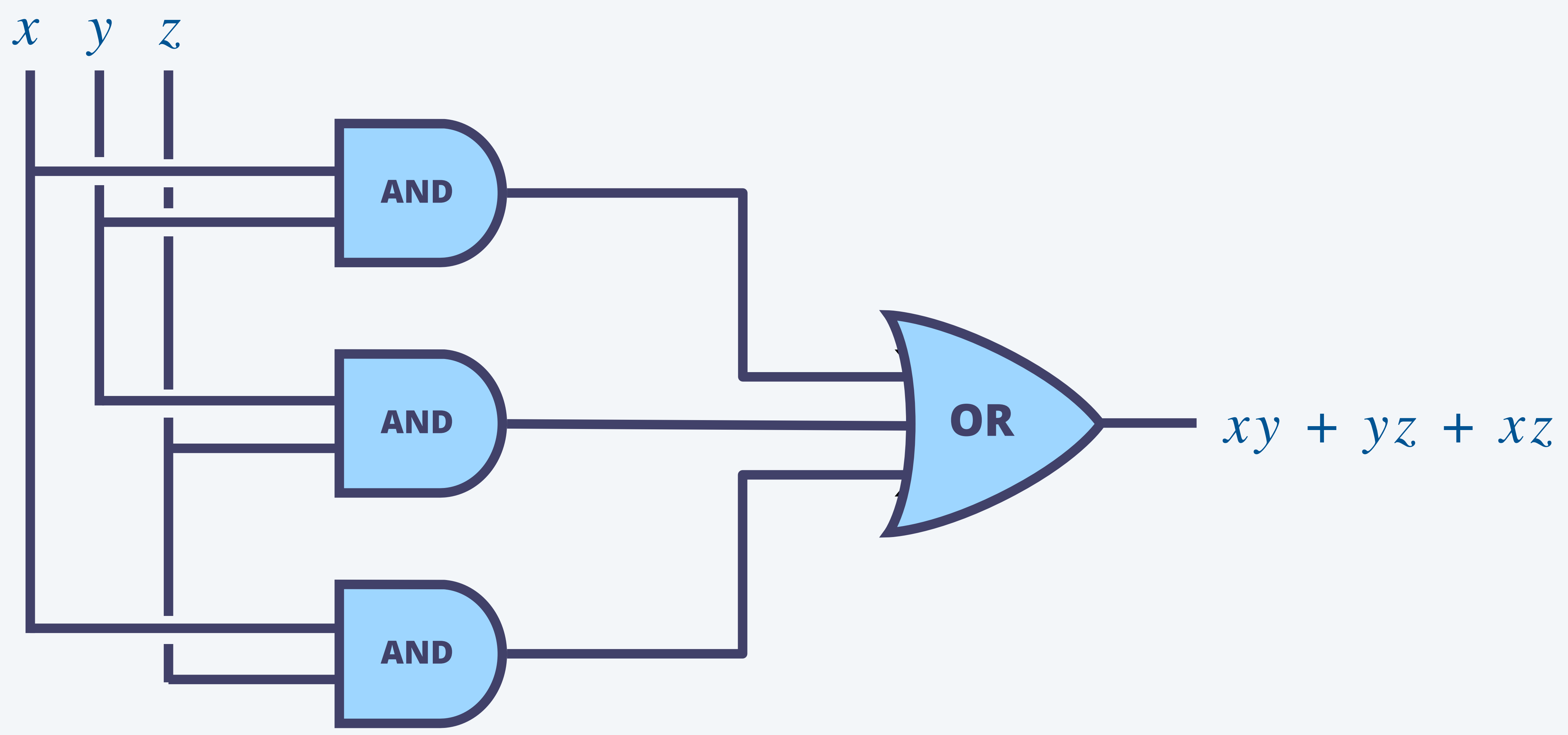
- Space = number of gates.
- Time = depth of circuit.

this course: we'll ignore such low-level optimization

Ex. Majority function.



3-way majority circuit (sum-of-products)



3-way majority circuit (optimized)

How many 3-way generalized AND gates are needed to build the sum-of-products circuit for the following truth table?

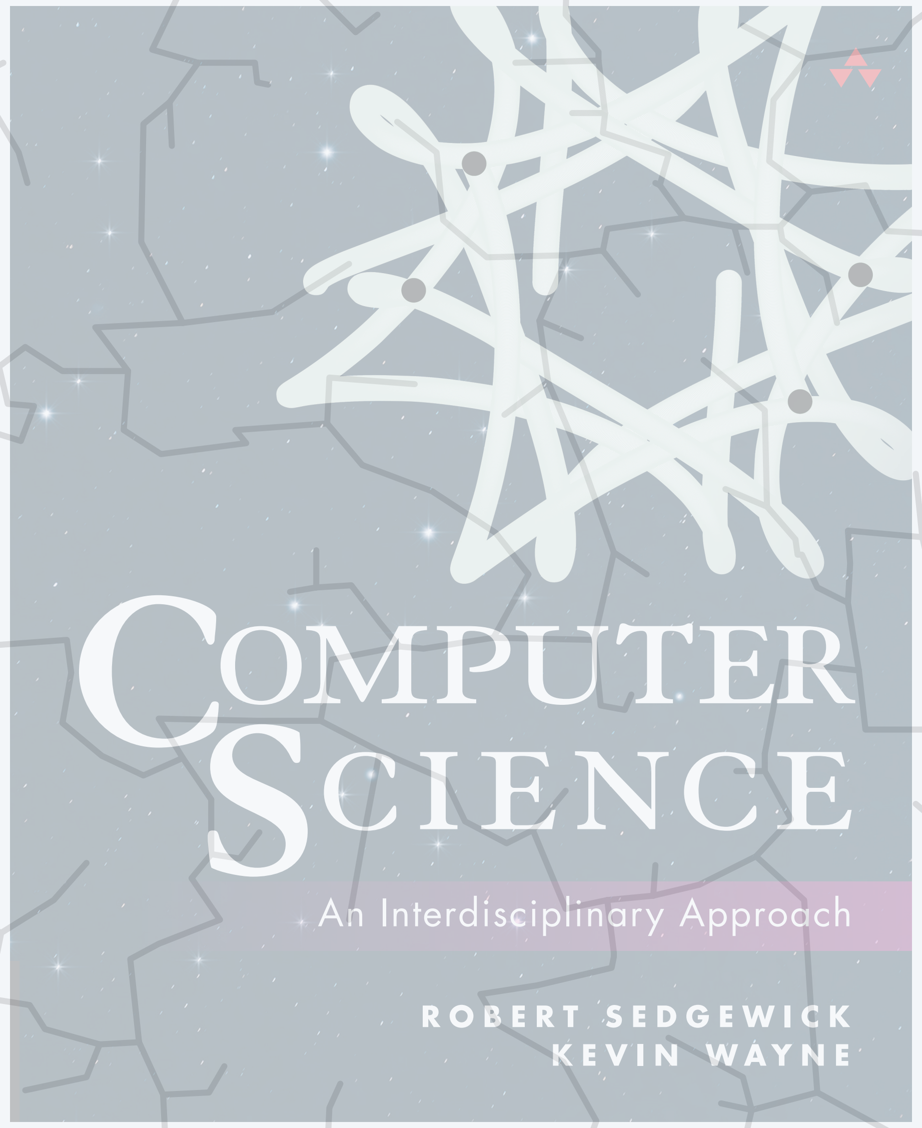
A. 1

B. 2

C. 3

D. 4

x	y	z	EQ
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1



https://introcs.cs.princeton.edu

7. DIGITAL CIRCUITS

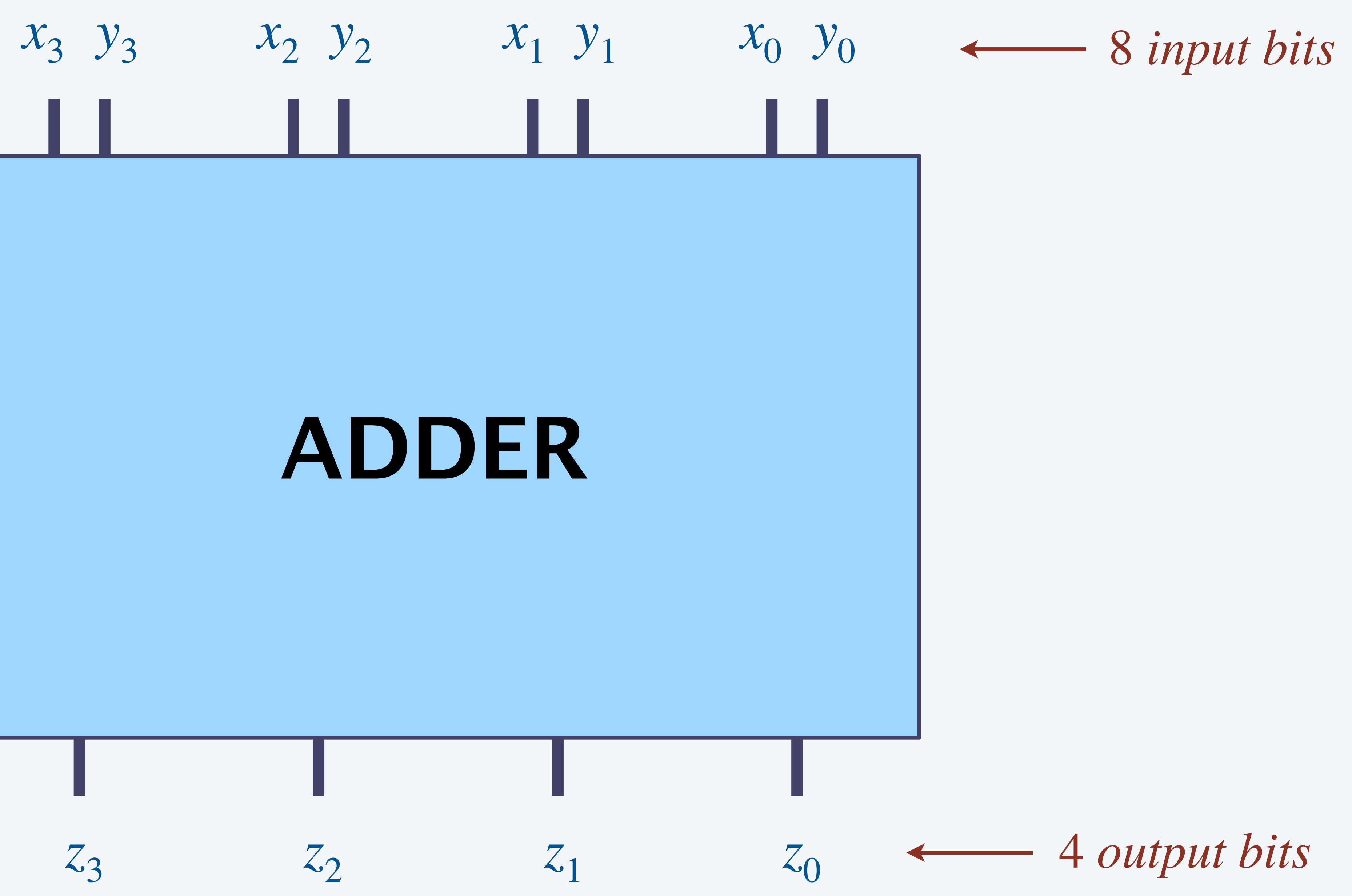
- *boolean algebra*
- *logic gates*
- *adder circuit*

Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers. ← *ignore integer overflow*

First step. Represent inputs and outputs in binary.

$$\begin{array}{ccccc} & x_3 & x_2 & x_1 & x_0 \\ + & y_3 & y_2 & y_1 & y_0 \\ \hline & z_3 & z_2 & z_1 & z_0 \end{array}$$

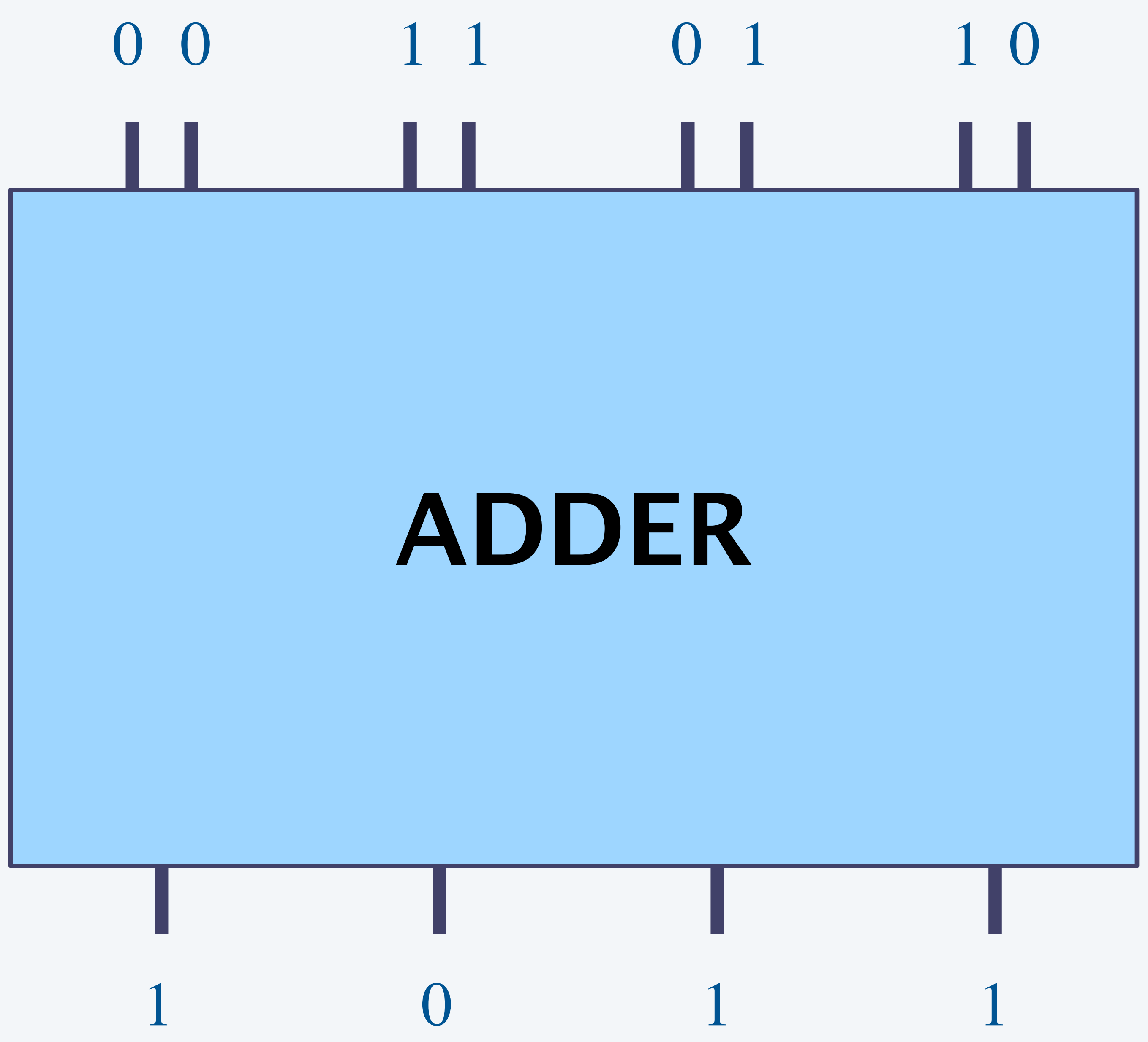


Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers. ← *ignore integer overflow*

First step. Represent inputs and outputs in binary.

$$
\begin{array}{ccccc}
 & 0 & 1 & 0 & 1 \\
+ & 0 & 1 & 1 & 0 \\
\hline
 & 1 & 0 & 1 & 1
\end{array}
$$



What is the binary sum 1011 + 0110?

A. 0001

B. 1001

C. 1101

D. 1121

E. 10001

$$
\begin{array}{rcccc}
 & 1 & 0 & 1 & 1 \\
+ & 0 & 1 & 1 & 0 \\
\hline
\end{array}
$$

Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers.

exceeds number of electrons in universe (!)

Straw-person solution. Build a truth table for each output bit.

Approach is not scalable! Truth table for 128-bit adder would have 2^{256} rows.

$$
\begin{array}{ccccc}
 & x_3 & x_2 & x_1 & x_0 \\
+ & y_3 & y_2 & y_1 & y_0 \\
\hline
 & z_3 & z_2 & z_1 & z_0
\end{array}
$$

x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	1	1	0	0	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	1	1	1	1	1	0	1	1	0	1
1	1	1	1	1	1	1	1	1	1	1	0

$2^8 = 512$ *rows*

truth table for 4-bit adder

Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers.

$c_0 = 0$

$$
\begin{array}{ccccc}
 & c_3 & c_2 & c_1 & c_0 \\
 & x_3 & x_2 & x_1 & x_0 \\
+ & y_3 & y_2 & y_1 & y_0 \\
\hline
 & z_3 & z_2 & z_1 & z_0
\end{array}
$$

Efficient solution. Do one bit at a time.

- Build truth table for each carry bit. ← *majority function* (!)
- Build truth table for each sum bit.

x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

truth table for carry bit

$c_{i+1} = MAJ(x_i, y_i, c_i)$

Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers.

$c_0 = 0$

$$\begin{array}{ccccc} & c_3 & c_2 & c_1 & c_0 \\ & x_3 & x_2 & x_1 & x_0 \\ + & y_3 & y_2 & y_1 & y_0 \\ \hline & z_3 & z_2 & z_1 & z_0 \end{array}$$

Efficient solution. Do one bit at a time.

- Build truth table for each carry bit ← *majority function* (!)
- Build truth table for each sum bit. ← *odd-parity function* (!)

x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

truth table for sum bit

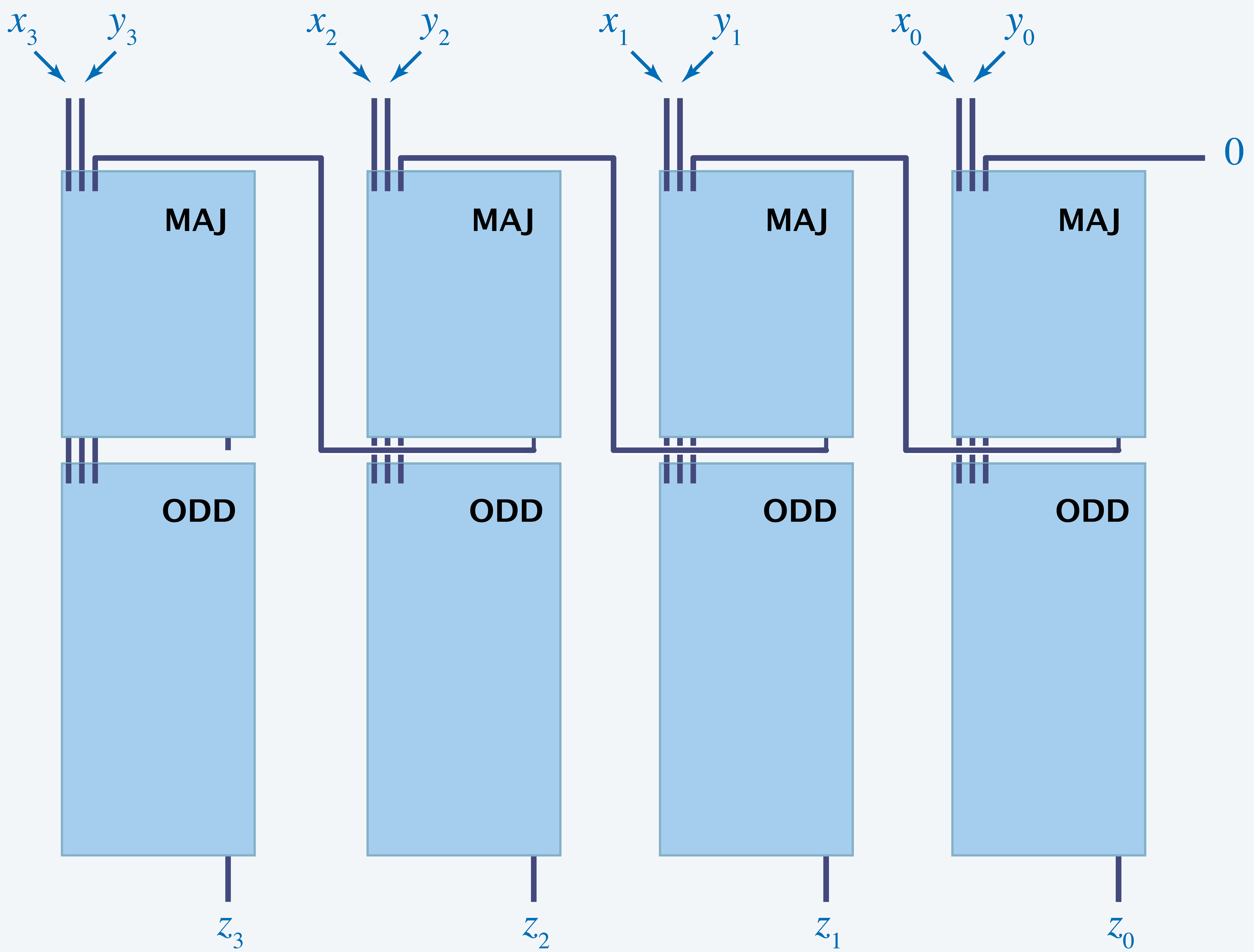
$z_i = ODD(x_i, y_i, c_i)$

Let's make an adder circuit!

Adder circuit. Compute $z = x + y$ for 4-bit binary integers.

$c_0 = 0$

$$
\begin{array}{rcccc}
 & c_3 & c_2 & c_1 & c_0 \\
 & x_3 & x_2 & x_1 & x_0 \\
+ & y_3 & y_2 & y_1 & y_0 \\
\hline
 & z_3 & z_2 & z_1 & z_0
\end{array}
$$



Efficient solution. Do one bit at a time.

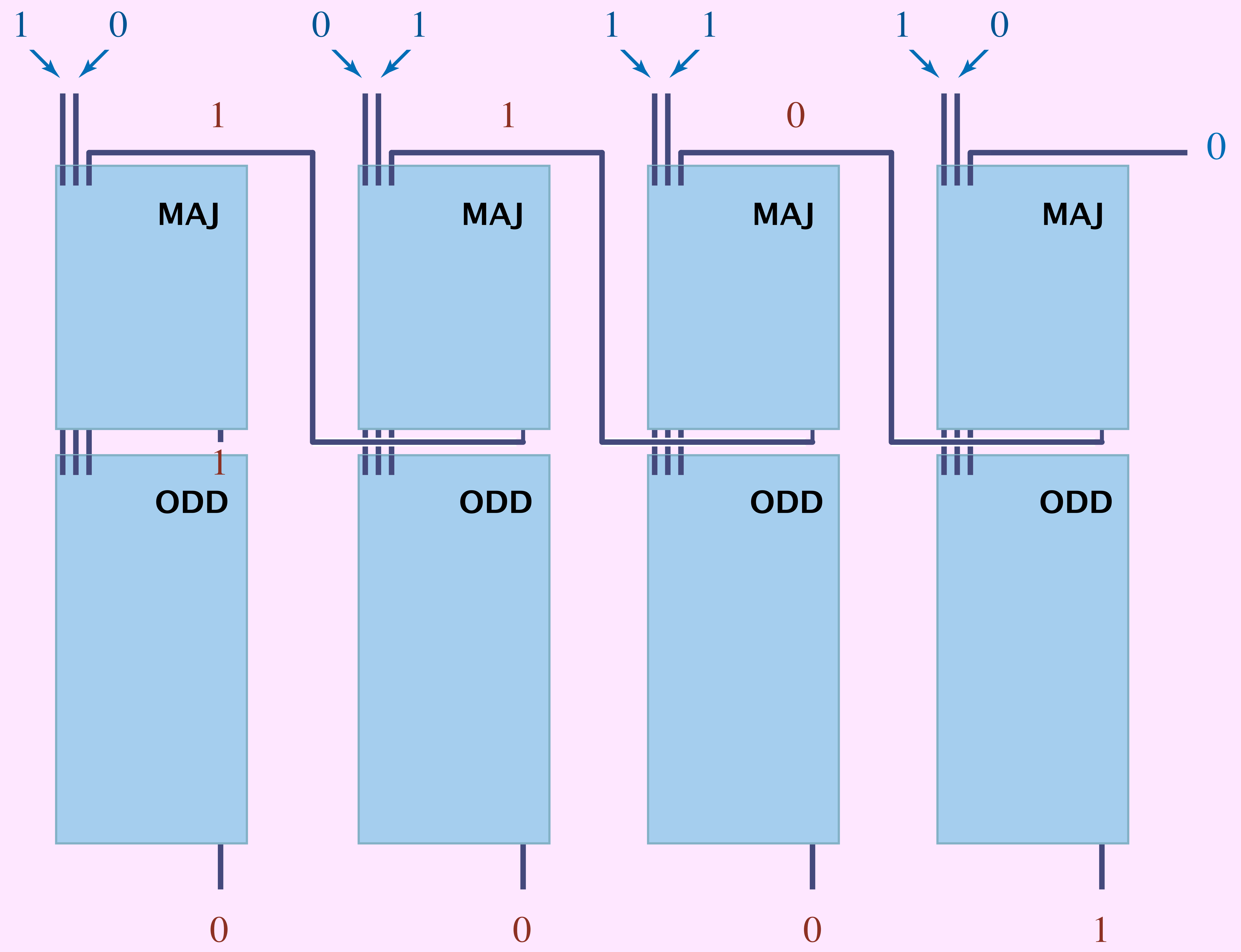
- Carry bit is *MAJ*.
- Sum bit is *ODD*.
- Chain 1-bit adders to “ripple” carries.

Size of circuit. $\Theta(n)$ gates for n-bit adder.

Adder circuit trace

Circuit trace. Trace the execution of the adder circuit on a given input.

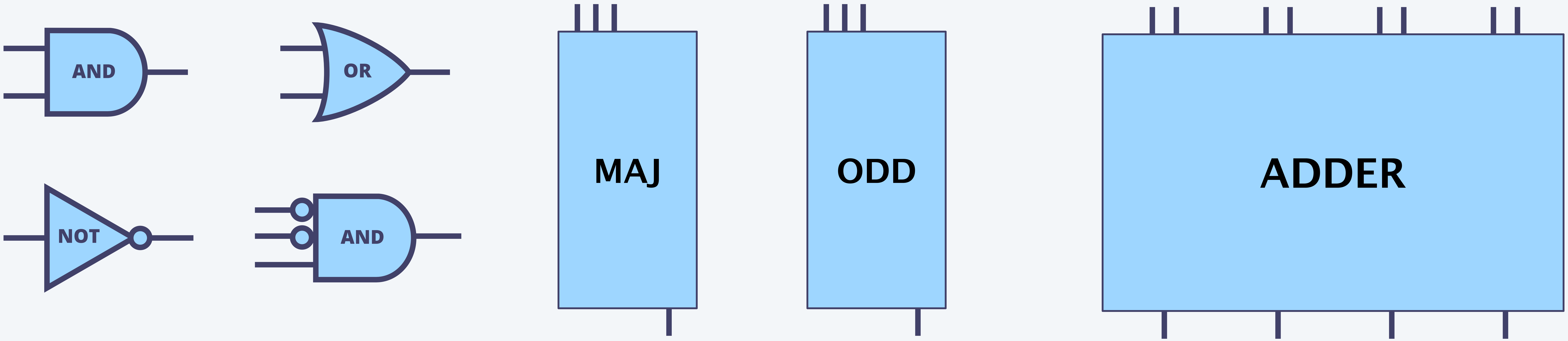
$$
\begin{array}{ccccc}
1 & 1 & 1 & & \\
 & 1 & 0 & 1 & 1 \\
+ & 0 & 1 & 1 & 0 \\
\hline
 & 0 & 0 & 0 & 1
\end{array}
$$



Encapsulation

Encapsulation in circuit design mirrors familiar software design principle.

- API describes behavior (input and outputs) of circuit.
- Implementation gives details of how to build it from wires and gates.
- Client uses circuit as a black box.

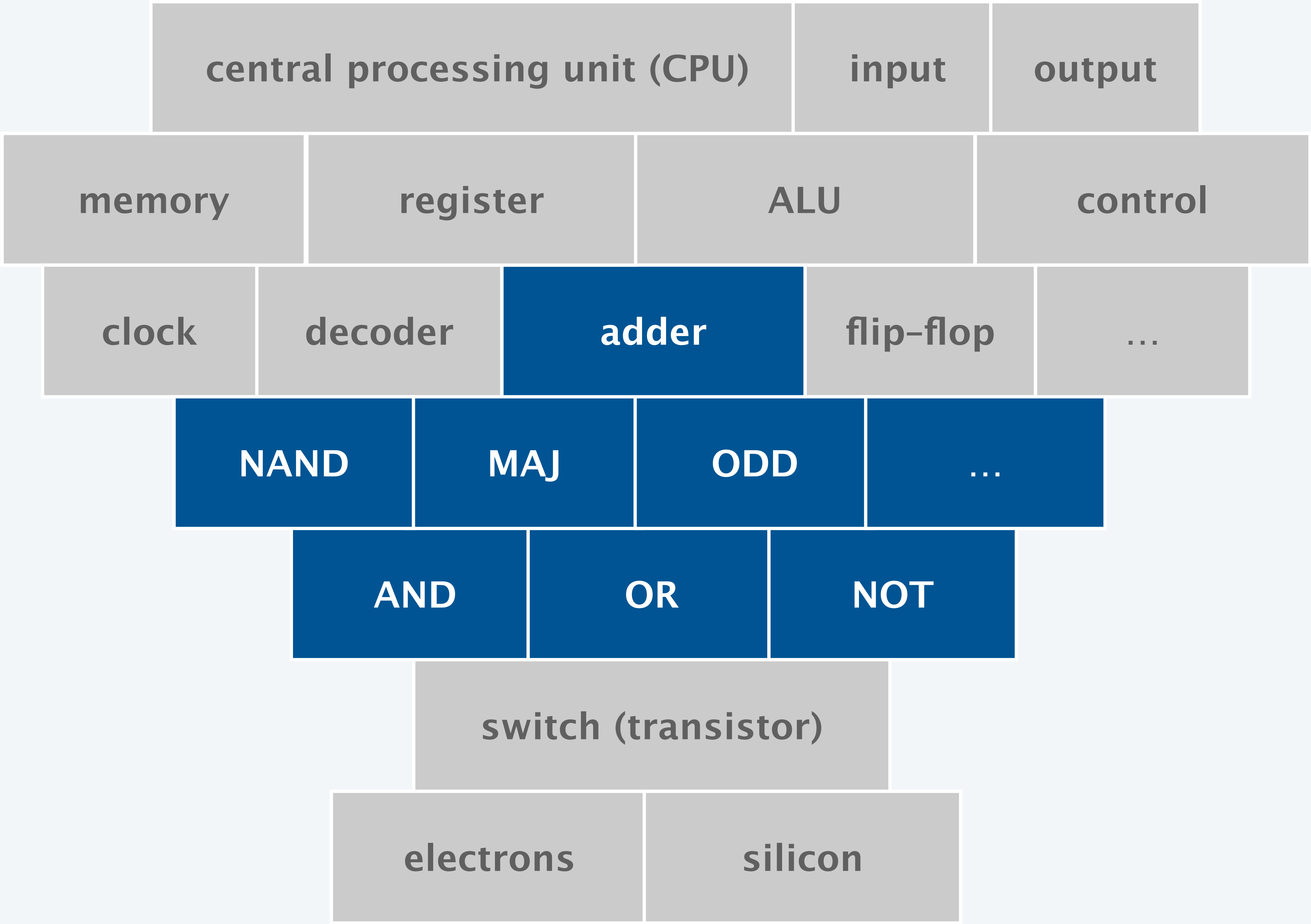


Bottom line. We manage complexity by encapsulating circuits.

Layers of abstraction

Layers of abstraction apply with a vengeance.

- On/off.
- Switch.
- Primitive gates (*AND*, *OR*, *NOT*).
- Composite gates (multiway *AND*/*OR*, *MAJ*, *ODD*).
- Adder circuit.
- Memory.
- Arithmetic logic unit (ALU).
- Central processing unit (CPU).
- Input and output.
- Your computer.



Want to learn more? See ECE 206 and ECE 365.

Credits

Co-instructors, course admin, and graduate student preceptors.

Alan Kaplan

Sebastian Caldas

Kobi Kaplan

Undergrad graders and lab TAs. Apply to be one next semester!

A final thought

Credits

image	source	license
Retro Telephone and Smartphone	Adobe Stock	education license
Macbook Pro	Apple	
Samsung Galaxy S23	Samsung	
Xbox One	Microsoft	
Cardiac Pacemaker	Adobe Stock	education license
Apple A16 Bionic Chip	Apple	
Boole is Coole	IrishPhilosophy	CC BY-NC-SA 2.0
Boole Orders Lunch	Sidney Harris	
From NAND to Tetris	nand2tetris.org	

Credits

image	source	license
Claude Shannon	Lucent Technologies	
Bit Player Theatrical Poster	thebitplayer.com	
John Hutton as Claude Shannon	thebitplayer.com	
Logic Gate Symbols	Adobe Stock	education license
Apple MacBook Pro	Adobe Stock	education license