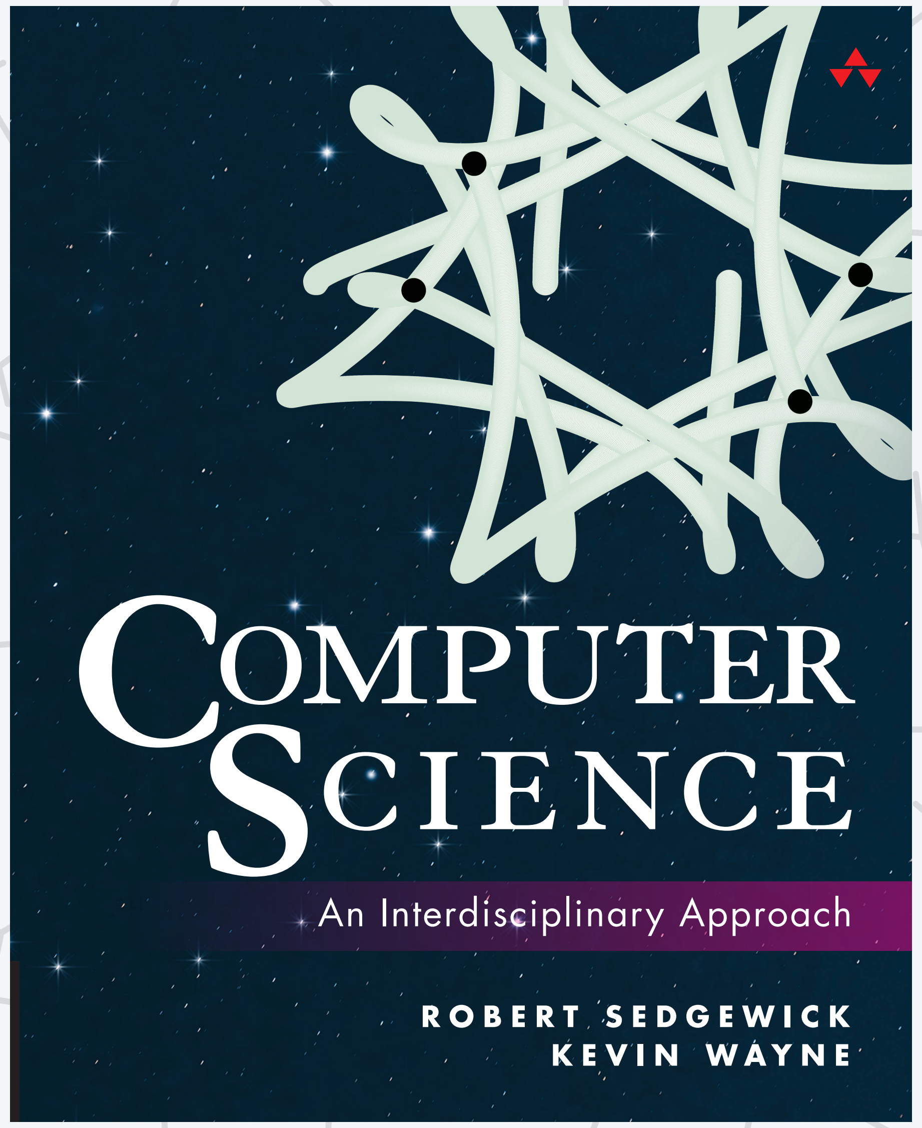
Robert Sedgewick | Kevin Wayne



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4.2 Algorithms

- *sequential search*
- *binary search*
- *insertion sort*
- *mergesort*

DRAFT

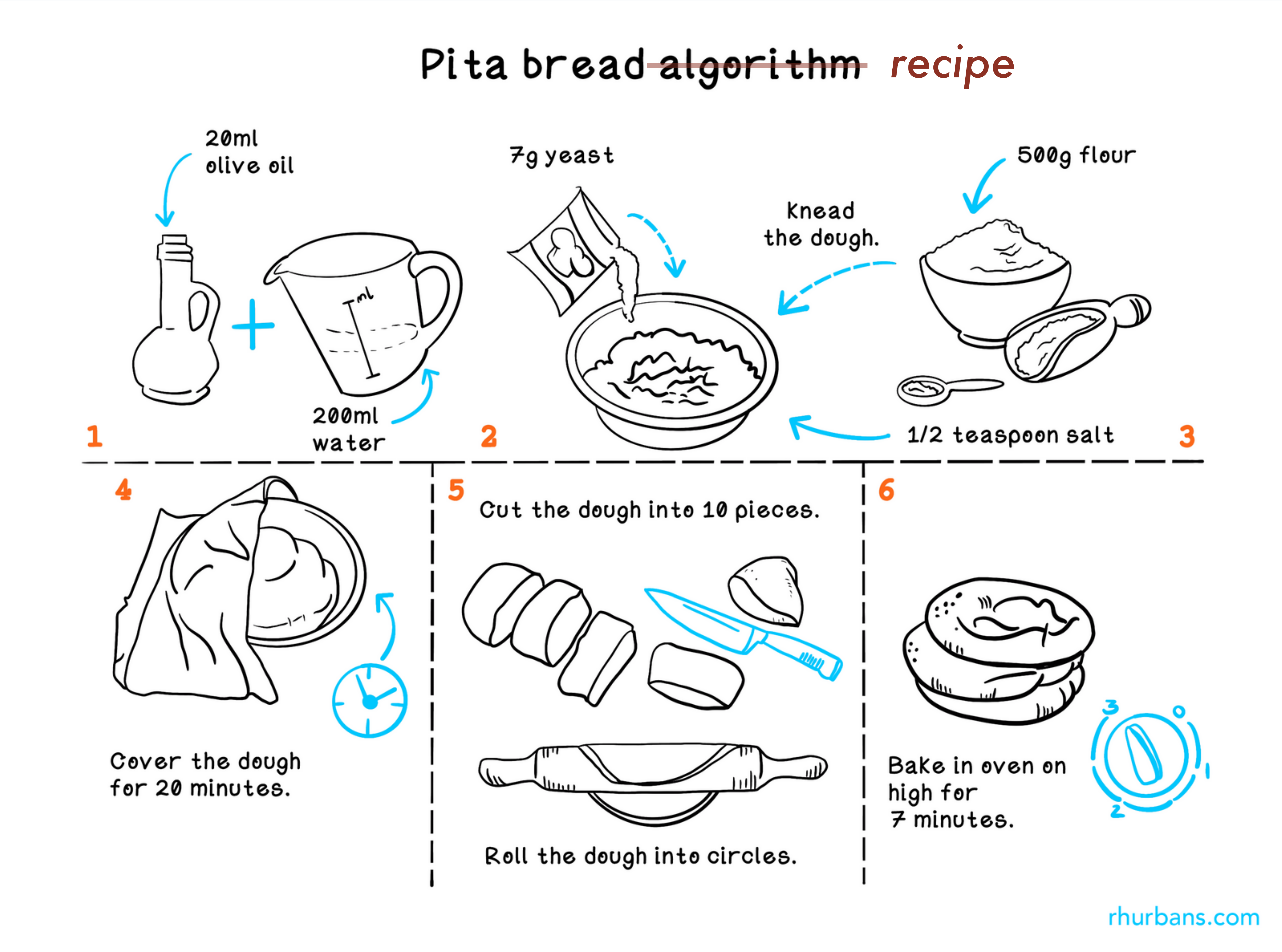
Algorithms


ALGO
RITHM


ALGORITHM
noun
1. A wird used by programmers when they don't want to explain what they did

Algorithms

Intuition. An algorithm is like a recipe. ← *but unambiguous and mechanically executable*



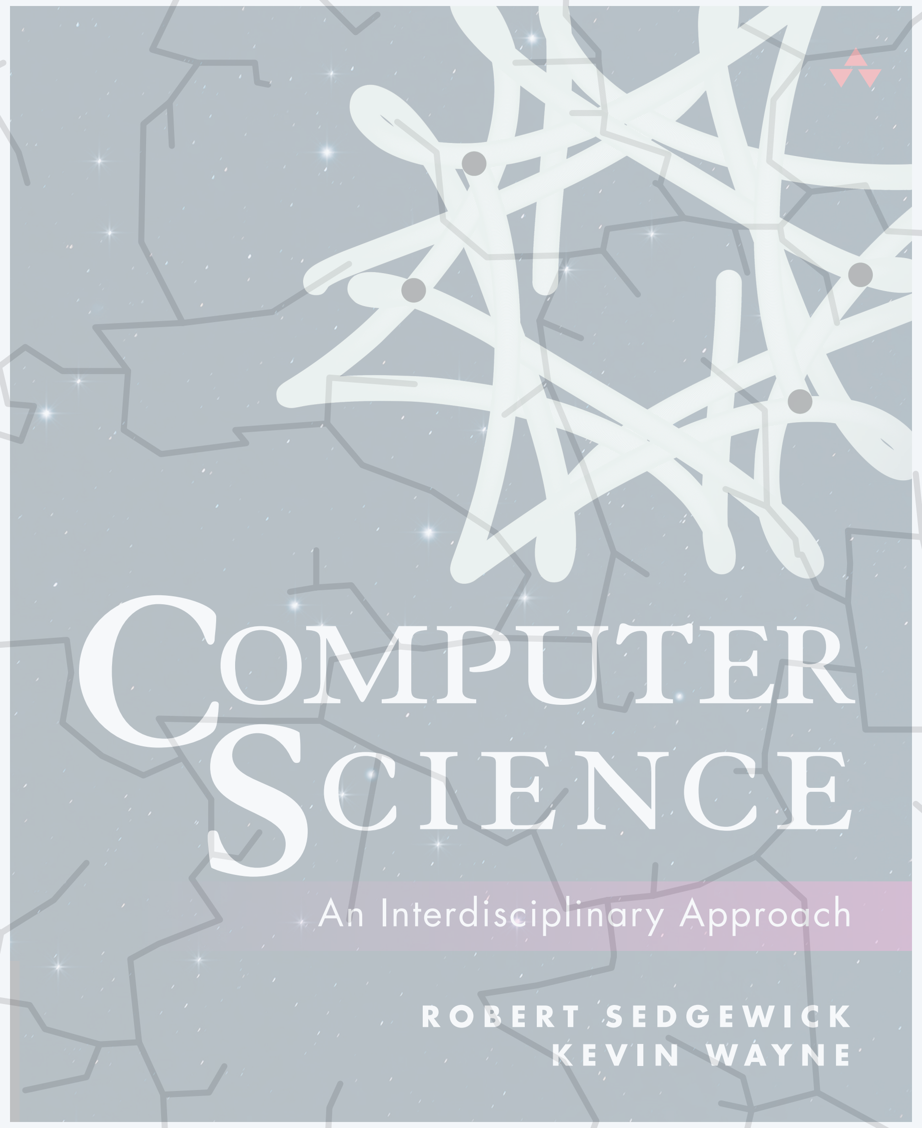
Algorithms

Algorithm. Step-by-step procedure for solving a problem.

- Takes input; produces output.
- Unambiguous and mechanically executable (e.g., in Java).

← *formalized by Turing machines (stay tuned)*

category	famous algorithms
historic	*Euclid's gcd algorithm, gradient descent, Newton's method*
sorting and searching	*binary search, insertion sort, mergesort* ← *today's lecture*
graphs	*DFS, BFS, Dijkstra, Kruskal, Ford–Fulkerson*
linear algebra	*Gaussian elimination, simplex method, QR method, PageRank*
scientific computing	*Smith–Waterman, Metropolis–Hastings, k-means, FFT*
machine learning	*A*-search, multiplicative weights, neural network, transformer*



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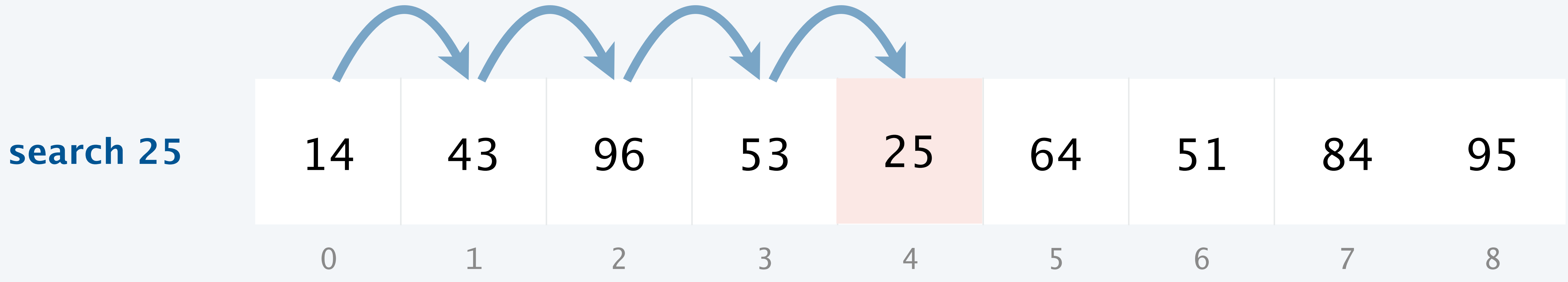
4.2 ALGORITHMS

- *sequential search*
- *binary search*
- *insertion sort*
- *mergesort*

Sequential search

Problem. Given an array of n elements and a search key, find index of search key in array.

Sequential search. Check each element in array until match is found.




```
public static int sequentialSearch(String[] a, String key) {
   int n = a.length;
   for (int i = 0; i < n; i++) {
      if (a[i].equals(key)) return i;
   }
   return -1;
}
```

we'll write code with `String` *keys*

return -1 to indicate key is not in array

In the worst case, how many equality tests (or array accesses) does sequential search make to search for a key in an array of length n?

A. $\Theta(1)$

B. $\Theta(\log n)$

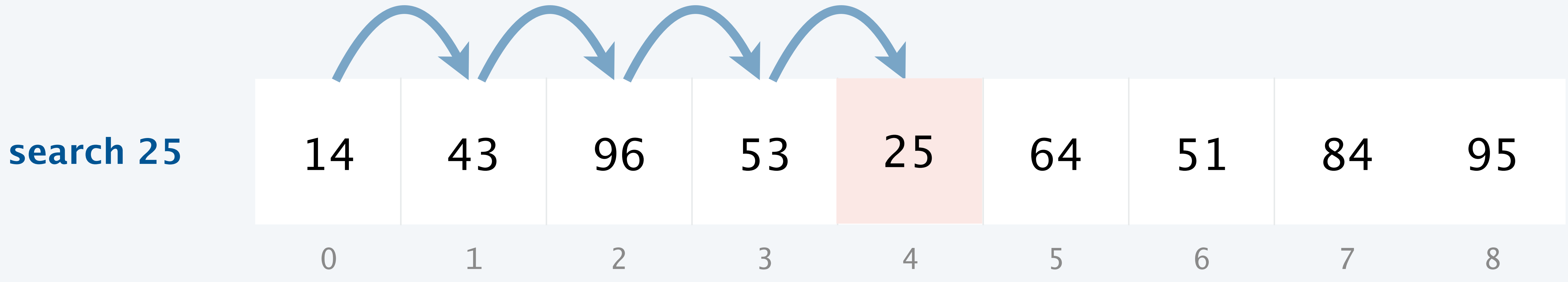
C. $\Theta(n)$

D. $\Theta(n^2)$

Sequential search

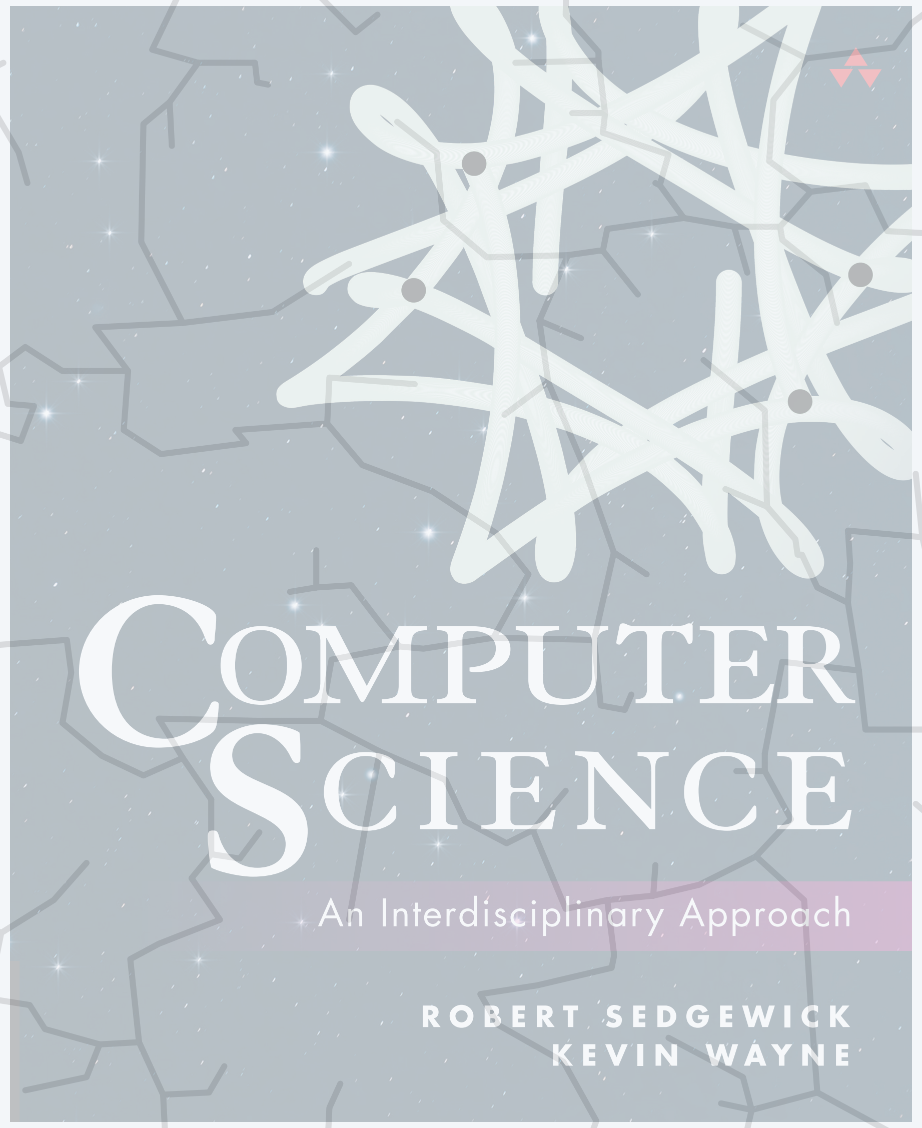
Problem. Given an array of n elements and a search key, find index of search key in array.

Sequential search (linear search). Check each element in array until match is found.



Cost model. Equality tests (or array accesses).

Performance. Sequential search solves the problem using $\leq n$ equality tests.



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4.2 ALGORITHMS

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Binary search

Problem. Given a sorted array of n elements and a search key, find index of search key in array.

Binary search. Compare search key with middle element.

- Too small, go left.
- Too big, go right.
- Equal, found.

sorted array

10	13	14	25	33	43	51	53	64	72	84	93	95	96	97
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
↑ *lo*														↑ *hi*

Binary search: Java implementation

Invariant. If key appears in array a[], then a[lo] ≤ key ≤ a[hi].

```
public static int binarySearch(String[] a, String key) {
   int lo = 0, hi = a.length - 1;
   while (lo <= hi) {
      int mid = lo + (hi - lo) / 2;
      int compare = key.compareTo(a[mid]);
      if      (compare < 0) hi = mid - 1;
      else if (compare > 0) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

essentially equivalent to
`mid = (lo + hi) / 2`
(but avoids arithmetic overflow)

return zero (if equal),
negative integer (if less),
positive integer (if greater)

In the worst case, how many compares (or array accesses) does binary search make to search for a key in a sorted array of length n?

- **A.** $\Theta(1)$
- **B.** $\Theta(\log n)$
- **C.** $\Theta(n)$
- **D.** $\Theta(n^2)$

Binary search: analysis

Problem. Given a sorted array of n elements and a search key, find index of search key in array.

Proposition. Binary search solves problem using $\leq 1 + \log_2 n$ compares.

Pf.

- Each iteration of `while` loop:
  - calls `compareTo()` once
  - decreases the length of remaining subarray by at least a factor of 2

← *can happen at most* $1 + \log_2 n$ *times. Why?*

$$n \to n/2 \to n/4 \to n/8 \to \cdots \to 2 \to 1$$

$1 + \log_2 n$

slightly better than 2×,
due to elimination of `a[mid]` *from subarray*
(or early termination of `while` *loop)*

Sequential search vs. binary search: empirical analysis

Running time and energy estimates (approximate):

- CPU core executes 10^8 compares/second.
- CPU core consumes 18 watts power.

Resources required. [on array of length $n = 10^9$]

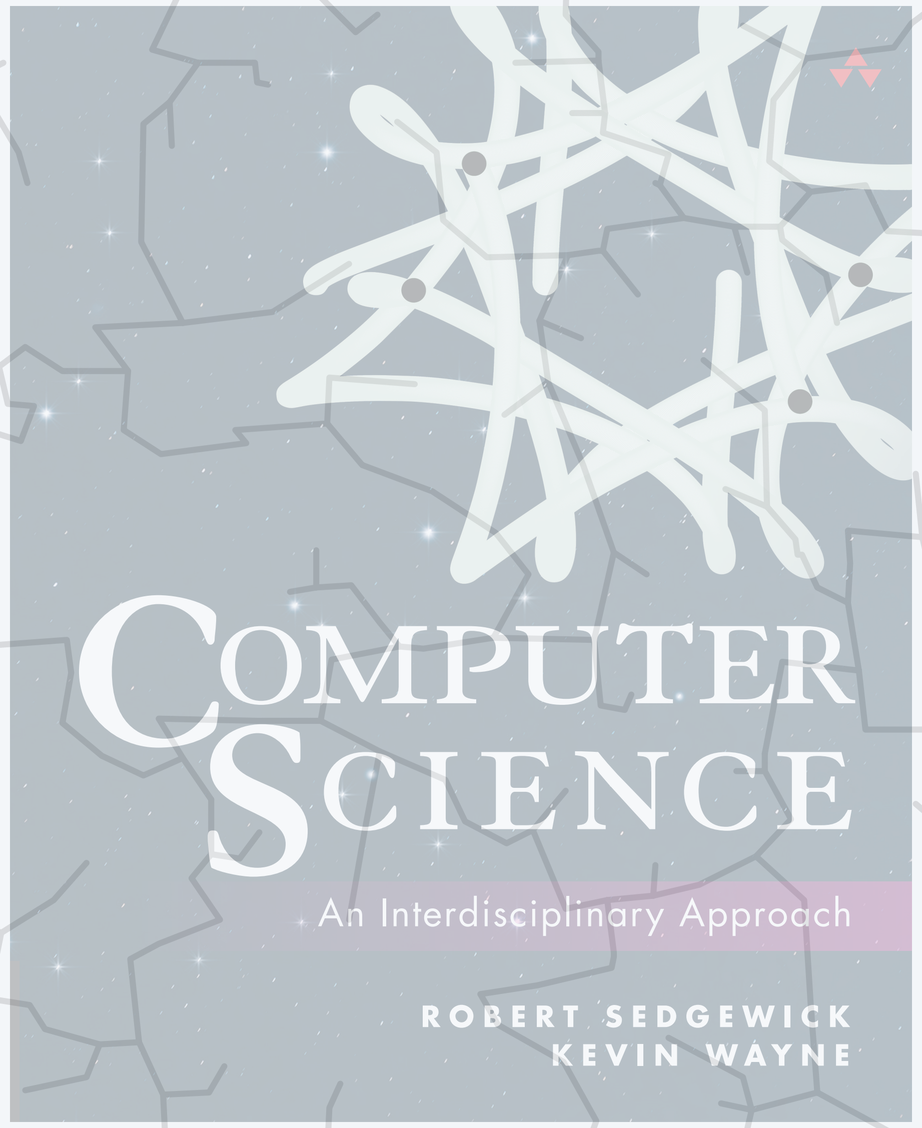
queries/hour	CPU cores	power
thousand	2.78	50 *watts*
million	2.78 *thousand*	50 *kilowatts*
billion	2.78 *million*	50 *megawatts*

sequential search $\Theta(n)$

queries/hour	CPU cores	power
million	–	–
billion	0.083	1.5 *watts*
trillion	83	1.5 *kilowatts*

binary search $\Theta(\log n)$ ← *about* 33M *times more efficient* (10^9 *vs*. $\log_2 10^9$)

Bottom line. Great algorithms can replace server farms. ← 500K *ChatGPT queries per hour* (≈ 1 *gigawatt*)



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4.2 ALGORITHMS

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- *binary search*
- *insertion sort*
- *mergesort*

Sorting problem

Problem. Given an array of n elements, rearrange in ascending order by key.

element → (Potter row)

key → (Malfoy)

Last ▾	First	House	Year
Longbottom	Neville	Gryffindor	1998
Weasley	Ron	Gryffindor	1998
Abbott	Hannah	Hufflepuff	1998
Potter	Harry	Gryffindor	1998
Chang	Cho	Ravenclaw	1997
Granger	Hermione	Gryffindor	1998
Malfoy	Draco	Slytherin	1998
Diggory	Cedric	Hufflepuff	1996
Weasley	Ginny	Gryffindor	1999
Parkinson	Pansy	Slytherin	1998

sorting hat

Sorting problem

Problem. Given an array of n elements, rearrange in ascending order by key.

Last ▾	First	House	Year
Abbott	Hannah	Hufflepuff	1998
Chang	Cho	Ravenclaw	1997
Granger	Hermione	Gryffindor	1998
Diggory	Cedric	Hufflepuff	1996
Longbottom	Neville	Gryffindor	1998
Malfoy	Draco	Slytherin	1998
Parkinson	Pansy	Slytherin	1998
Potter	Harry	Gryffindor	1998
Weasley	Ron	Gryffindor	1998
Weasley	Ginny	Gryffindor	1999

key → (Malfoy)

element → (Potter row)

↑ *sorted by key*

sorting hat

Sorting applications

International Departures

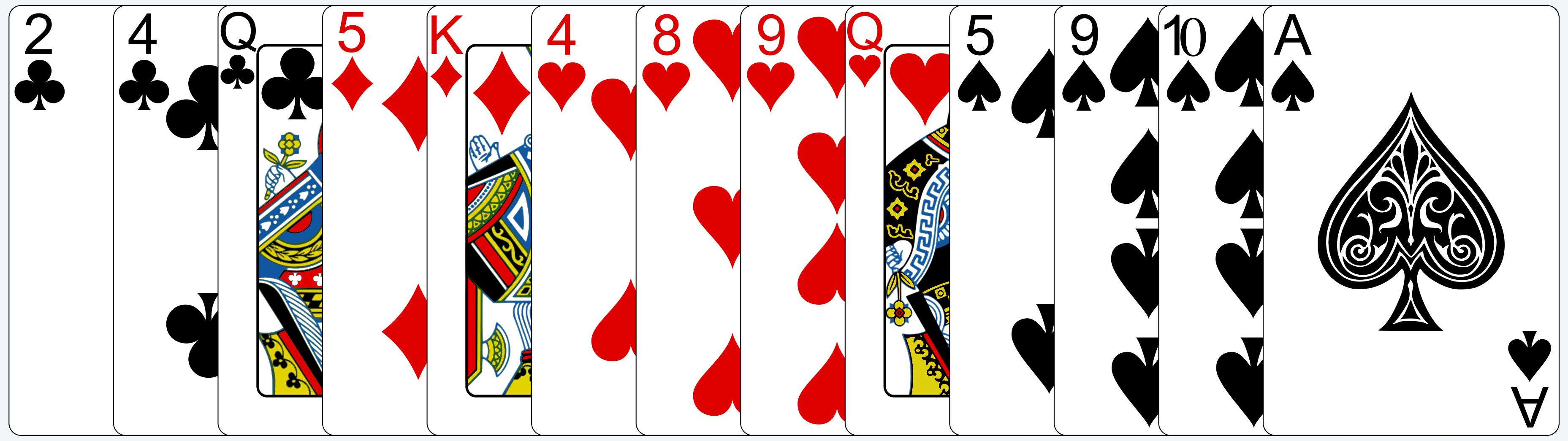
Flight No	Destination	Time	Gate	Remarks
CX7183	Berlin	7:50	A-11	Gate closing
QF3474	London	7:50	A-12	Gate closing
BA372	Paris	7:55	B-10	Boarding
AY6554	New York	8:00	C-33	Boarding
KL3160	San Francisco	8:00	F-15	Boarding
BA8903	Manchester	8:05	B-12	Gate lounge open
BA710	Los Angeles	8:10	C-12	Check-in open
QF3371	Hong Kong	8:15	F-10	Check-in open
MA4866	Barcelona	8:15	F-12	Check-in at kiosks
CX7221	Copenhagen	8:20	G-32	Check-in at kiosks

chronological order

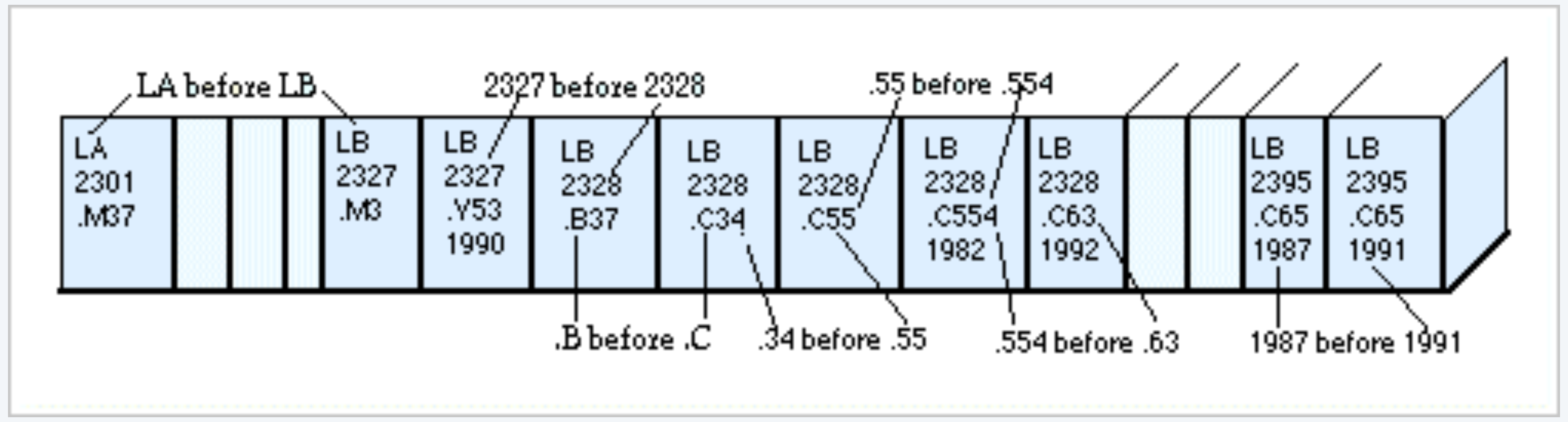
No.	Video name	Views (billions)
1.	"Baby Shark Dance"[3]	10.15
2.	"Despacito"[6]	7.73
3.	"Johny Johny Yes Papa"[12]	6.15
4.	"Shape of You"[13]	5.61
5.	"See You Again"[15]	5.41

numerical order (descending)

alphabetical order



suit and rank order

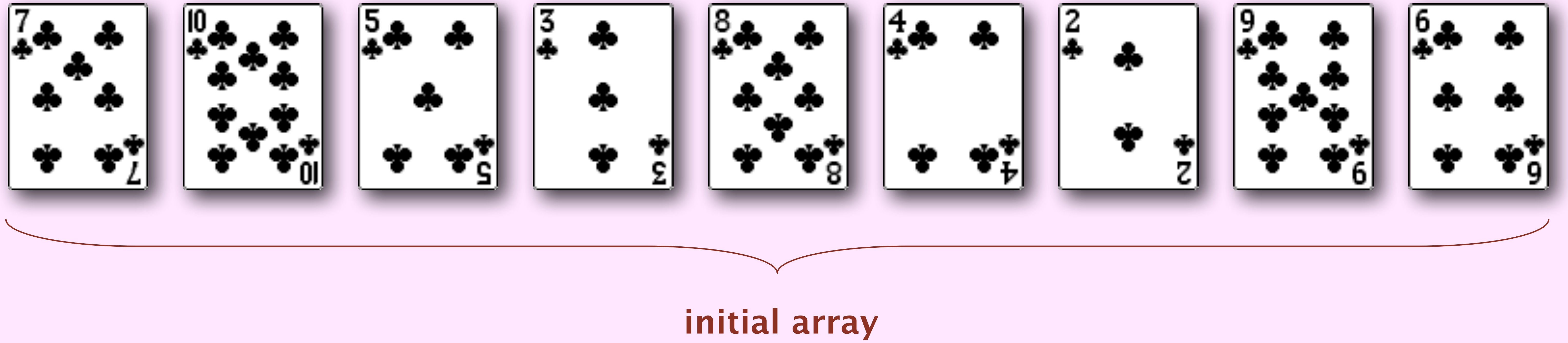


Library of Congress order

Insertion sort demo

Algorithm. For each index $i = 0$ to $n - 1$:

- Let x be the element at index i.
- Repeatedly exchange x with each larger element to its immediate left.



Insertion sort demo (Romanian folk dance)

Algorithm. For each index $i = 0$ to $n - 1$:

- Let x be the element at index i.
- Repeatedly exchange x with each larger element to its left.

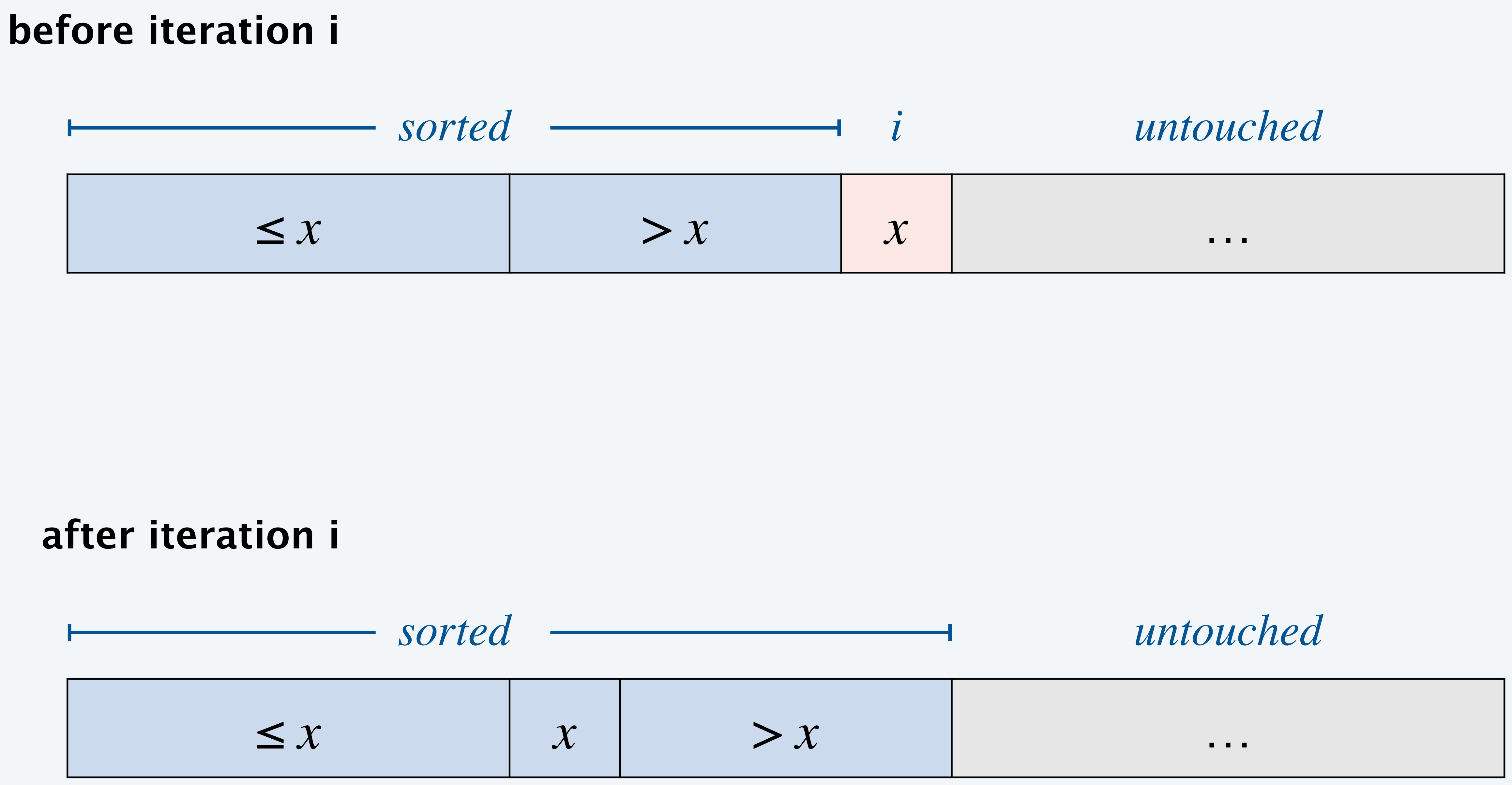
https://www.youtube.com/watch?v=ROalU379l3U

Insertion sort

Algorithm. For each index $i = 0$ to $n - 1$:

- Let x be the element at index i.
- Repeatedly exchange x with each larger element to its left.

Invariants.



Insertion sort: Java implementation

```
public class Insertion {

   public static void sort(String[] a) {
      int n = a.length;
      for (int i = 0; i < n; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;   ← breaks out of innermost loop
   }

   private static boolean less(String v, String w) {
      return v.compareTo(w) < 0;
   }

   private static void exch(String[] a, int i, int j) {
      String temp = a[i];
      a[i] = a[j];
      a[j] = temp;
   }

}
```

How many compares to insertion sort an array of n distinct keys in reverse order?

A. $\Theta(1)$

B. $\Theta(\log n)$

C. $\Theta(n)$

D. $\Theta(n^2)$

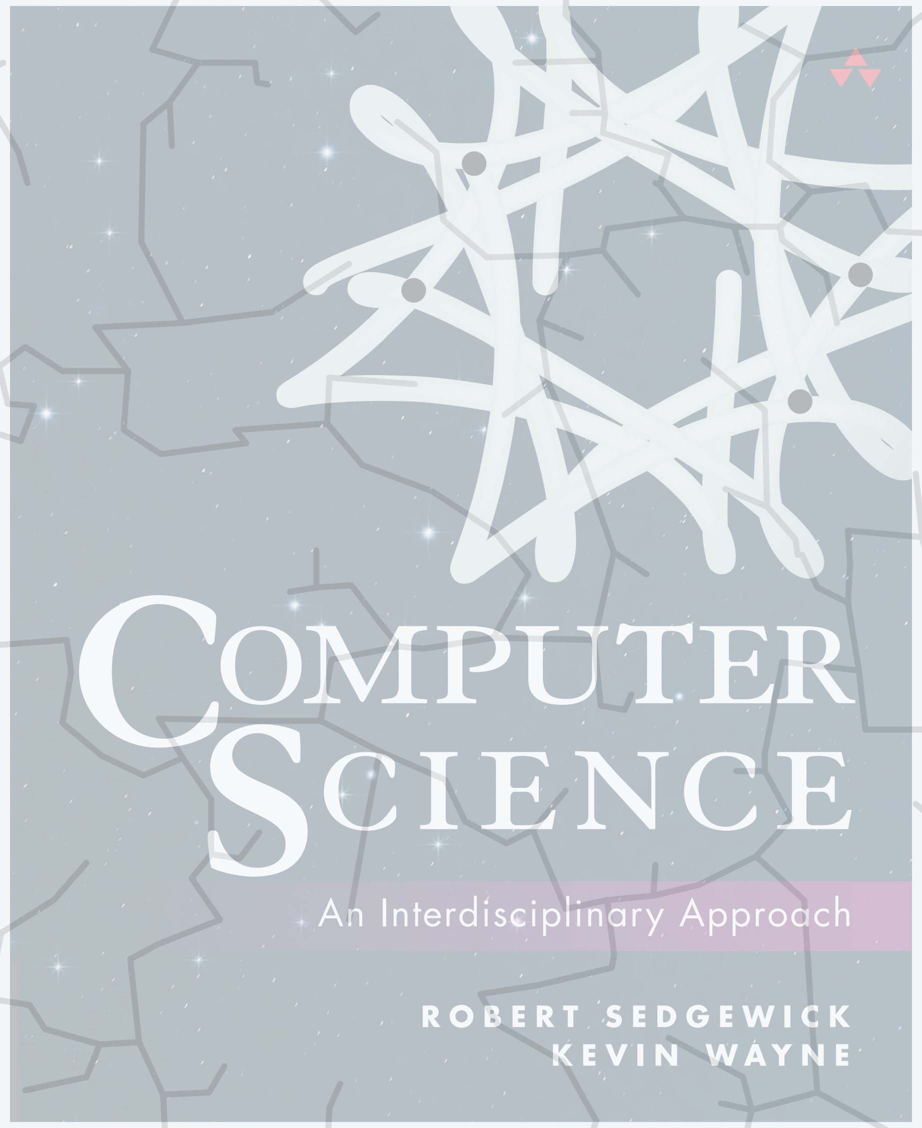
Insertion sort: analysis

Sorting cost model. Number of compares (or array accesses).

Proposition. Insertion sort never makes more than $\sim \frac{1}{2} n^2$ compares to sort an array of length n.

Pf.

- The worst case is a reverse-sorted array of n distinct keys.
- In iteration i, insertion sort makes i compares.
- Total number of compares = $0 + 1 + 2 + \ldots + (n-1) = \frac{1}{2} n(n-1)$.



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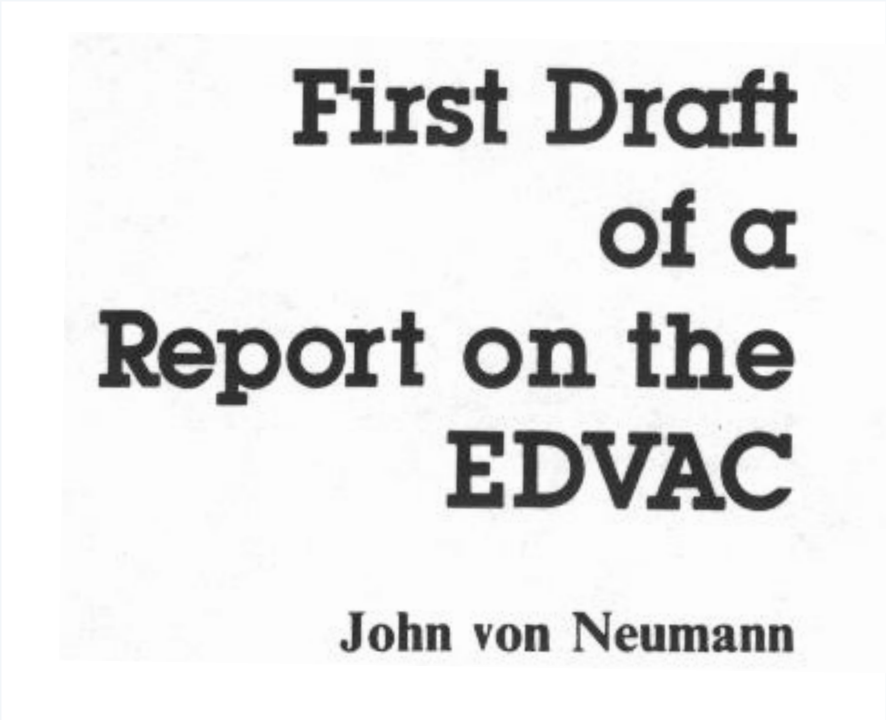
4.2 ALGORITHMS

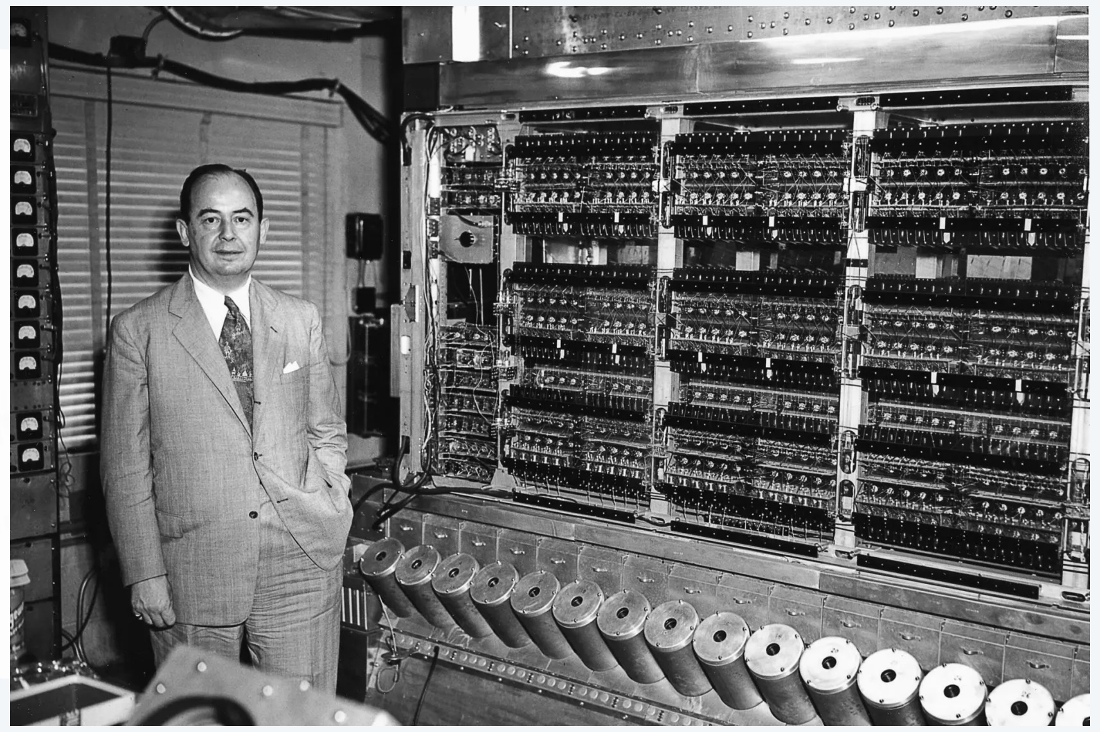
- *sequential search*
- *binary search*
- *insertion sort*
- *mergesort*

Mergesort overview

Basic plan.

- Divide array into two halves.
- Recursively sort left half.
- Recursively sort right half.
- Merge two sorted halves.

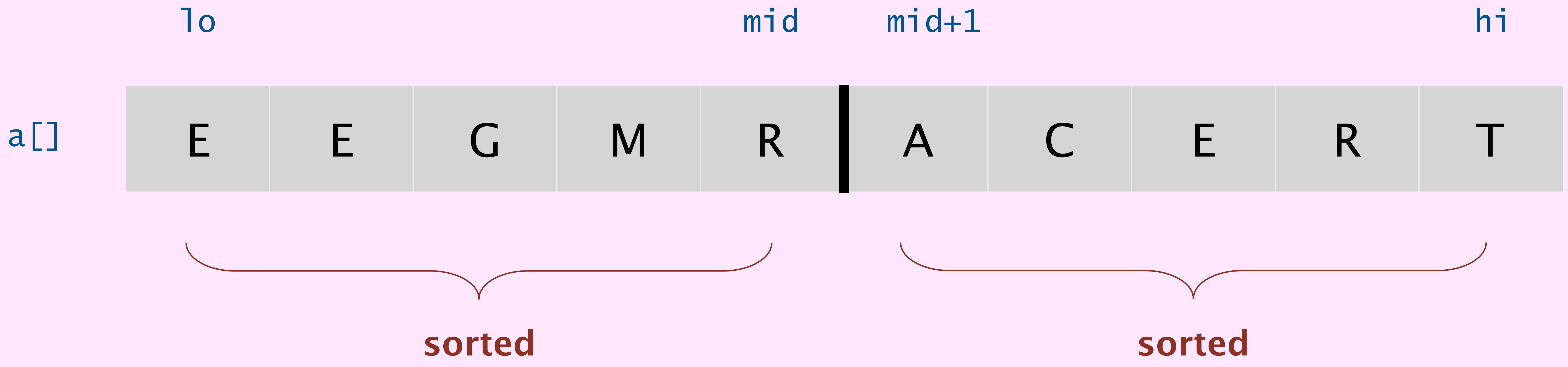




input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Abstract in-place merge demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.

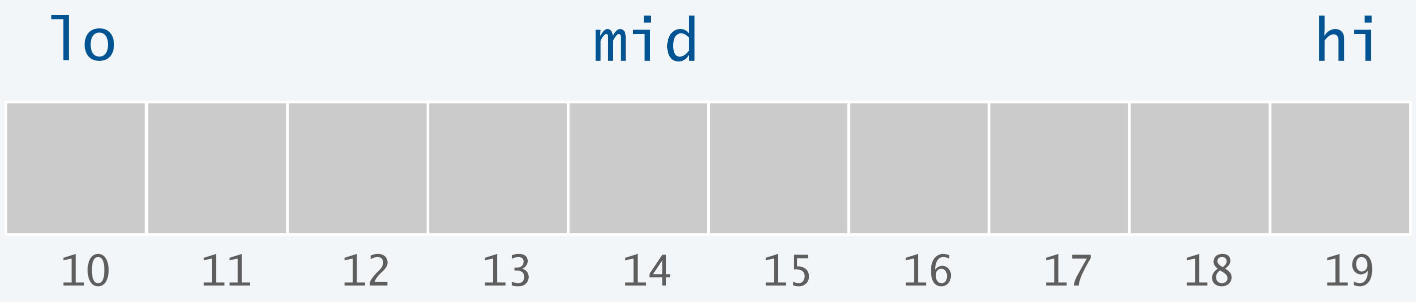


Merging: Java implementation

```
private static void merge(String[] a, String[] aux, int lo, int mid, int hi) {
   for (int k = lo; k <= hi; k++)        copy
      aux[k] = a[k];

   int i = lo, j = mid+1;                                          merge
   for (int k = lo; k <= hi; k++) {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

Mergesort: Java implementation

```
public class Merge {
   private static void merge(...) {
      /* as before */
   }

   private static void sort(String[] a, String[] aux, int lo, int hi) {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(String[] a) {
      String[] aux = new String[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort: trace

a[]

merge(a, aux, lo, mid, hi)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

lo → first argument; hi → last argument

← *result after recursive call*

Insertion sort vs. mergesort: empirical analysis

Running time estimates (approximate):

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

n	laptop	super
thousand	*instant*	*instant*
million	2.8 *hours*	1 *second*
billion	317 *years*	1 *week*

insertion sort

n	laptop	super
thousand	*instant*	*instant*
million	1 *second*	*instant*
billion	18 *minutes*	*instant*

mergesort

Bottom line. Great algorithms are better than supercomputers.

Algorithms: quiz 4

In the worst case, how many compares to merges two sorted subarrays of length $n / 2$ to produce a sorted array of length n ?

A. $n / 2$

B. $n - 1$

C. $\Theta(n \log n)$

D. $\Theta(n^2)$

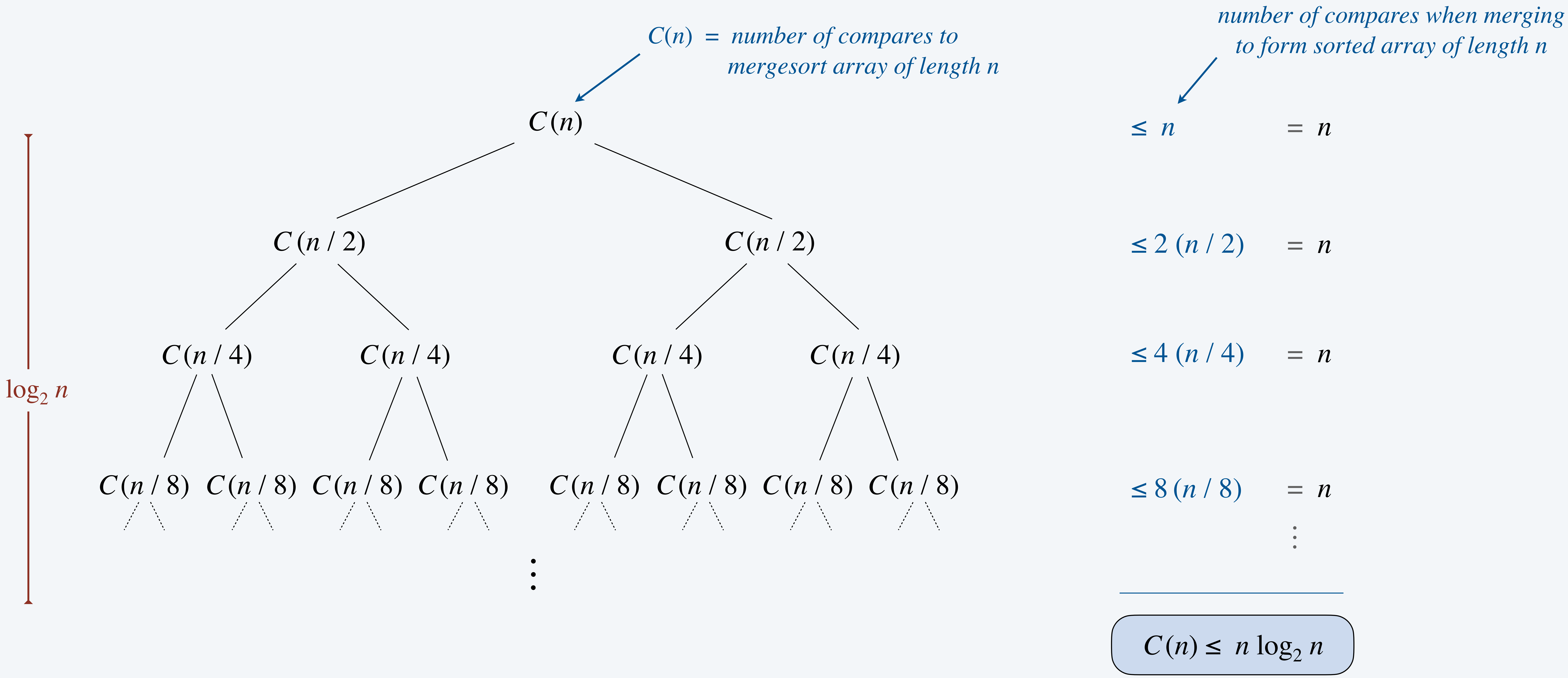
merging two sorted arrays, each of length n/2

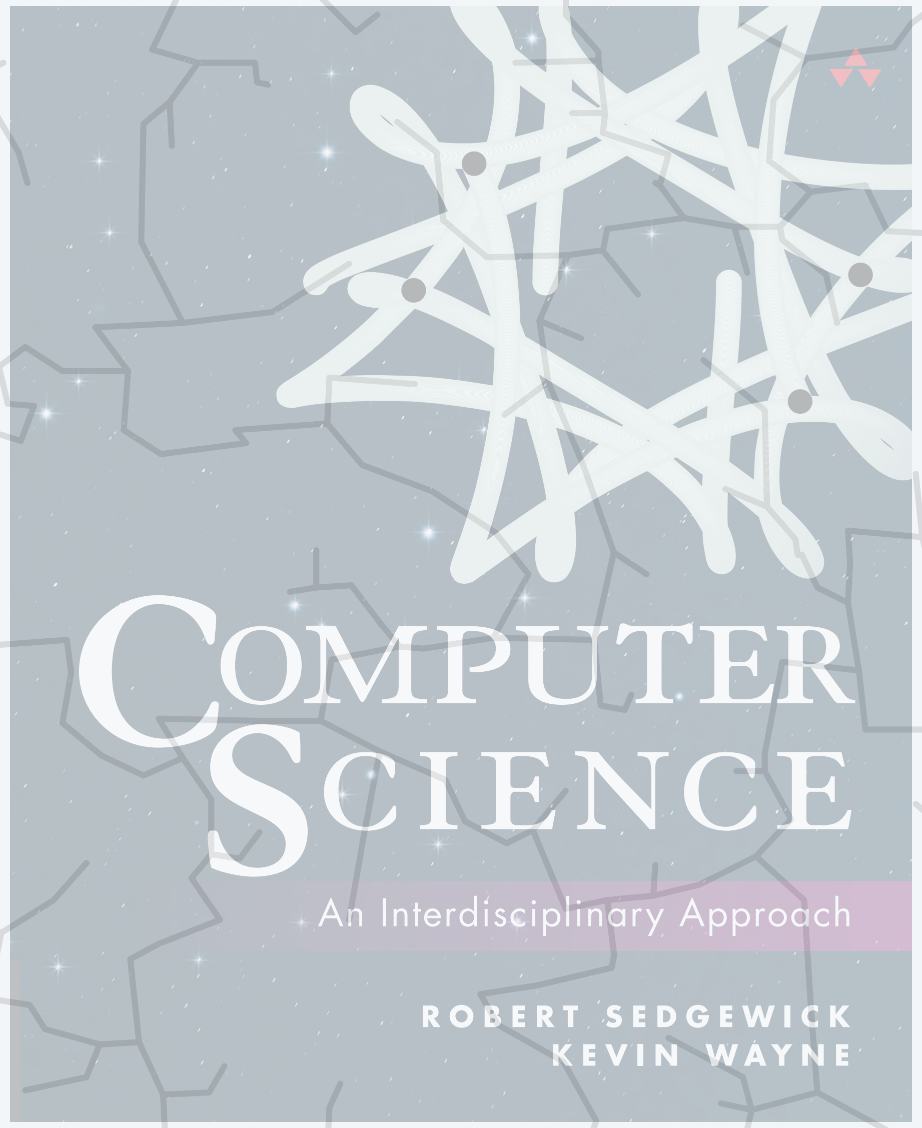
a_0	a_1	a_2	a_3		b_0	b_1	b_2	b_3

Mergesort analysis

Proposition. Mergesort makes $\leq n \log_2 n$ compares to sort any array of length n.

Pf by picture. [assuming n is a power of 2]





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4.2 ALGORITHMS

- *sequential search*
- *binary search*
- *insertion sort*
- *mergesort*
- *system libraries*

Arrays Javadoc

`java.util.Arrays` contains static methods for manipulating arrays. ← *sorting, searching, comparing, …*

Module java.base
Package java.util

Class Arrays

java.lang.Object
    java.util.Arrays

```
public class Arrays
extends Object
```

This class contains various methods for manipulating arrays (such as sorting and searching). This class also contains a static factory that allows arrays to be viewed as lists.

The methods in this class all throw a `NullPointerException`, if the specified array reference is null, except where noted.

The documentation for the methods contained in this class includes brief descriptions of the *implementations*. Such descriptions should be regarded as *implementation notes*, rather than parts of the *specification*. Implementors should feel free to substitute other algorithms, so long as the specification itself is adhered to. (For example, the algorithm used by `sort(Object[])` does not have to be a MergeSort, but it does have to be *stable*.)

https://docs.oracle.com/en/java/javase/11/docs/api/java.base/java/util/Arrays.html

System sort and binary search

`Arrays.sort()` and `Arrays.binarySearch()` each refer to several overloaded methods.

- For primitive types.
- For reference types. ← *must be* `Comparable` (*such as* `String`, *stay tuned*)
- For subarrays.

to access methods defined in `java.util.Arrays`

```
import java.util.Arrays;

public class TestSort {
   public static void main(String[] args) {
      Arrays.sort(args);
      StdOut.println(Arrays.toString(args));
   }
}
```

useful function to print elements in an array (separated by commas, enclosed in square braces)

```
~/cos126/algorithms> java-introcs TestSort S O R T M E
[E, M, O, R, S, T]
```

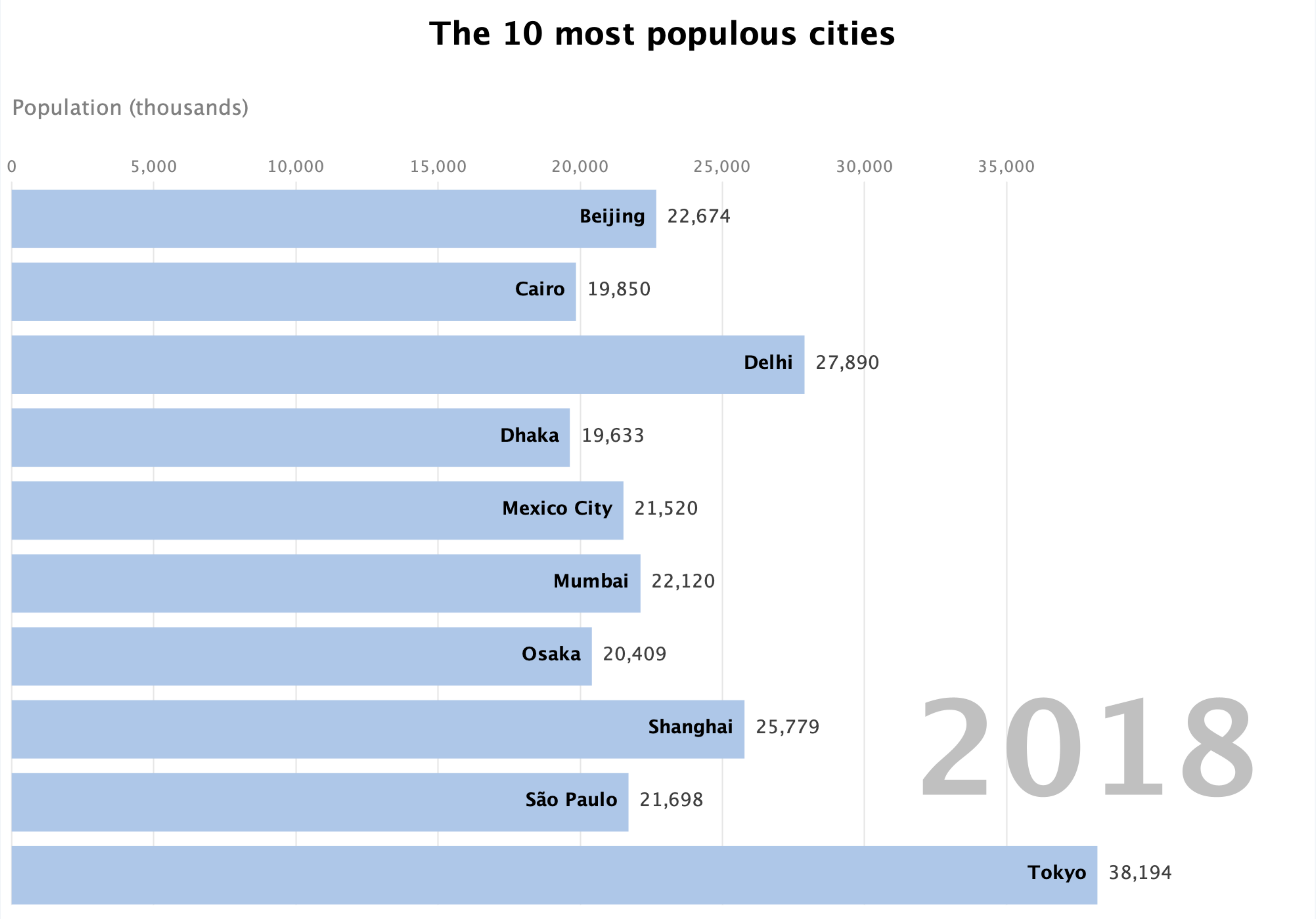
Bar charts

BarChart is a data type (with simple API) to draw bar charts.

```
// create the bar chart
String title = "The 10 most populous cities";
String xLabel = "Population (thousands)";
String source = "Source: United Nations";
BarChart chart = new BarChart(title, xLabel, source);

// add the bars and caption to the bar chart
chart.add("Beijing",     22674, "East Asia");
chart.add("Cairo",       19850, "Middle East");
chart.add("Delhi",       27890, "South Asia");
chart.add("Dhaka",       19633, "South Asia");
chart.add("Mexico City", 21520, "Latin America");
chart.add("Mumbai",      22120, "South Asia");
chart.add("Osaka",       20409, "East Asia");
chart.add("Shanghai",    25779, "East Asia");
chart.add("São Paulo",   21698, "Latin America");
chart.add("Tokyo",       38194, "East Asia");
chart.setCaption("2018");

// draw the bar chart
chart.draw();
```



Sorted bar chart

Sort the bars in descending order by value (e.g., population).

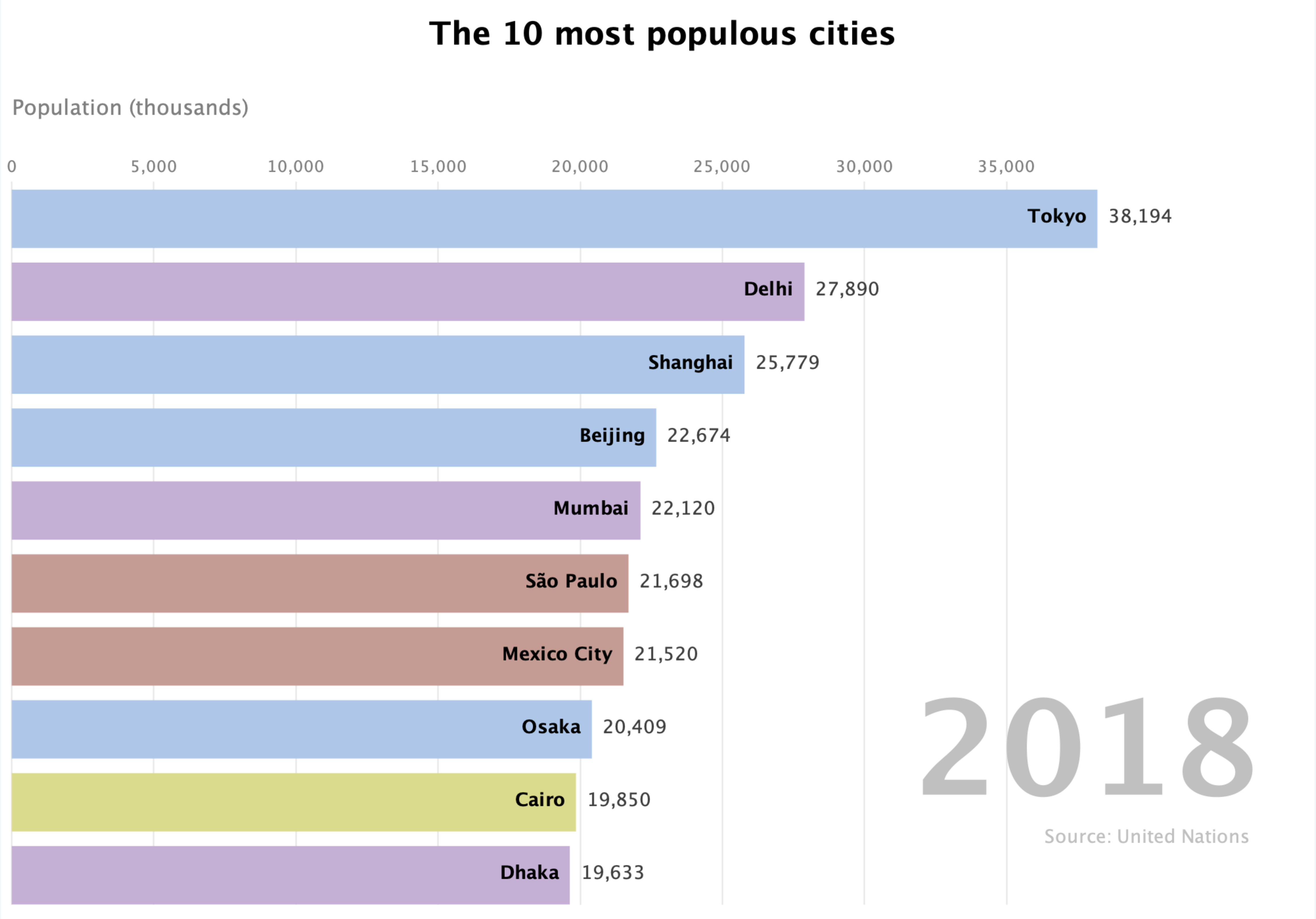
Color the bars by category (e.g., region).

name and region

Challenge. Maintain auxiliary info when sorting.

Solution. Define helper `Bar` data type.

Latin America



bars appear in descending order by population

Bar data type

To create a user-defined data type for use with `Arrays.sort()`:

- Declare the data type to be `Comparable`.
- Include a `compareTo()` method that compares two objects.

```
public class Bar implements Comparable<Bar>  {
   private final String name;
   private final int value;
   private final String category;

   public Bar(String name, int value, String category) {
      this.name = name;
      this.value = value;
      this.category = category;
   }

   public int compareTo(Bar that) {
      if (this.value < that.value) return -1;
      if (this.value > that.value) return +1;
      return 0;
   }
   ...
}
```

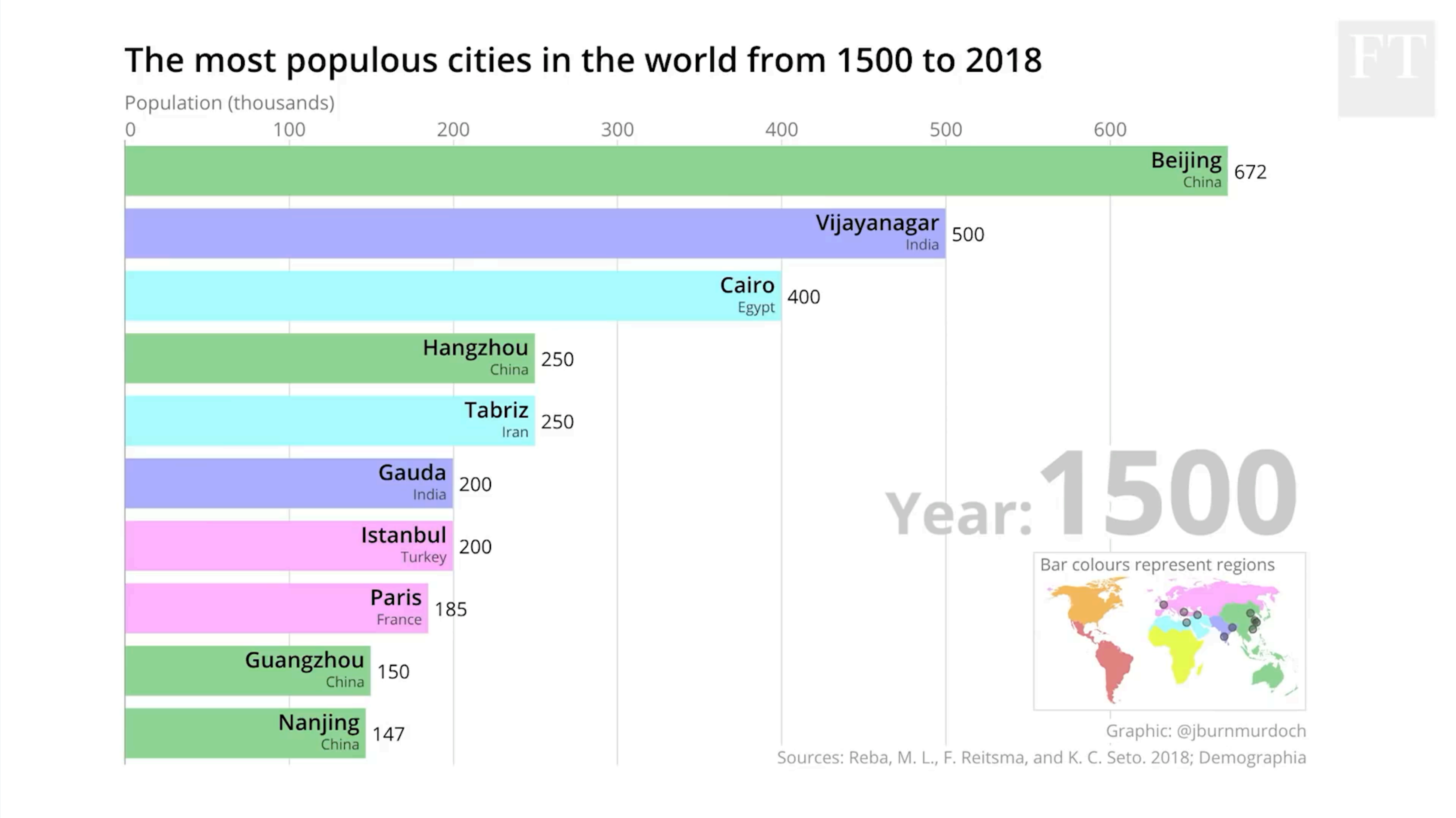
return a negative integer (smaller), positive integer (larger), or zero (equal)

```
Bar[] bars = new Bar[10];
...
Arrays.sort(bars);
```

sort bars in ascending order by value

Bar chart race

Animated bar chart. Draw sorted bar chart (largest $k = 10$) for each year.



https://www.youtube.com/watch?v=pMs5xapBewM

Credits

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