Parallel Sequences

COS 326

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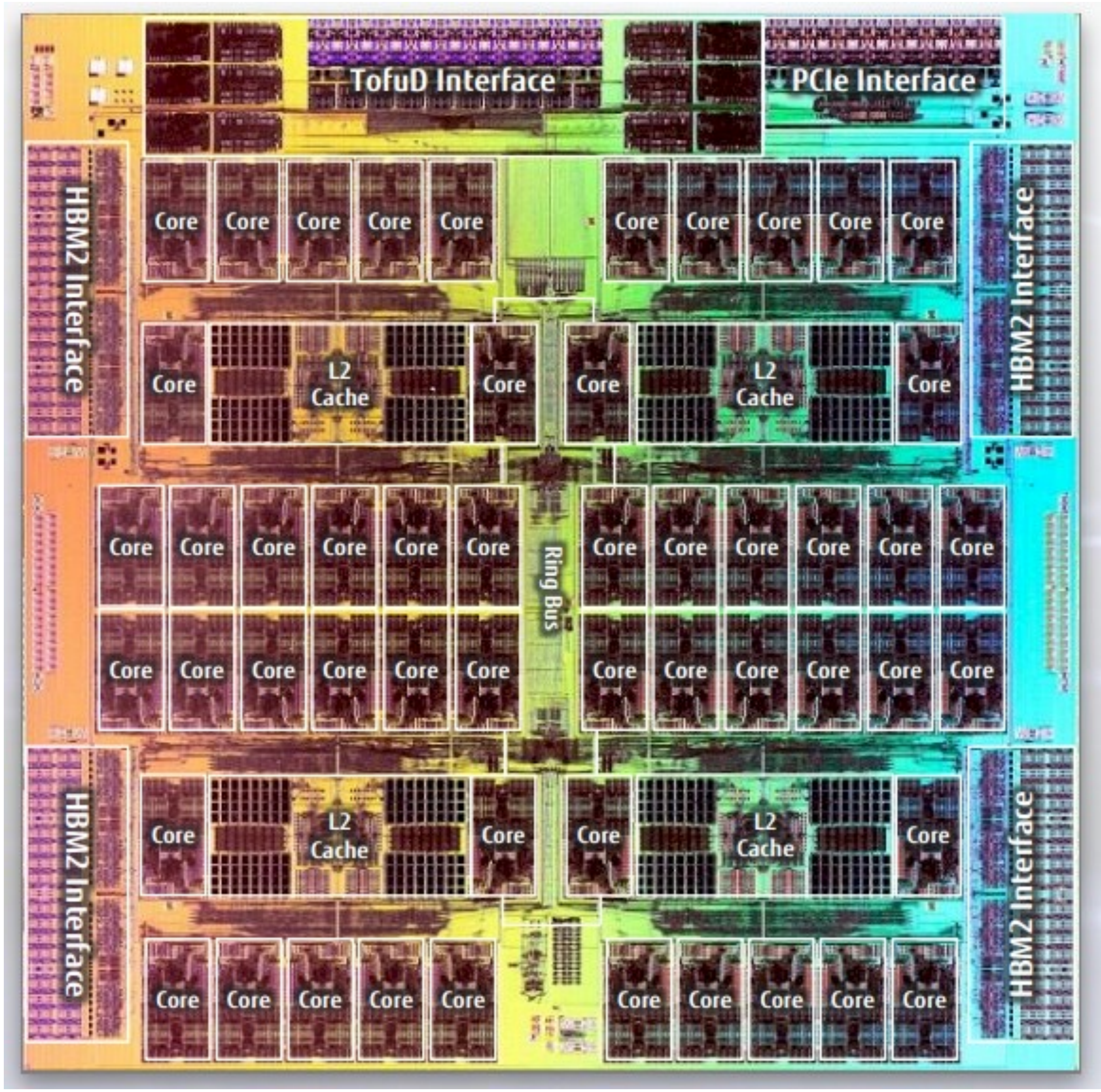
Parallel Programming



Fujitsu A64FX (48 ARM cores)

Programming with shared mutable data is very hard!

How can we leverage

- pure functions
- immutable data
- function composition

to write large-scale parallel programs?

What if you had a really big job to do?

Example: Create an index of every web page on the planet.

- Google does that regularly!
- There are billions of them!

Example: Search facebook for a friend or twitter for a tweet

To get big jobs done, we typically need 1000s of computers, but:

- how do we distribute work across all those computers?
- you definitely can't use shared-memory parallelism because the computers don't share memory!
- when you use 1 computer, you just hope it doesn't fail. If it does, you go to the store, buy a new one and restart the job.
- when you use 1000s of computers at a time, failures become the norm. what to do when 1 of 1000 computers fail? Start over?

Big Jobs ---> Better Abstractions

Need high-level interfaces to shield application programmers from the complex details. Complex implementations solve the problems of distribution, fault tolerance and performance.

Common abstraction: Parallel collections

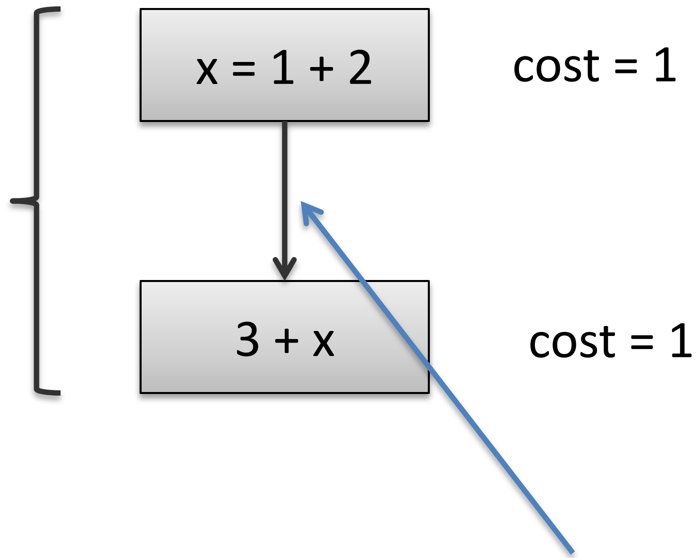
Example collections: sets, tables, dictionaries, sequences

Example bulk operations: create, map, reduce, join, filter

COMPLEXITY OF PARALLEL ALGORITHMS

Visualizing Computational Costs

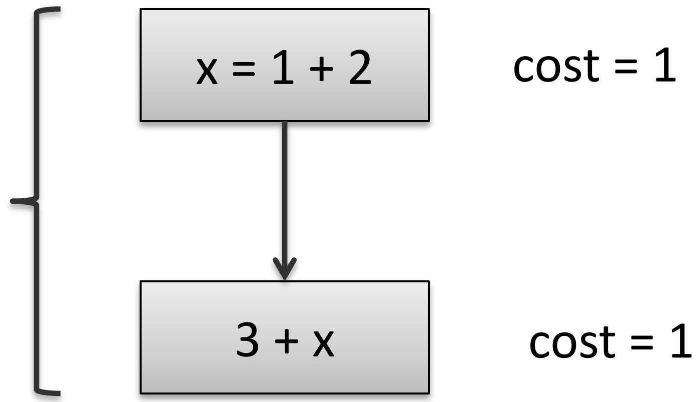
let x = 1 + 2 in
3 + x



dependence:
x = 1 + 2 *happens before* 3 + x

Visualizing Computational Costs

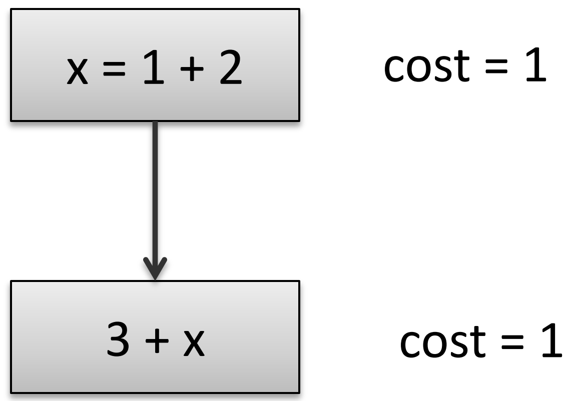
let x = 1 + 2 in
3 + x



<u>**Execution of dependency diagrams:**</u> A processor can only begin executing the computation associated with a block when the computations of all of its predecessor blocks have been completed.

Visualizing Computational Costs

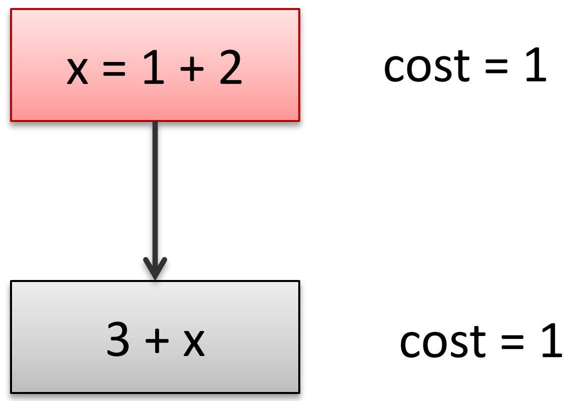
step 1:
execute first block



Cost so far: 0

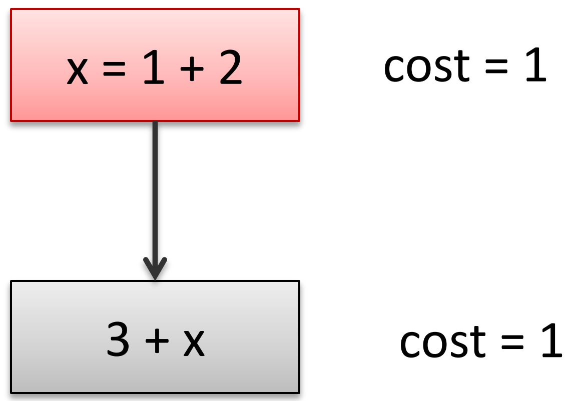
Visualizing Computational Costs

step 1:
execute first block



Cost so far: 1

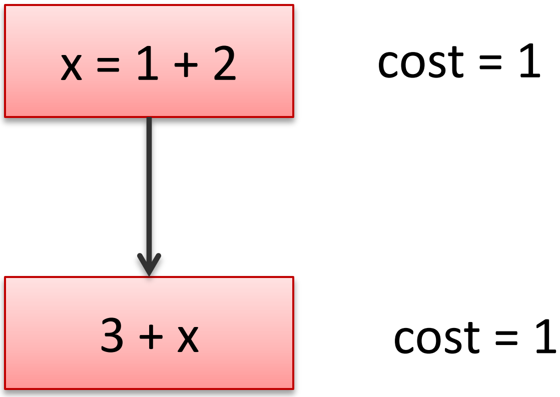
Visualizing Computational Costs



step 2:
execute second block
because all of its
predecessors have
been completed

Cost so far: 1

Visualizing Computational Costs

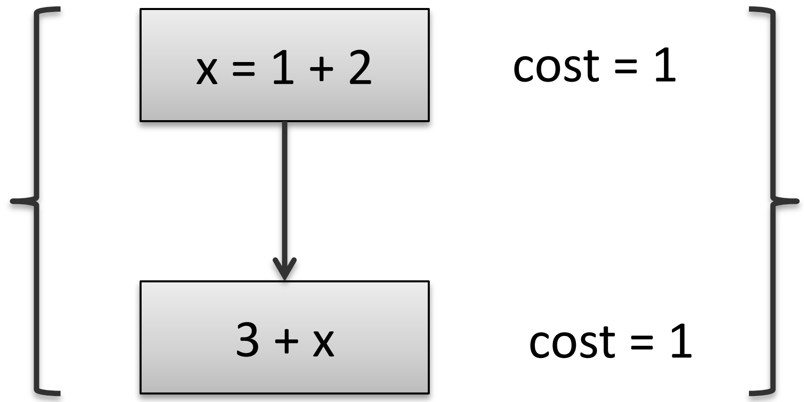


step 2:
execute second block because all of its predecessors have been completed

Cost so far: 1 + 1

Visualizing Computational Costs

let x = 1 + 2 in
3 + x



total cost
= 1 + 1
= 2

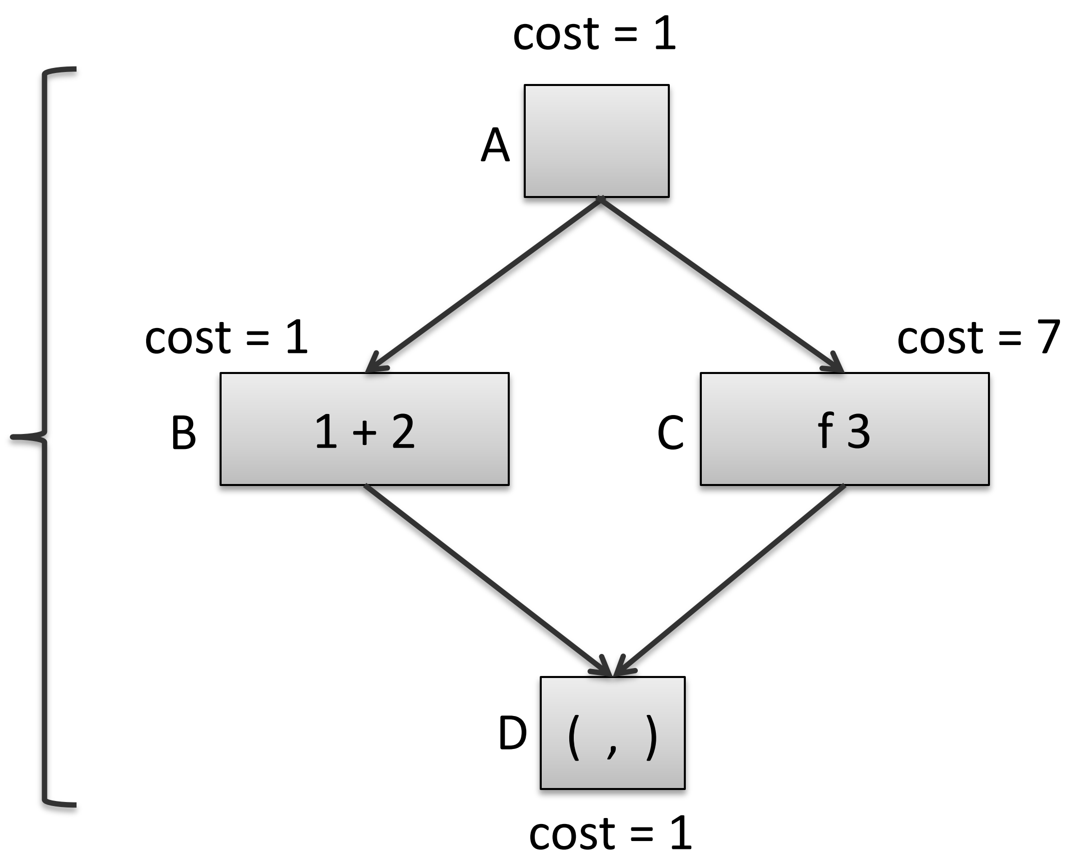
Visualizing Computational Costs

(1 + 2 || f 3)

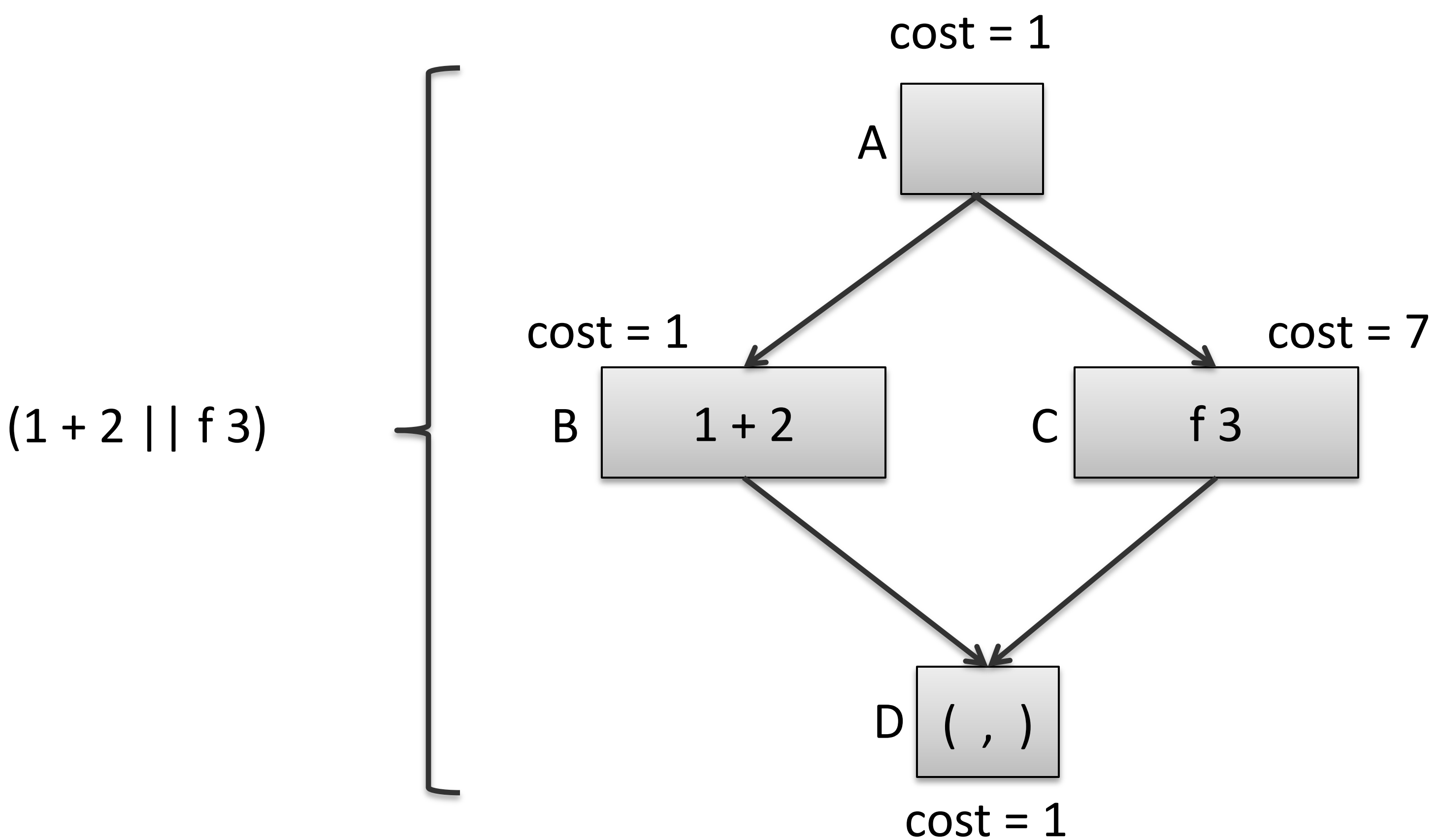
parallel pair:
compute both left and right-hand sides independently
return pair of values
(easy to implement using futures)

Visualizing Computational Costs

(1 + 2 || f 3)



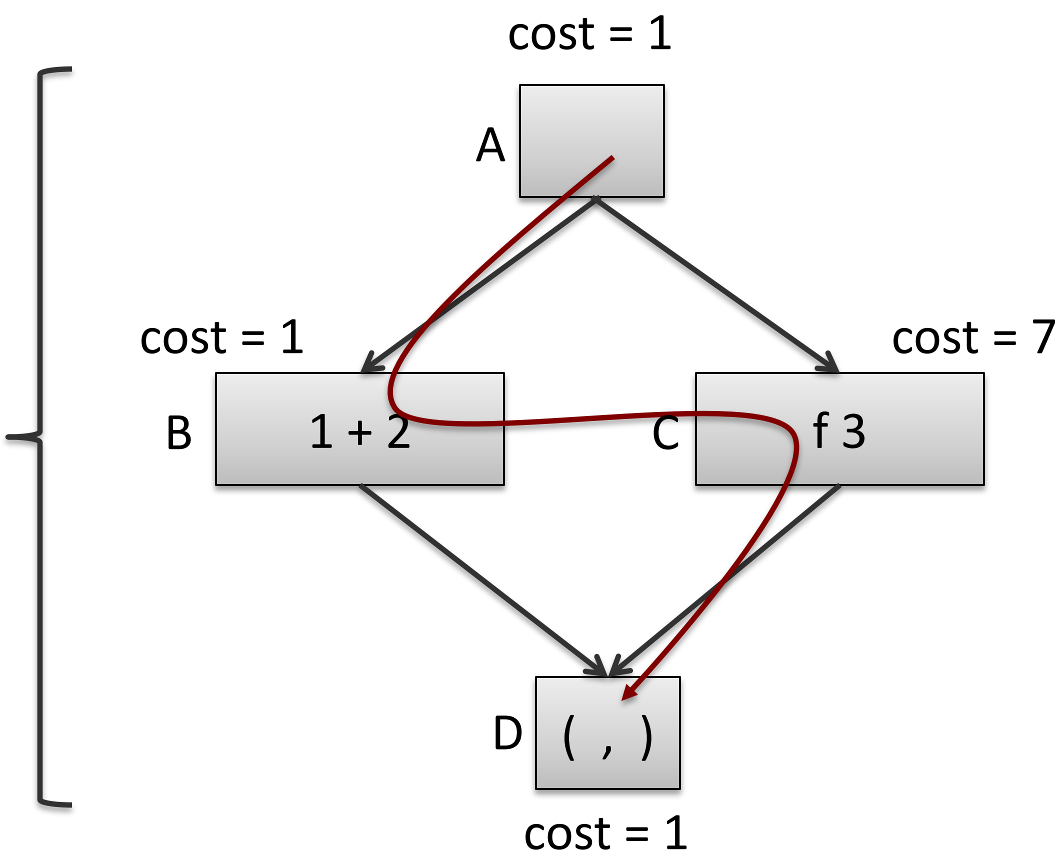
Visualizing Computational Costs



Suppose we have 1 processor. How much time does this computation take?

Visualizing Computational Costs

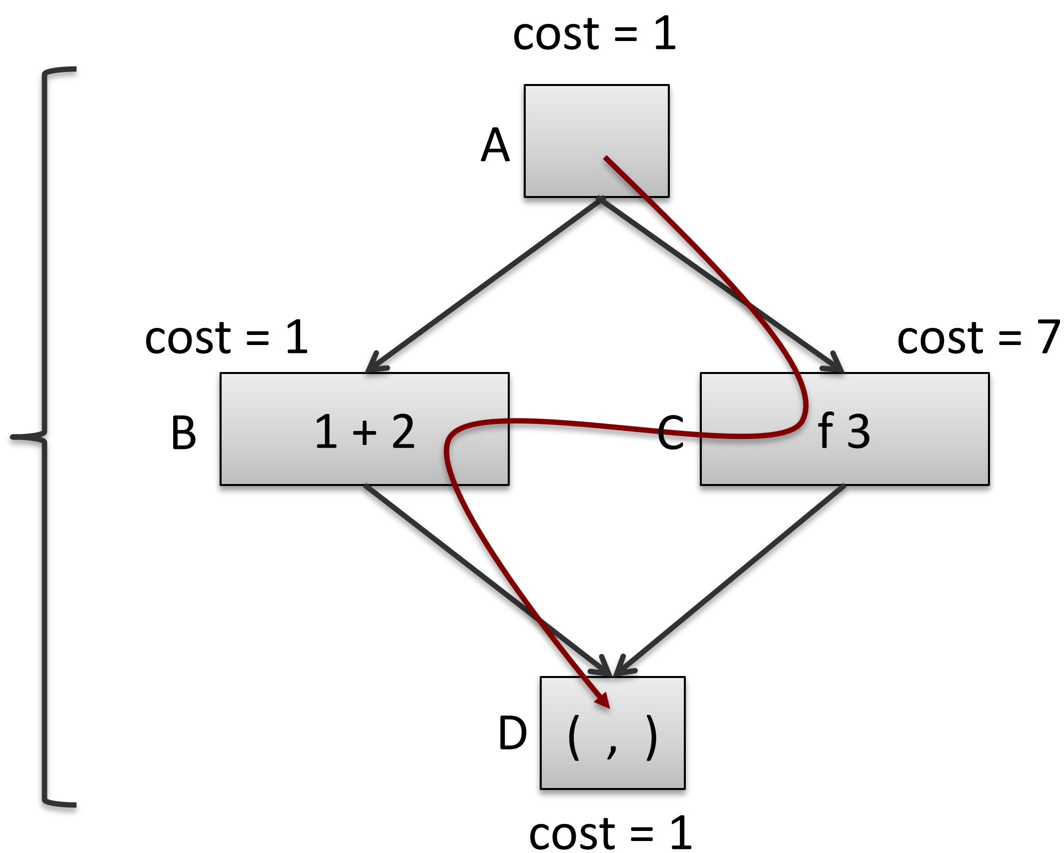
(1 + 2 || f 3)



Suppose we have 1 processor. How much time does this computation take?
Schedule A-B-C-D: 1 + 1 + 7 + 1

Visualizing Computational Costs

(1 + 2 || f 3)



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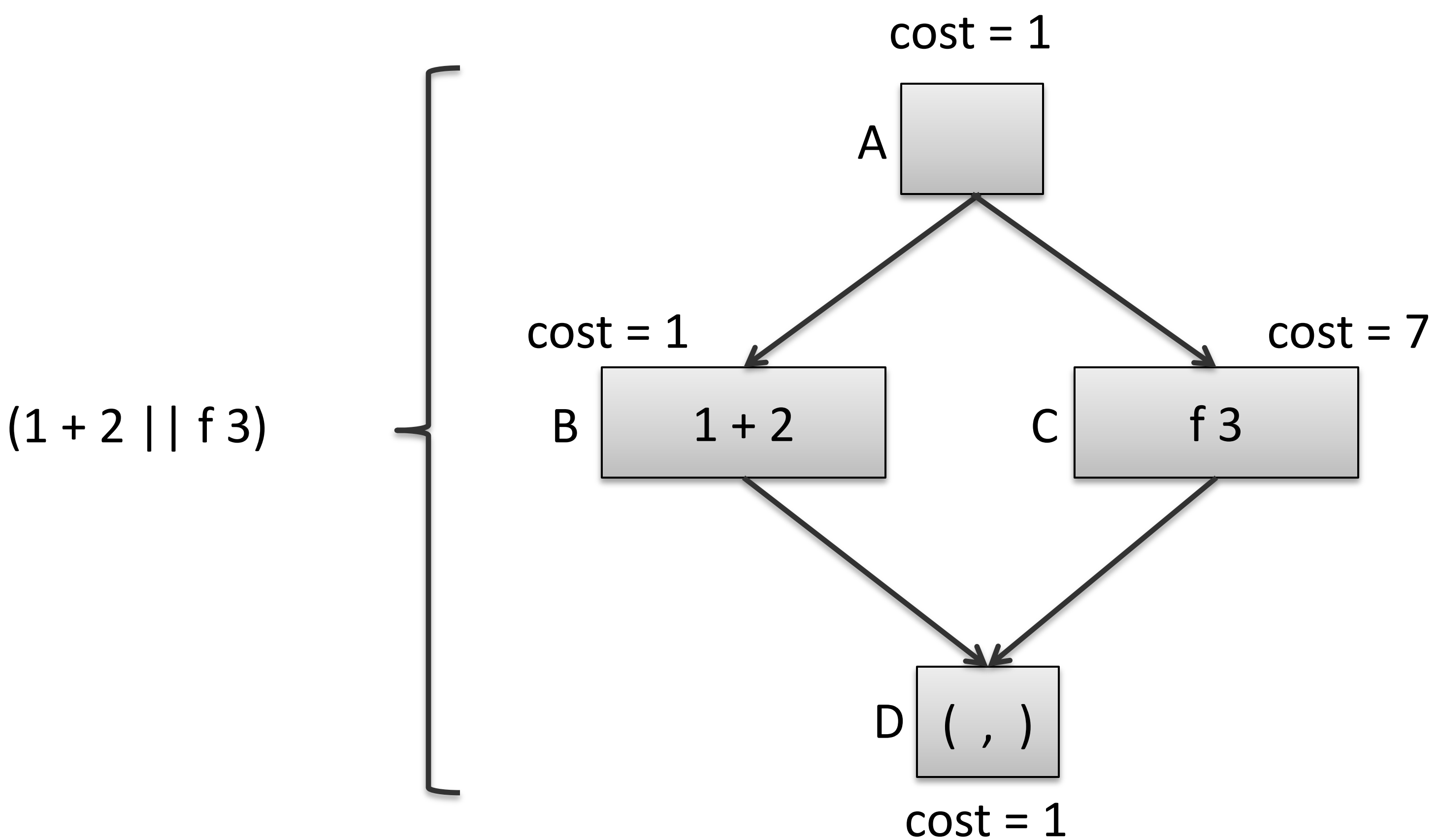
Schedule A-C-B-D: 1 + 1 + 7 + 1

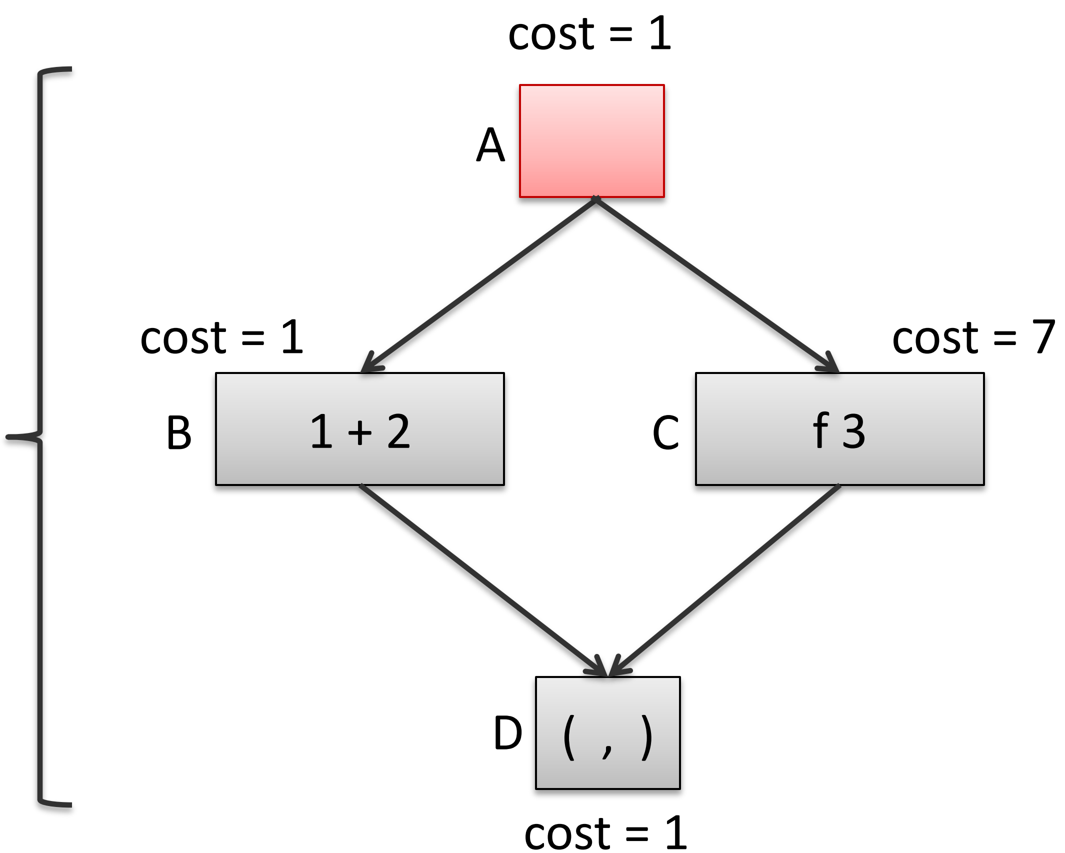
Visualizing Computational Costs



Suppose we have 2 processors. How much time does this computation take?

Visualizing Computational Costs

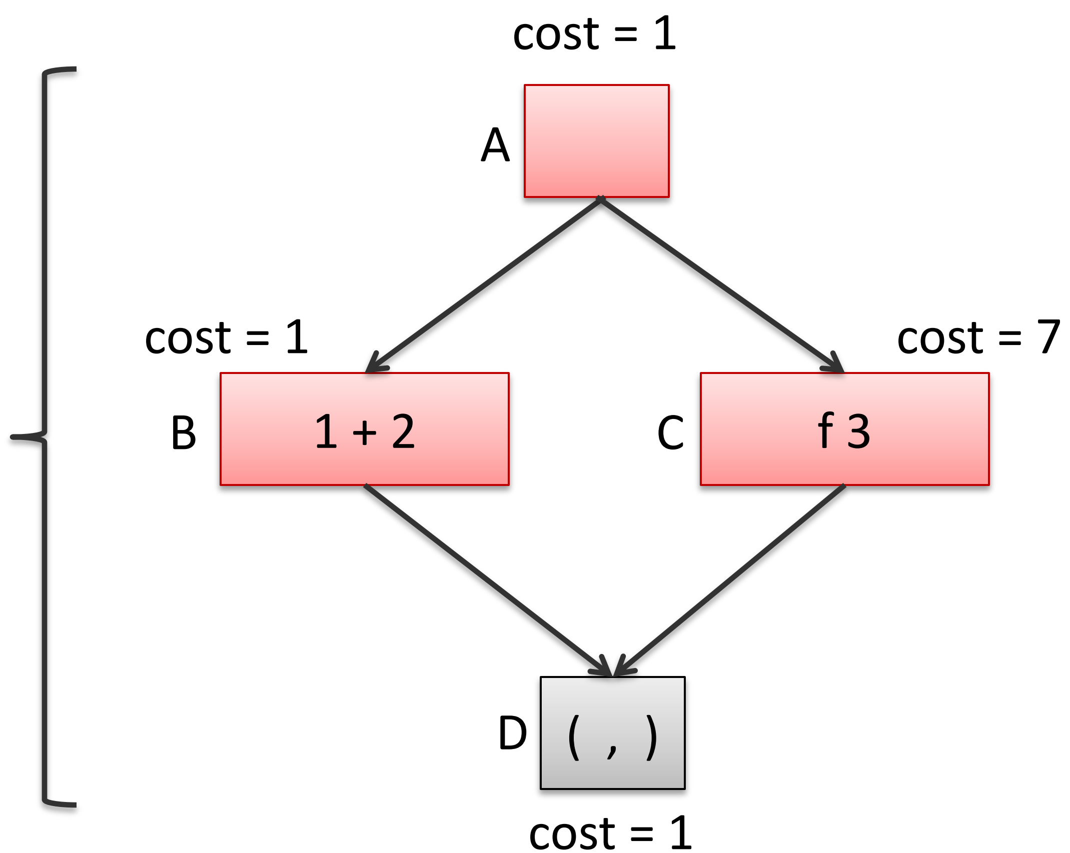
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1

Visualizing Computational Costs

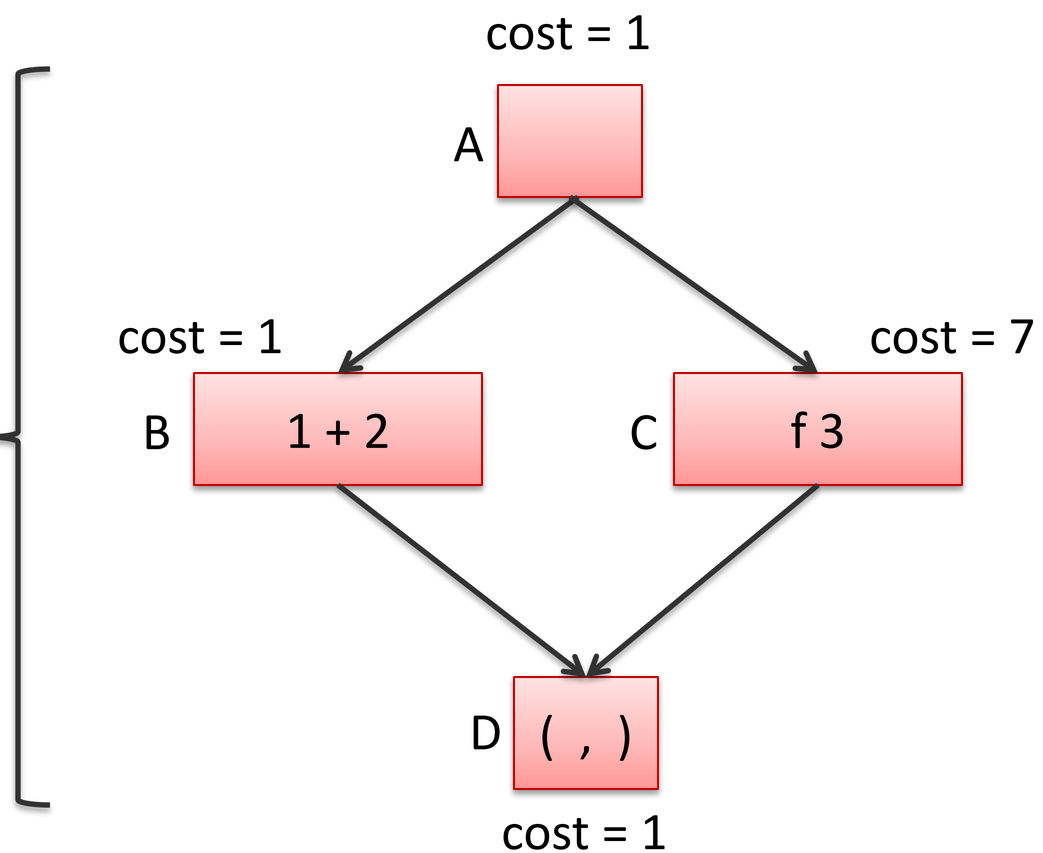
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1 + max(1,7)

Visualizing Computational Costs

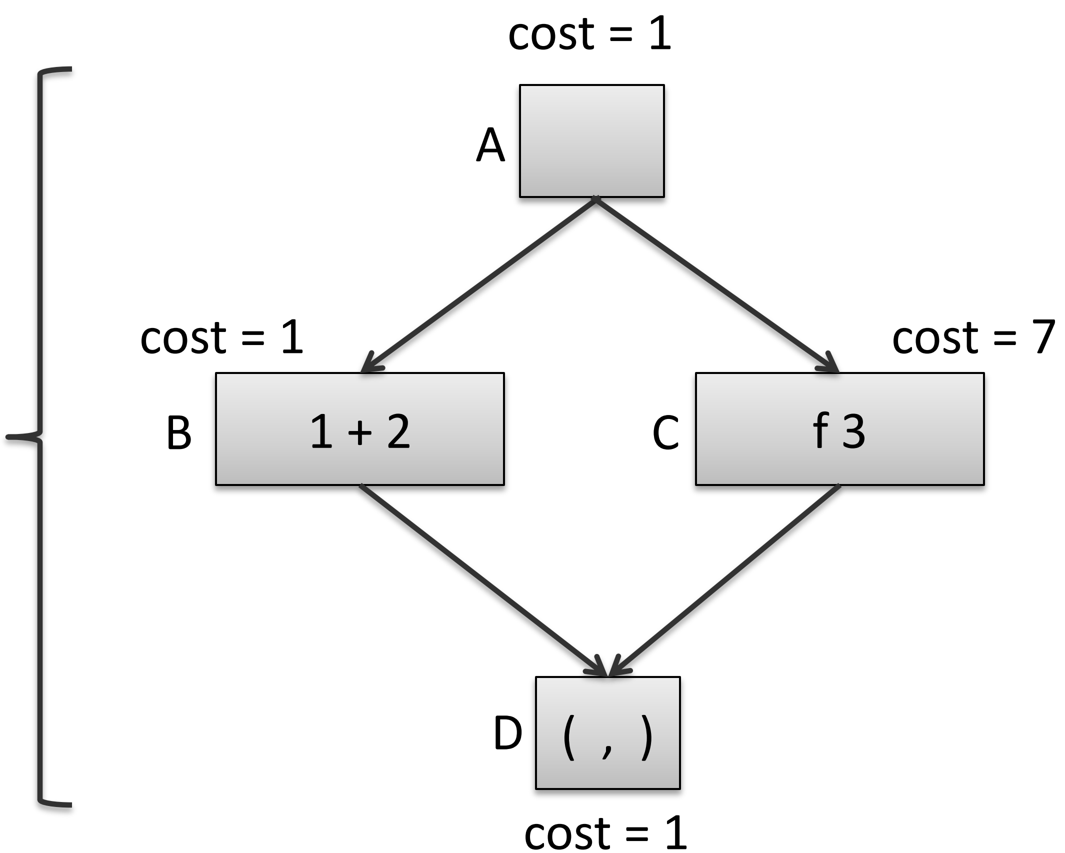
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1 + max(1,7) + 1

Visualizing Computational Costs

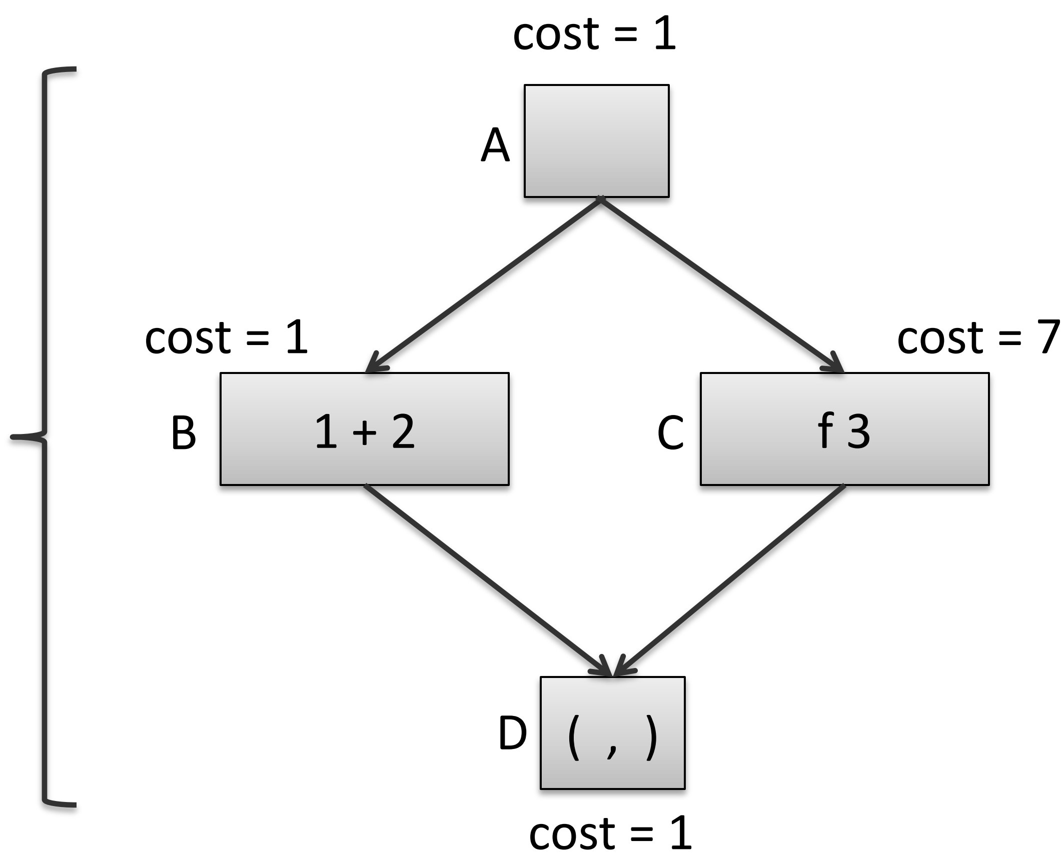
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Total cost: 1 + max(1,7) + 1. We say the *schedule* we used was: A-CB-D

Visualizing Computational Costs

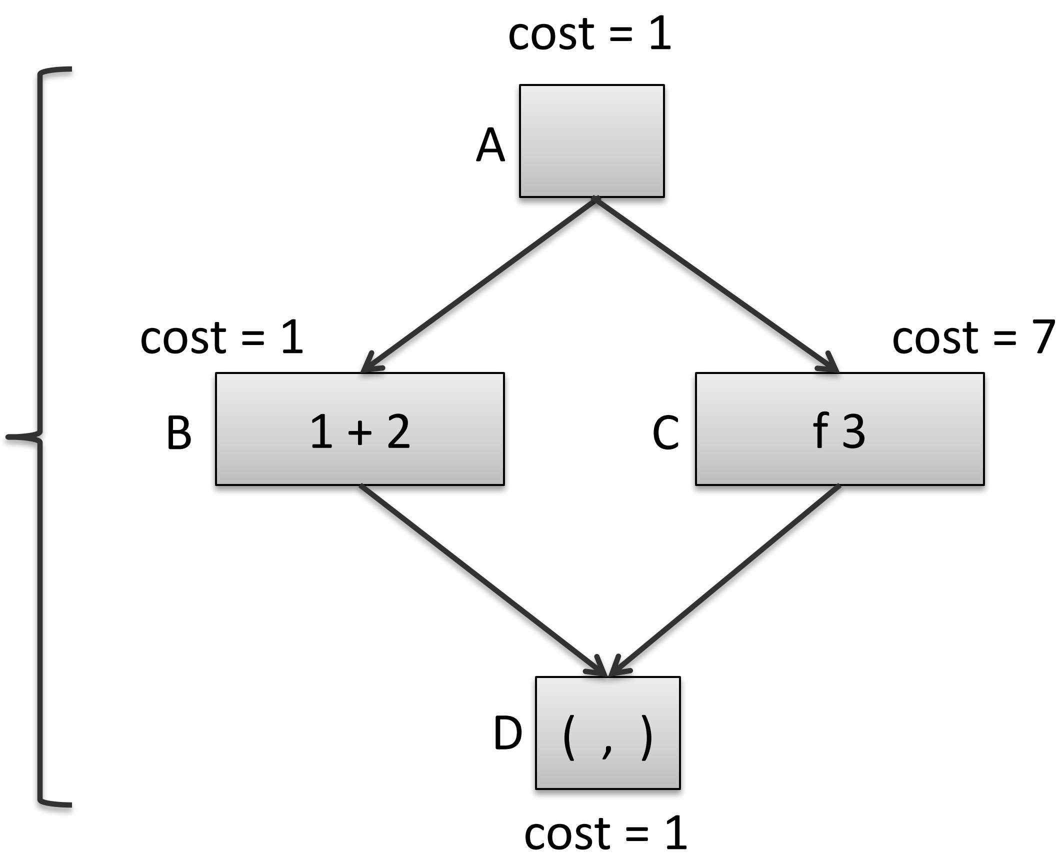
(1 + 2 || f 3)



Suppose we have 3 processors. How much time does this computation take?

Visualizing Computational Costs

(1 + 2 || f 3)

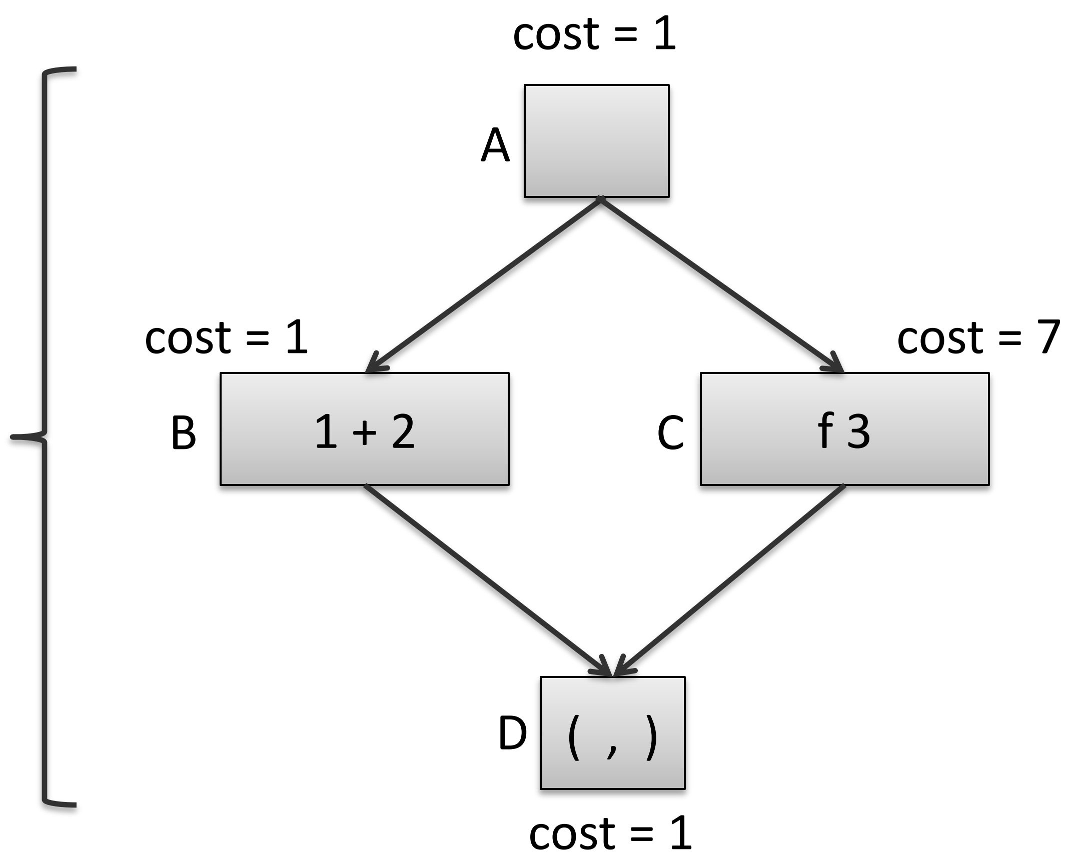


Suppose we have 3 processors. How much time does this computation take?

Schedule A-BC-D: 1 + max(1,7) + 1 = 9

Visualizing Computational Costs

(1 + 2 || f 3)



Suppose we have infinite processors. How much time does this computation take?
Schedule A-BC-D: 1 + max(1,7) + 1 = 9

Work and Span

Understanding the complexity of a parallel program is a little more complex than a sequential program

- the number of processors has a significant effect

One way to *approximate* the cost is to consider a parallel algorithm independently of the machine it runs on is to consider *two* metrics:

- Work: The cost of executing a program with just 1 processor.
- Span: The cost of executing a program with an infinite number of processors

Always good to minimize work

- Every instruction executed consumes energy
- Minimize span as a second consideration
- Communication costs are also crucial (we are ignoring them)

Parallelism

The parallelism of an algorithm is an estimate of the maximum number of processors an algorithm can profit from.

- parallelism = work / span

If work = span then parallelism = 1.

- We can only use 1 processor
- It's a sequential algorithm

If span = ½ work then parallelism = 2

- We can use up to 2 processors

If work = 100, span = 1

- All operations are independent & can be executed in parallel
- We can use up to 100 processors

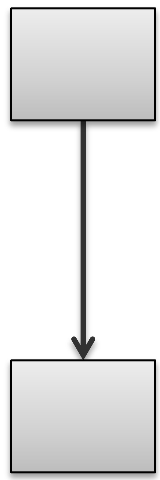
Related concept:

“speedup”

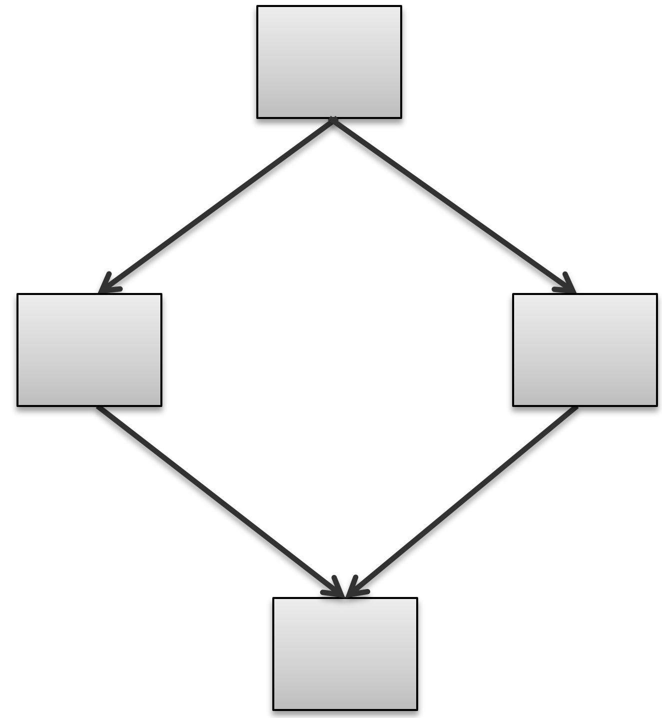
How much faster is the n-processor version *in practice,* not just in theory

Series-Parallel Graphs

one operation



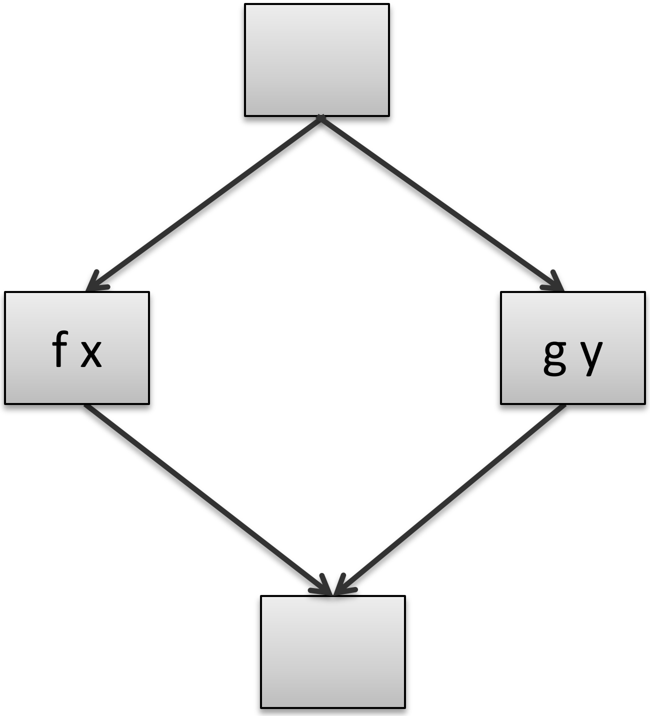
two operations
in sequence
e1; e2



two operations
in parallel
(e1 || e2)

Series-parallel graphs arise from execution of functional programs with parallel pairs. Also known as well-structured, nested parallelism.

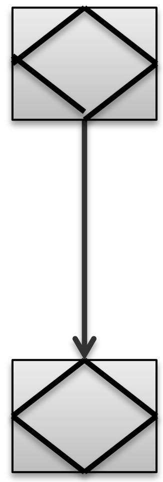
Parallel Pairs



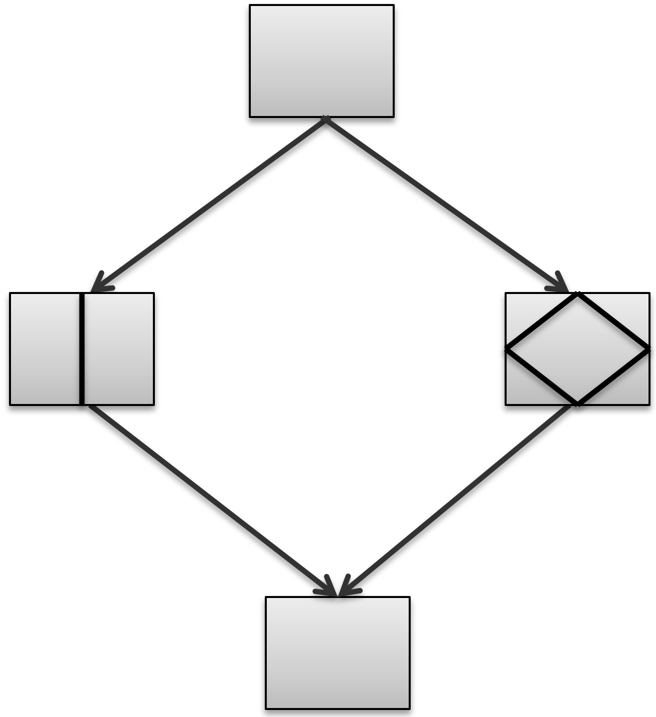

```
let both f x g y =
  let ff = future f x in
  let gv = g y in
  (force ff, gv)
```

Series-Parallel Graphs Compose

one operation



two graphs
in sequence

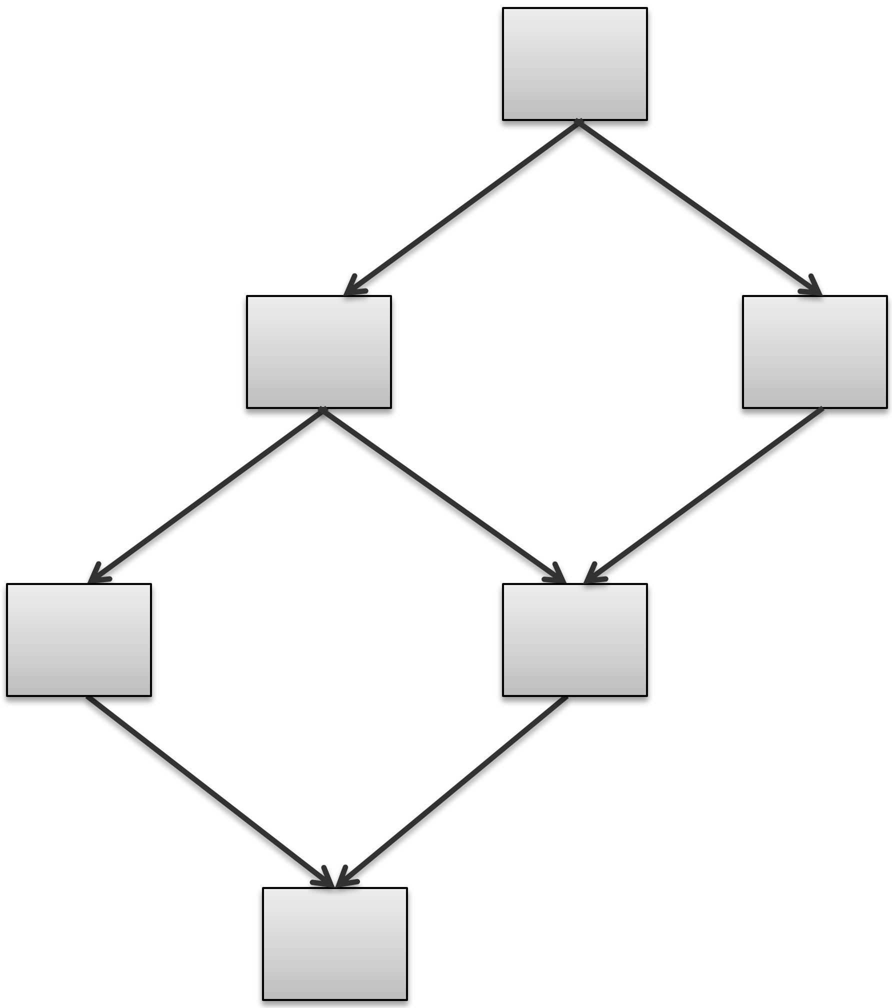


two graphs
in parallel

In general, a series-parallel graph has a source and a sink and is:

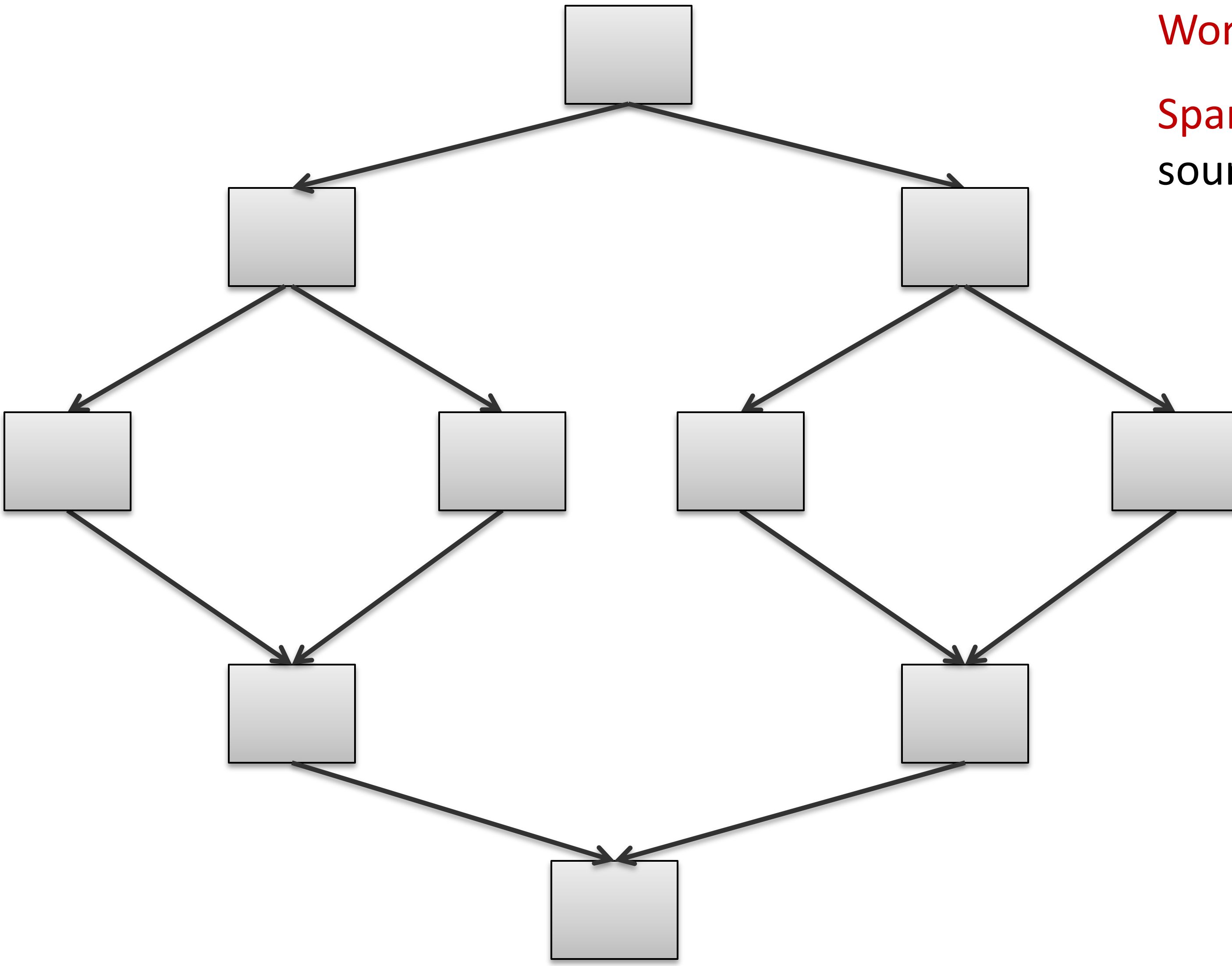
- a single node, or
- two series-parallel graphs in sequence, or
- two series-parallel graphs in parallel

Not a Series-Parallel Graph



However:
The results about greedy schedulers (next few slides) do apply to DAG schedules as well as series-parallel schedules!

Work and Span of Acyclic Graphs

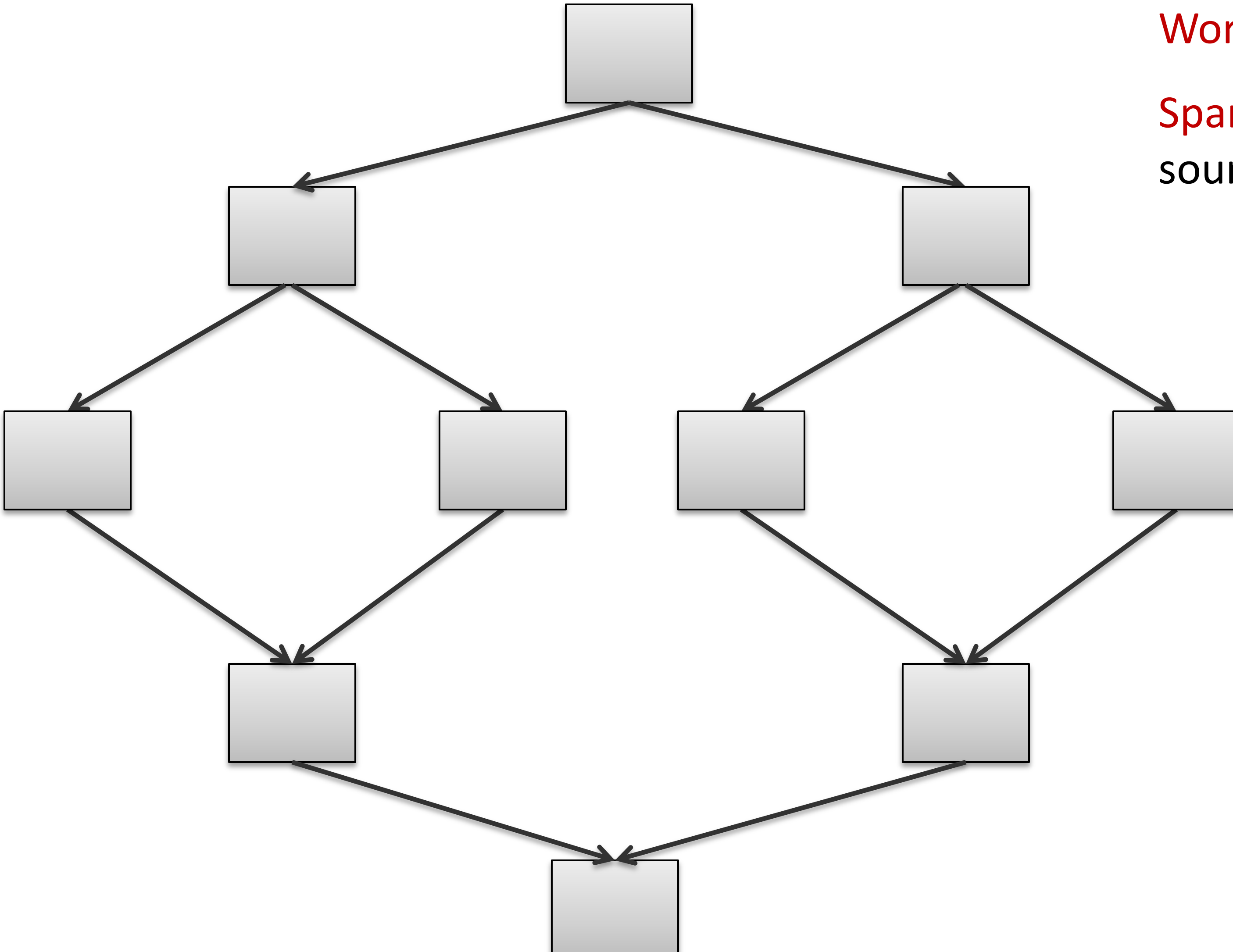


Let's assume each node costs 1.

Work: sum the nodes.

Span: longest path from source to sink.

Work and Span of Acyclic Graphs



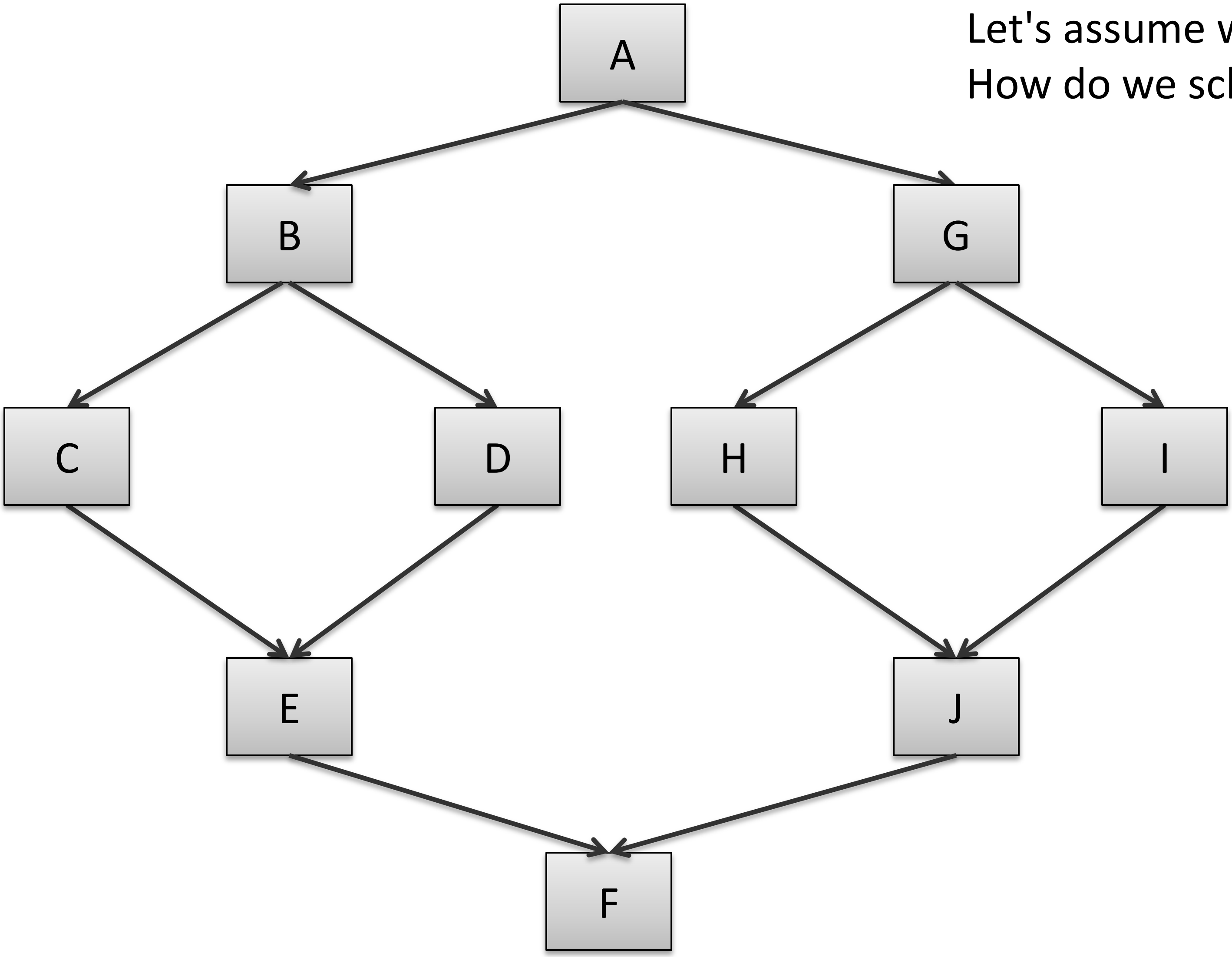
Let's assume each node costs 1.

Work: sum the nodes.

Span: longest path from source to sink.

work = 10
span = 5

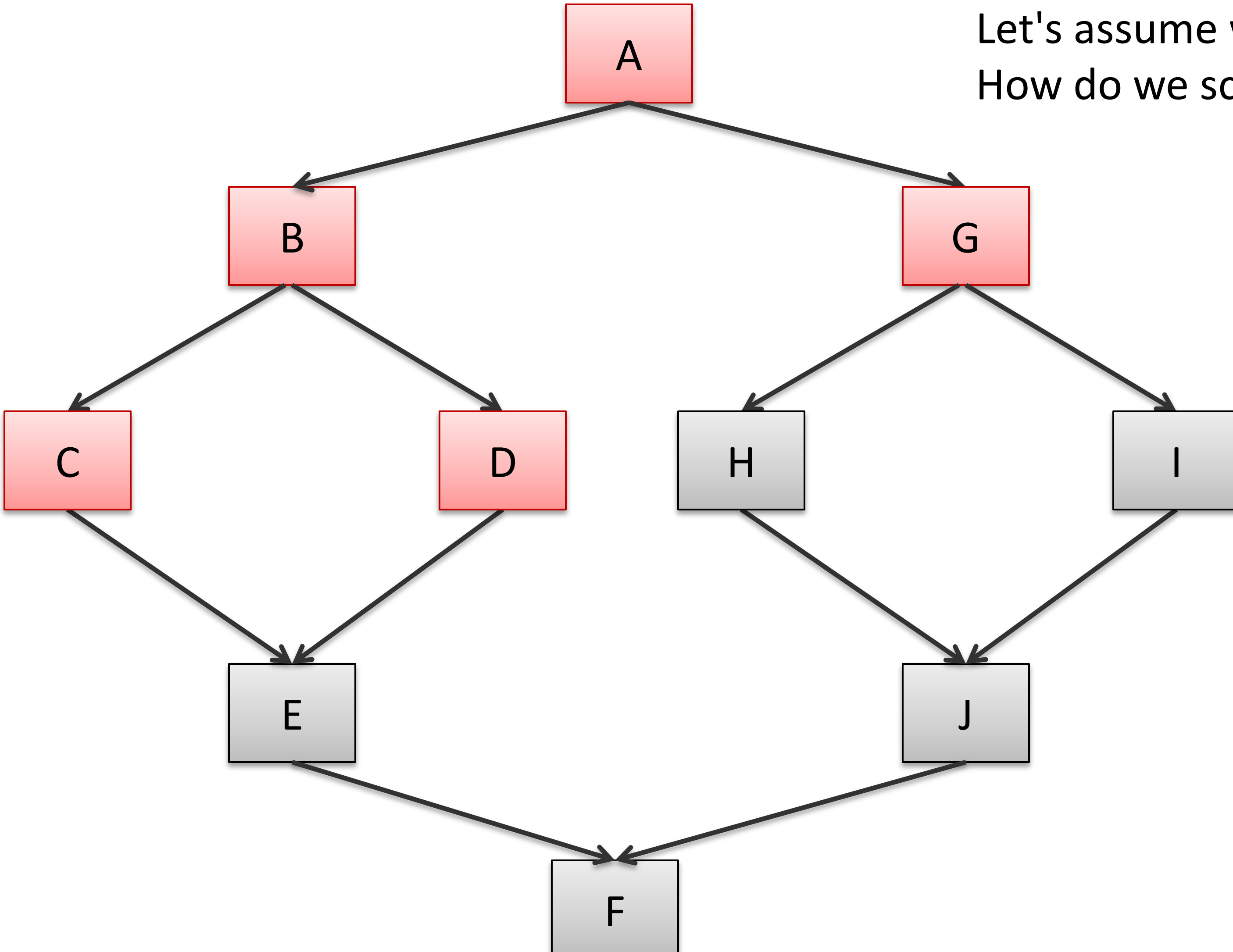
Scheduling



Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Scheduling



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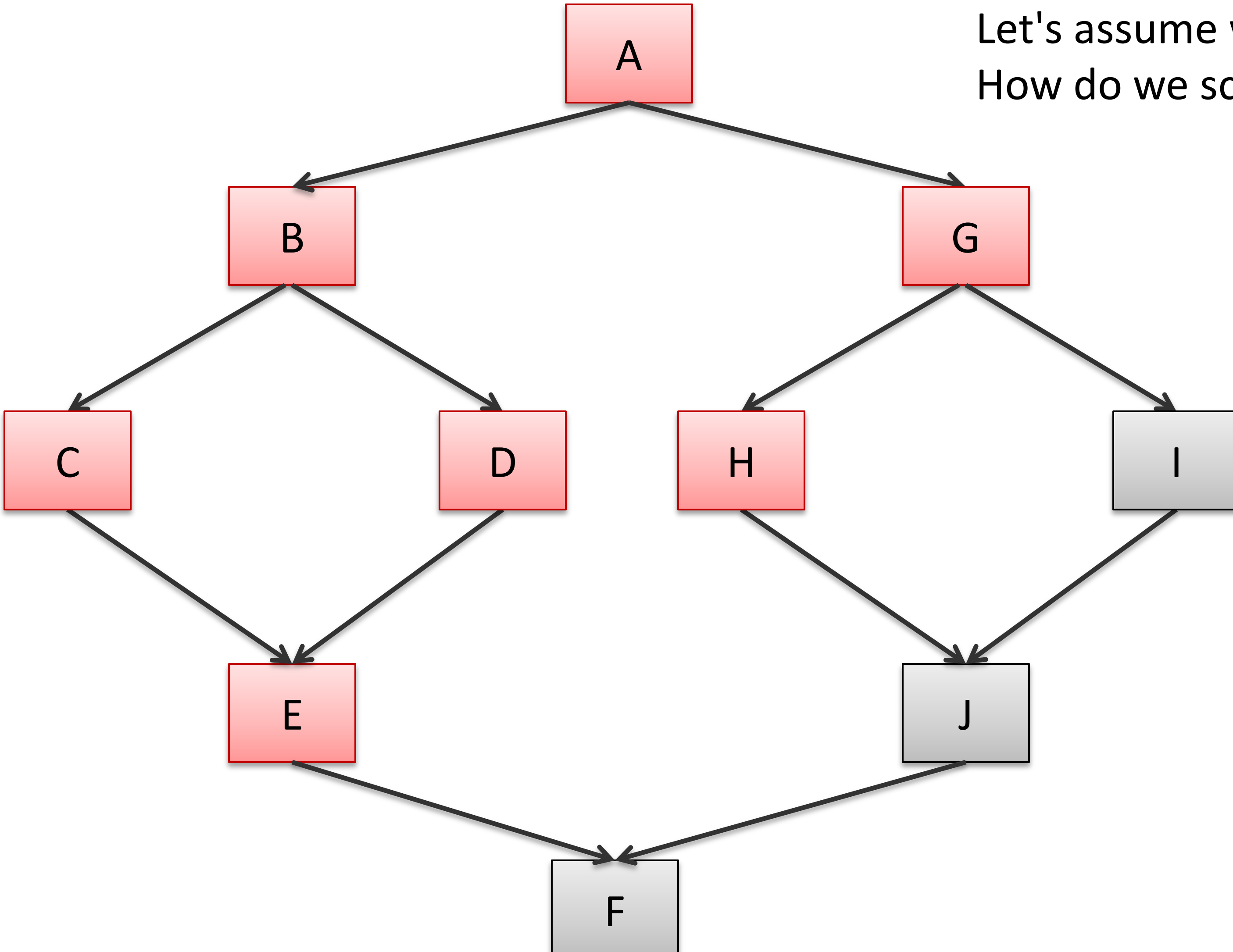
Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D

Scheduling

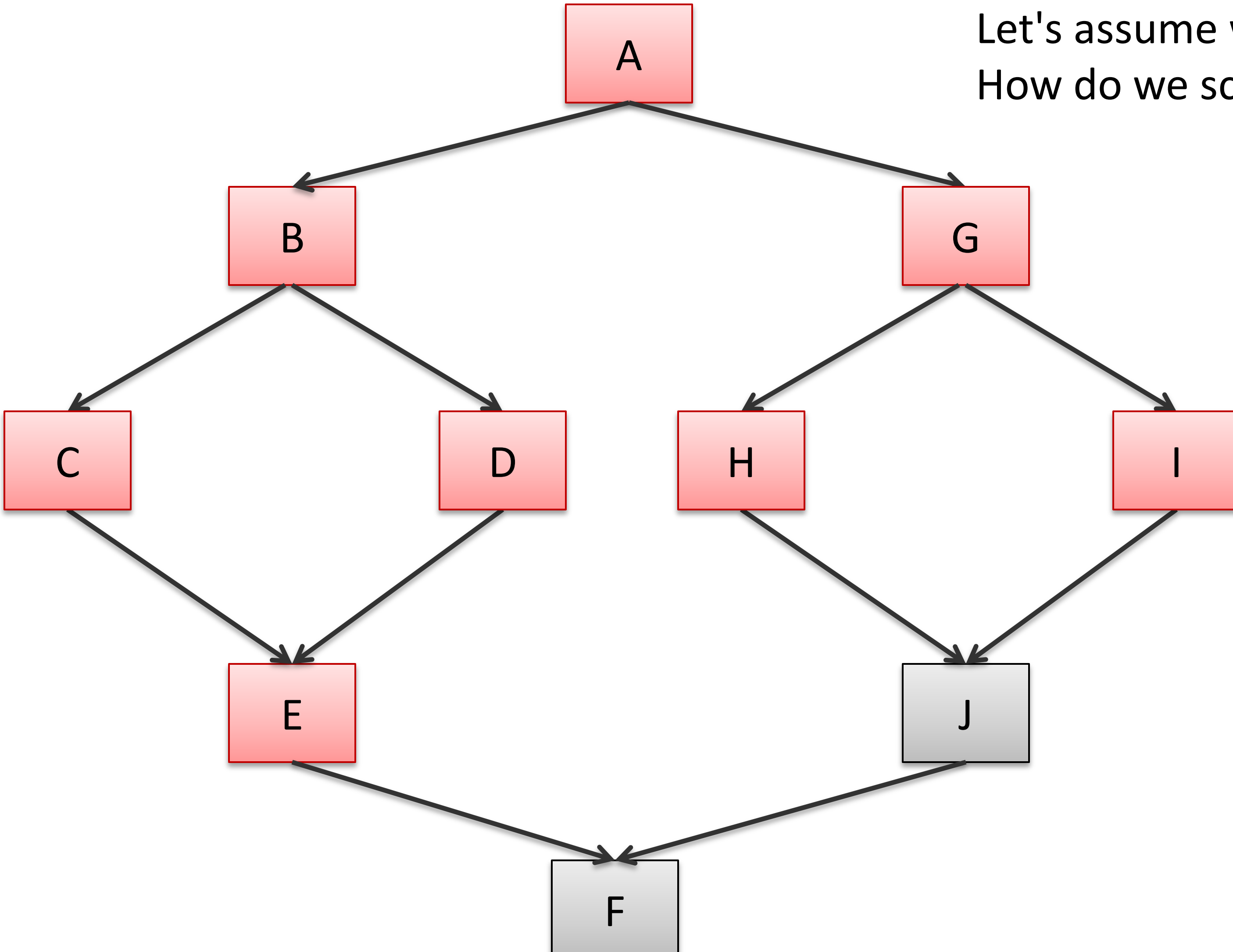
Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?



Option 1:
A
B G
C D
E H

Scheduling



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Option 1:
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I

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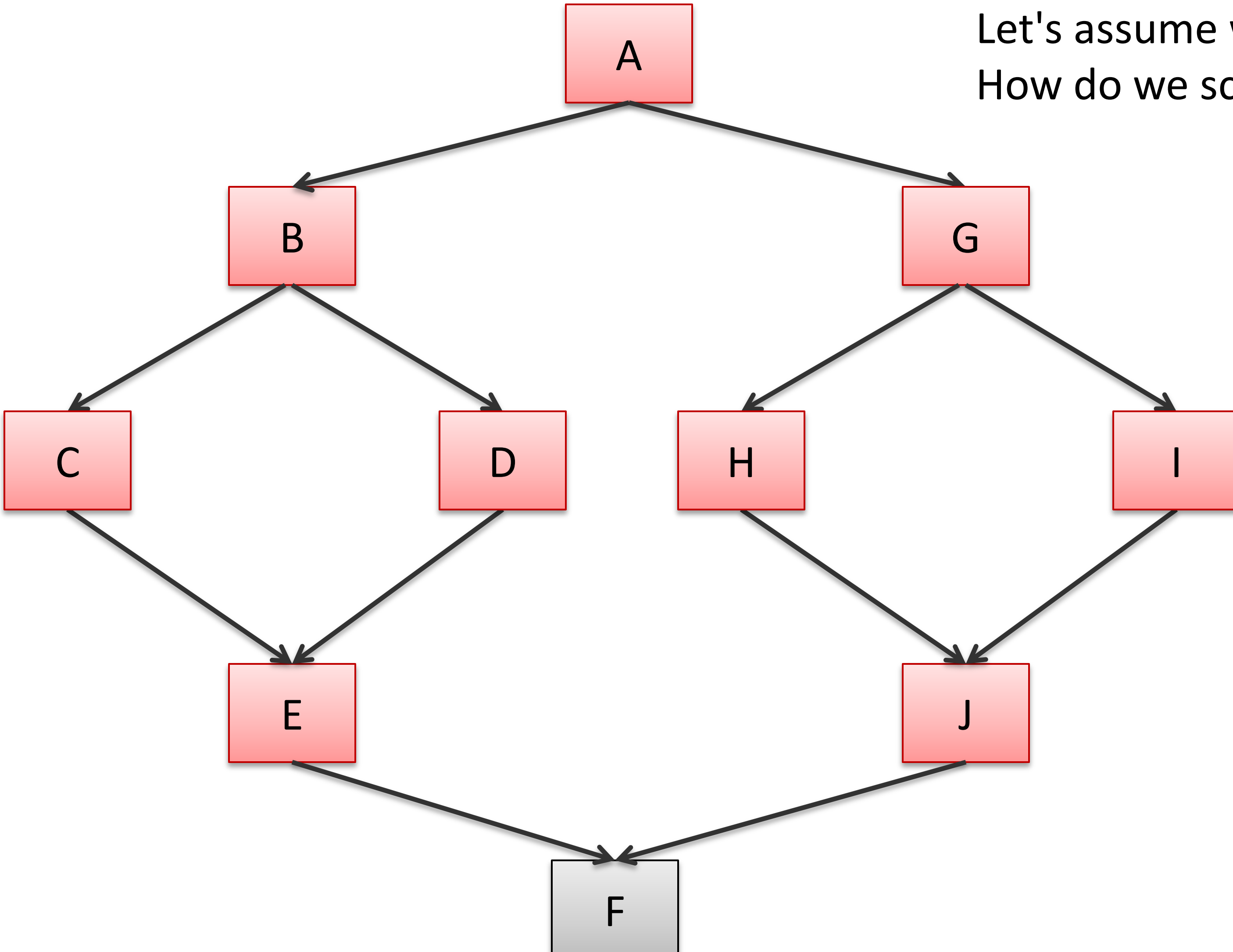
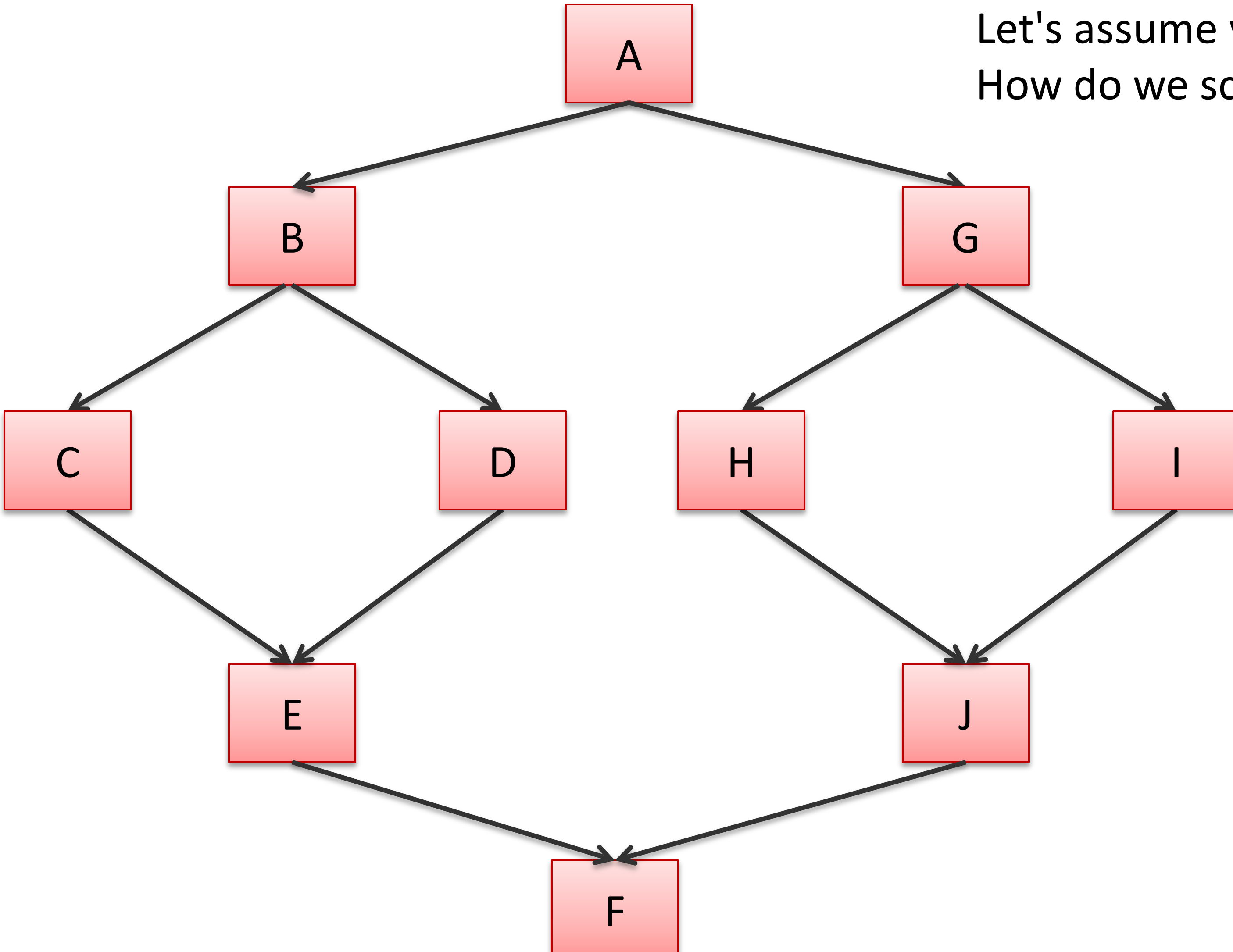


Option 1:
A
B G
C D
E H
I
J

Scheduling

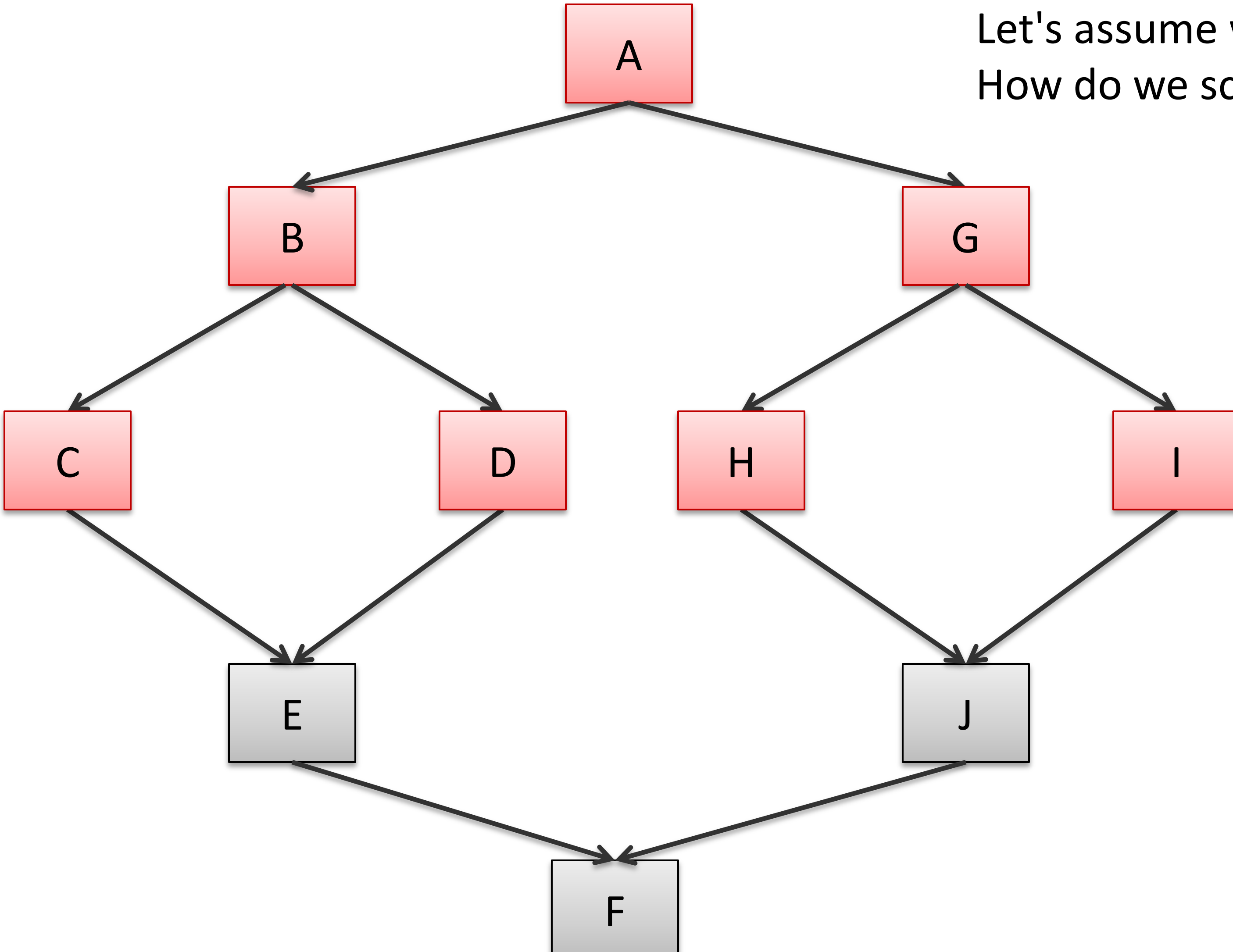
Let's assume each node costs 1.



Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
E H
I
J
F

Scheduling

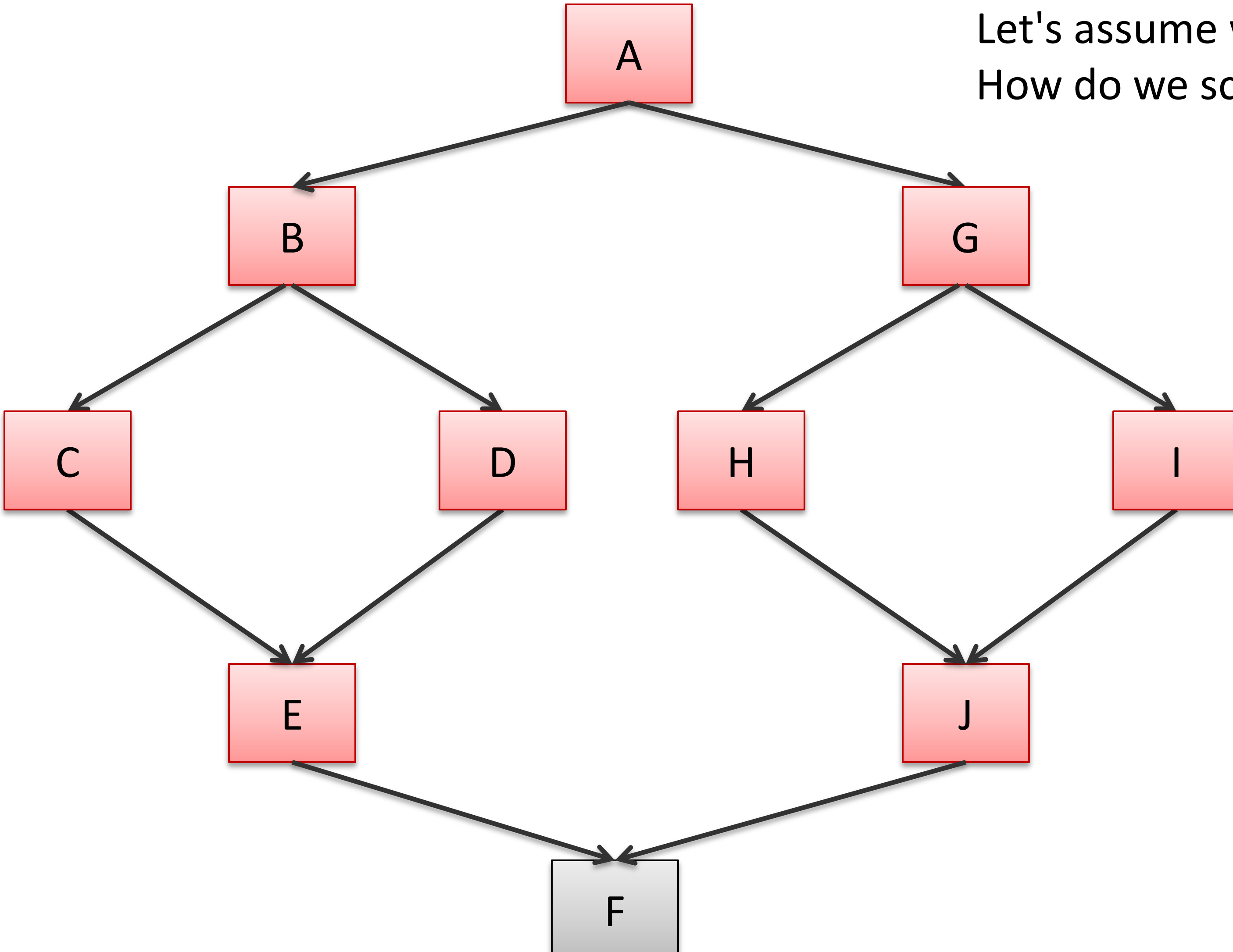


Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
~~E H~~ H I
~~I~~
~~J~~
~~F~~

Scheduling



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Option 1:
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~~J~~
~~F~~

Scheduling

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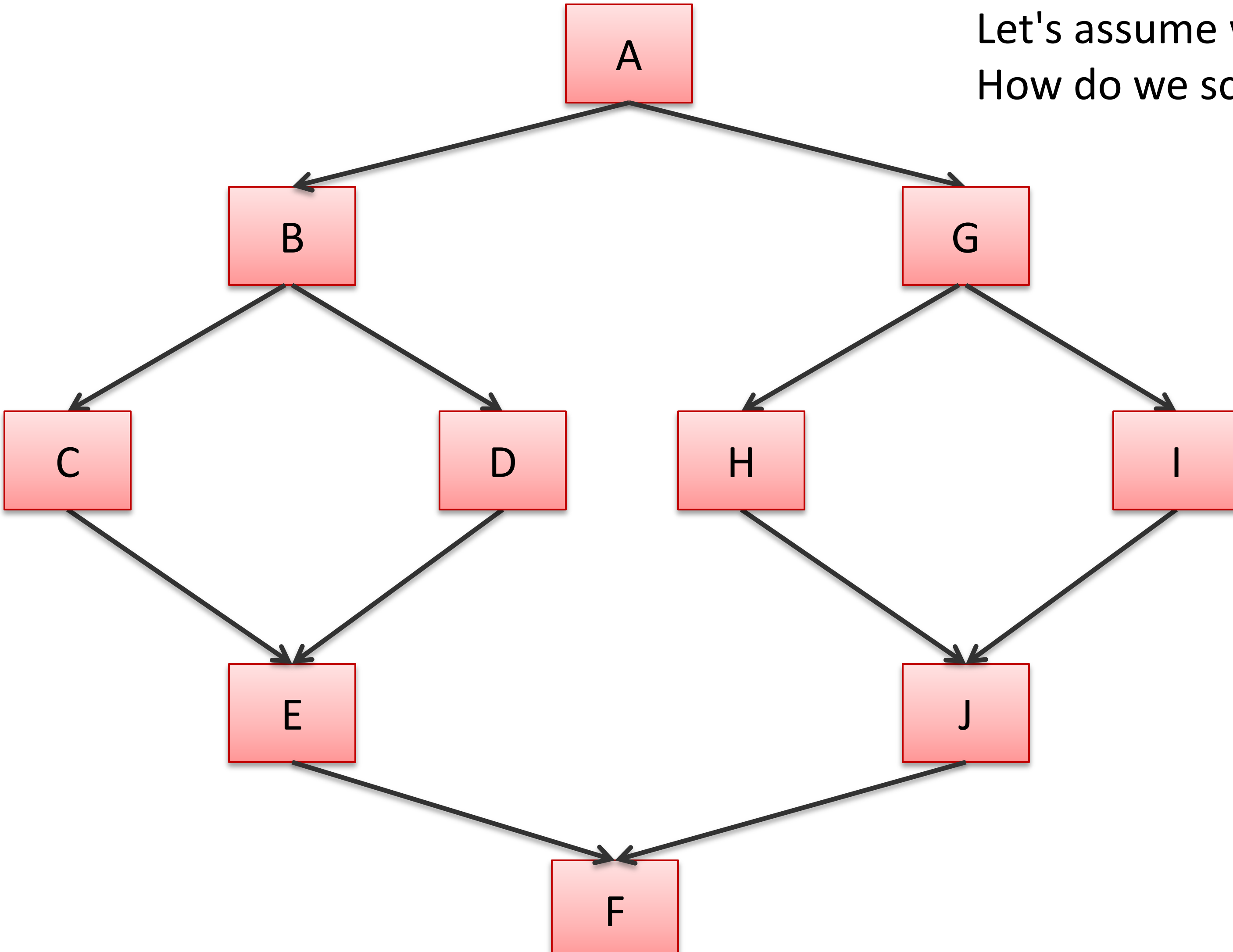
How do we schedule computation?



Option 1:

A

B G

C D

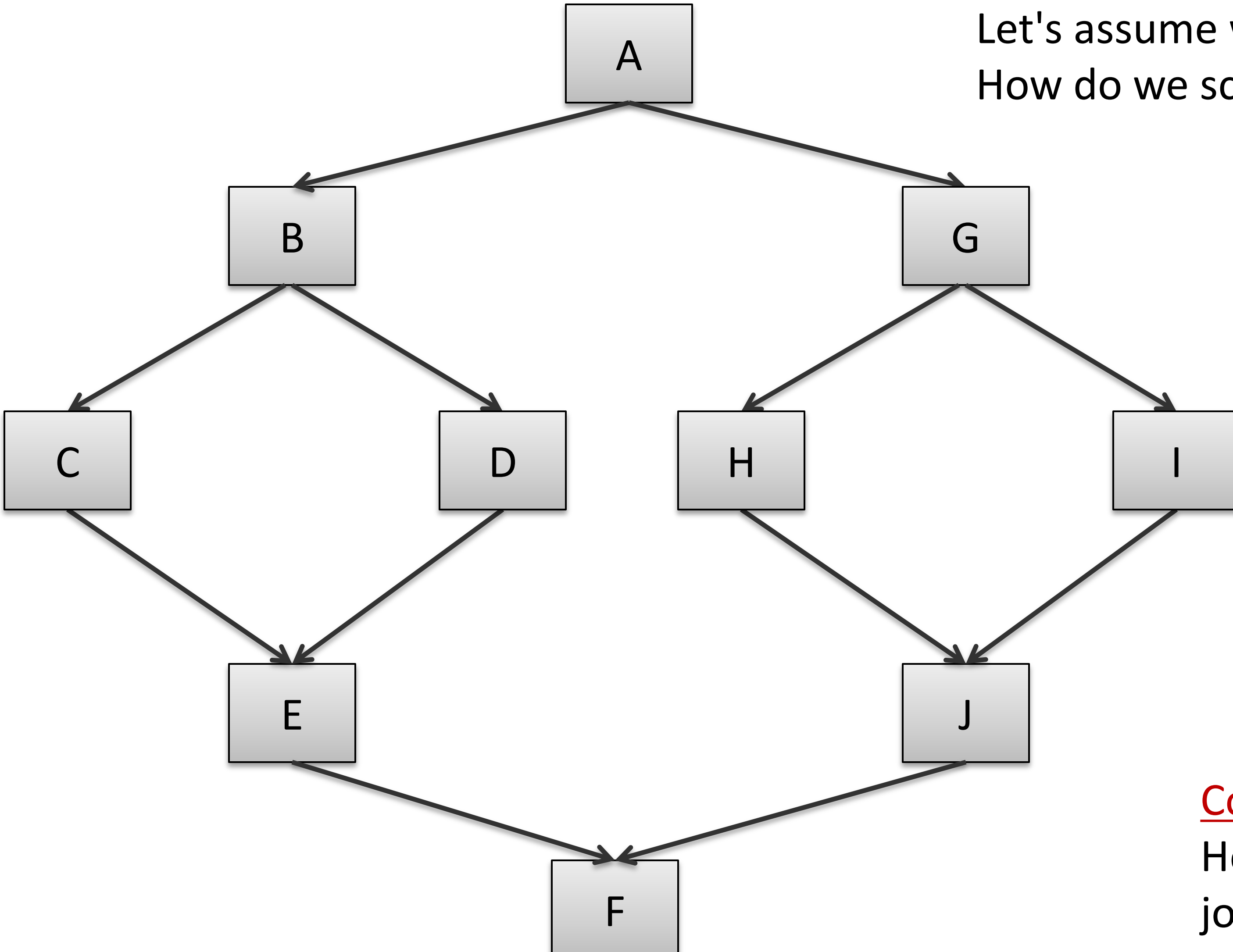
~~E H~~ H I

~~I~~ E J

~~J~~ F

~~F~~

Scheduling



Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:

A	
B G	
C D	
~~E H~~	H I
~~I~~	E J
~~J~~	F
~~F~~	

Conclusion:
How you schedule jobs can have an impact on performance

Greedy Schedulers

Greedy schedulers will schedule some task to a processor as soon as that processor is free.

- Doesn't sound so smart!

Greedy Schedulers

Greedy schedulers will schedule some task to a processor as soon as that processor is free.

- Doesn't sound so smart!

Properties (for p processors):

- T(p) < work/p + span
  - won't be worse than dividing up the data perfectly between processors, except for the last little bit, which causes you to add the span on top of the perfect division
- T(p) >= max(work/p, span)
  - can't do better than perfect division between processors (work/p)
  - can't be faster than span

Greedy Schedulers

Properties (for p processors):

max(work/p, span) <= T(p) < work/p + span

Consequences:

- as span gets small relative to work/p
  - work/p + span ==> work/p
  - max(work/p, span) ==> work/p
  - so T(p) ==> work/p greedy schedulers converge to the optimum!

- if span approaches the work
  - work/p + span ==> span
  - max(work/p, span) ==> span
  - so T(p) ==> span greedy schedulers converge to the optimum!

And therefore

Even though greedy schedulers are simple to implement,

they can be effective in building a parallel programming system.

and

This *supports* the idea that **work and span** are useful ways to reason about the cost of parallel programs.

PARALLEL SEQUENCES

Parallel Sequences

Parallel sequences

$$< e_0 , e_1 , e_2 , \ldots , e_{n-1} >$$

Operations:

- creation (called **tabulate**)
- indexing an element in constant span
- map
- scan -- like a fold: $<u, u + e_0, u + e_0 + e_1, \ldots>$ log n span!

Languages:

- Nesl [Blelloch]
- Data-parallel Haskell

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
work = O(n)          span = O(1)
```

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
work = O(n)             span = O(1)
```

```
nth : 'a seq -> int -> 'a

nth <e0, e1, ..., e(n-1)> i == ei
work = O(1)             span = O(1)
```

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
work = O(n)          span = O(1)
```

```
nth : 'a seq -> int -> 'a

nth <e0, e1, ..., e(n-1)> i == ei
work = O(1)          span = O(1)
```

```
length : 'a seq -> int

length <e0, e1, ..., e(n-1)> == n
work = O(1)          span = O(1)
```

Example Problems

Write a function that creates the sequence <0, ..., n-1>

with Span = O(1) and Work = O(n).

Operations:

	Work	Span
`tabulate f n`	n	1
`nth i s`	1	1
`length s`	1	1

Example Problems

Write a function that creates the sequence <0, ..., n-1>

with Span = O(1) and Work = O(n).

```
(* create n == <0, 1, ..., n-1> *)
let create n =
```

Operations:

```
                  Work   Span
tabulate f n      n      1
nth i s           1      1
length s          1      1
```

Example Problems

Write a function that creates the sequence <0, ..., n-1>

with Span = O(1) and Work = O(n).

```
(* create n == <0, 1, ..., n-1> *)
let create n =
  tabulate (fun i -> i) n
```

Operations:

```
                   Work   Span
tabulate f n       n      1
nth i s            1      1
length s           1      1
```

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, maps f over each element of the sequence with Span = O(1) and Work = O(n), returning the new sequence (if f is constant work)

Operations:

	Work	Span
`tabulate f n`	n	1
`nth i s`	1	1
`length s`	1	1

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, maps f over each element of the sequence with Span = O(1) and Work = O(n), returning the new sequence (if f is constant work)

```
(* map f <v0, ..., vn-1> == <f v0, ..., f vn-1> *)
let map f s =
```

Operations:

	Work	Span
`tabulate f n`	n	1
`nth i s`	1	1
`length s`	1	1

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, maps f over each element of the sequence with Span = O(1) and Work = O(n), returning the new sequence (if f is constant work)

```
(* map f <v0, ..., vn-1> == <f v0, ..., f vn-1> *)
let map f s =
  tabulate (fun i -> f (nth s i)) (length s)
```

Operations:

```
                 Work   Span
tabulate f n     n      1
nth i s          1      1
length s         1      1
```

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, reverses the sequence. with Span = O(1) and Work = O(n)

Operations:

	Work	Span
`tabulate f n`	n	1
`nth i s`	1	1
`length s`	1	1

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, reverses the sequence. with Span = O(1) and Work = O(n)

```
(* reverse <v0, ..., vn-1> == <vn-1, ..., v0> *)
let reverse s =
```

Operations:

```
                 Work   Span
tabulate f n     n      1
nth i s          1      1
length s         1      1
```

Example Problems

Write a function such that given a sequence <v0, ..., vn-1>, reverses the sequence. with Span = O(1) and Work = O(n)

```
(* reverse <v0, ..., vn-1> == <vn-1, ..., v0> *)
let reverse s =
  let n = length s in
  tabulate (fun i -> nth s (n-i-1)) n
```

Operations:

	Work	Span
`tabulate f n`	n	1
`nth i s`	1	1
`length s`	1	1

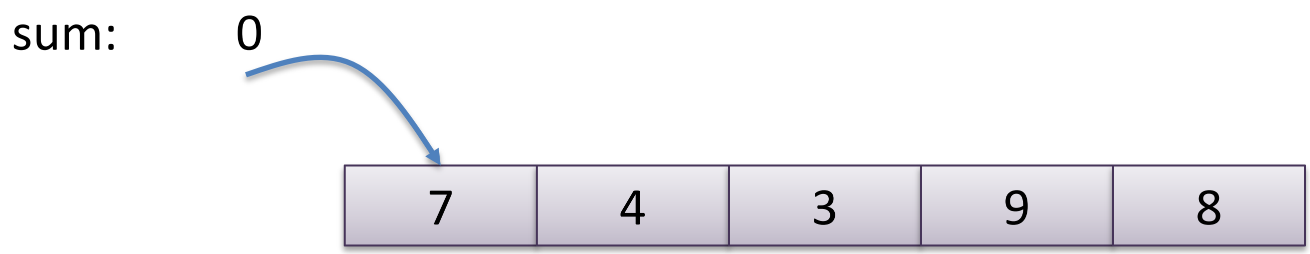
A Parallel Sequence API

	Work	Span
`type 'a seq`		
`tabulate : (int -> 'a) -> int -> 'a seq`	O(N)	O(1)
`length : 'a seq -> int`	O(1)	O(1)
`nth : 'a seq -> int -> 'a`	O(1)	O(1)
`append : 'a seq -> 'a seq -> 'a seq` (can build this from tabulate, nth, length)	O(N+M)	O(1)
`split : 'a seq -> int -> 'a seq * 'a seq`	O(N)	O(1)

For efficient implementations, see this paper by Andrew Tao '24:
https://icfp23.sigplan.org/details/ocaml-2023-papers/2/Parallel-Sequences-in-Multicore-OCaml

Fold and Reduce

We have seen many sequential algorithms can be programmed succinctly using fold or reduce. Eg: sum all elements:



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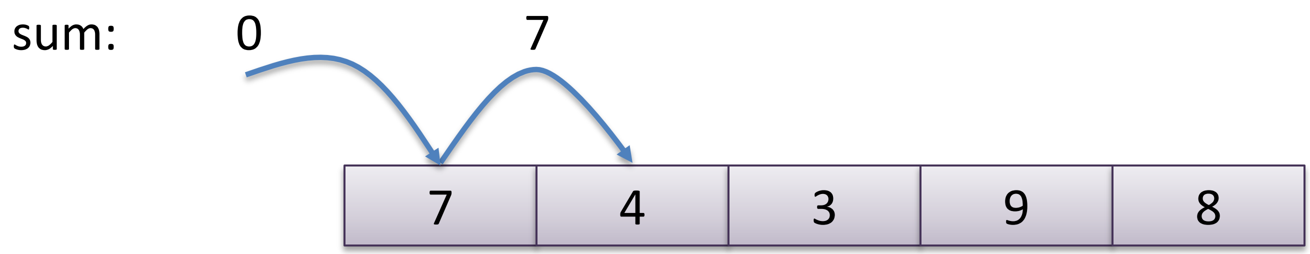

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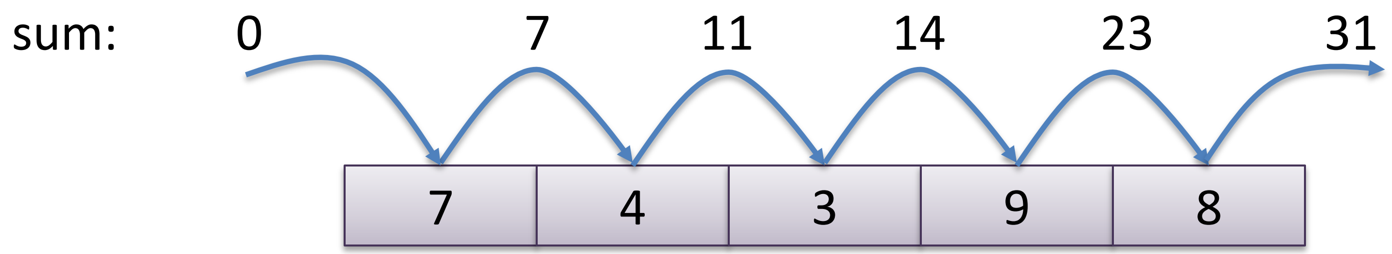

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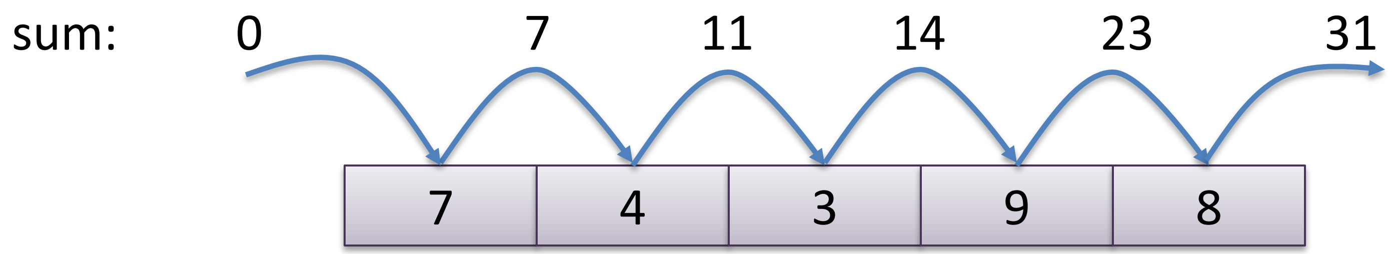


```
let sum_all (l:int list) = reduce (+) 0 l
```

Fold and Reduce

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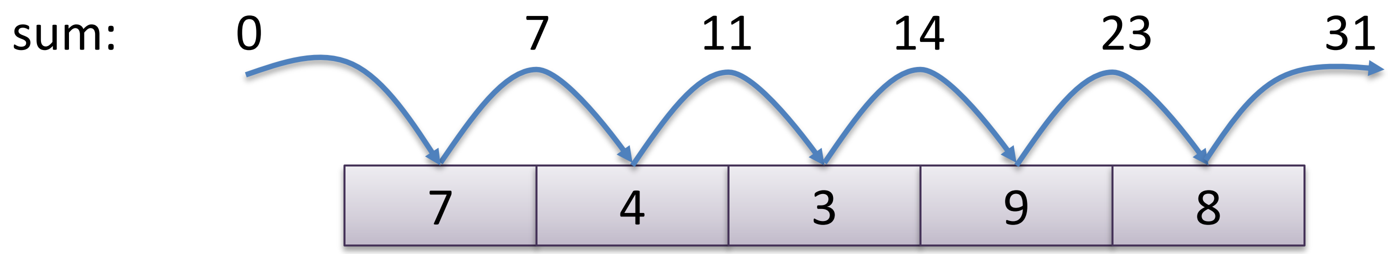



```
let sum_all (l:int list) = reduce (+) 0 l
```

<u>Key to parallelization:</u> Notice that because sum is an *associative* operator, we do not have to add the elements strictly left-to-right:

$$(((((init + v1) + v2) + v3) + v4) + v5) \;==\; ((init + v1) + v2) + ((v3 + v4) + v5)$$

add on processor 1 add on processor 2

Side Note

The key is *associativity*:

(((((init + v1) + v2) + v3) + v4) + v5) == ((init + v1) + v2) + ((v3 + v4) + v5)

add on processor 1

add on processor 2

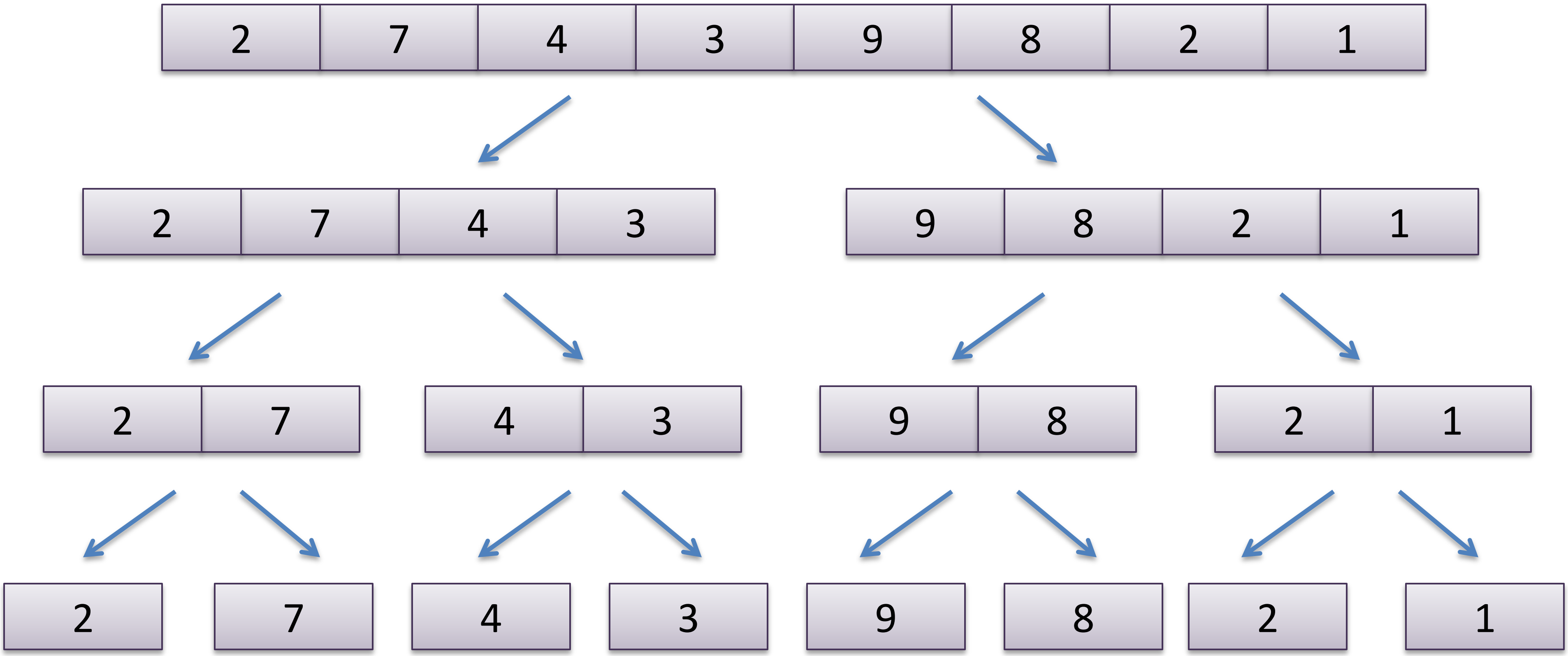
Commutativity not needed!

Commutativity allows us to reorder the elements:

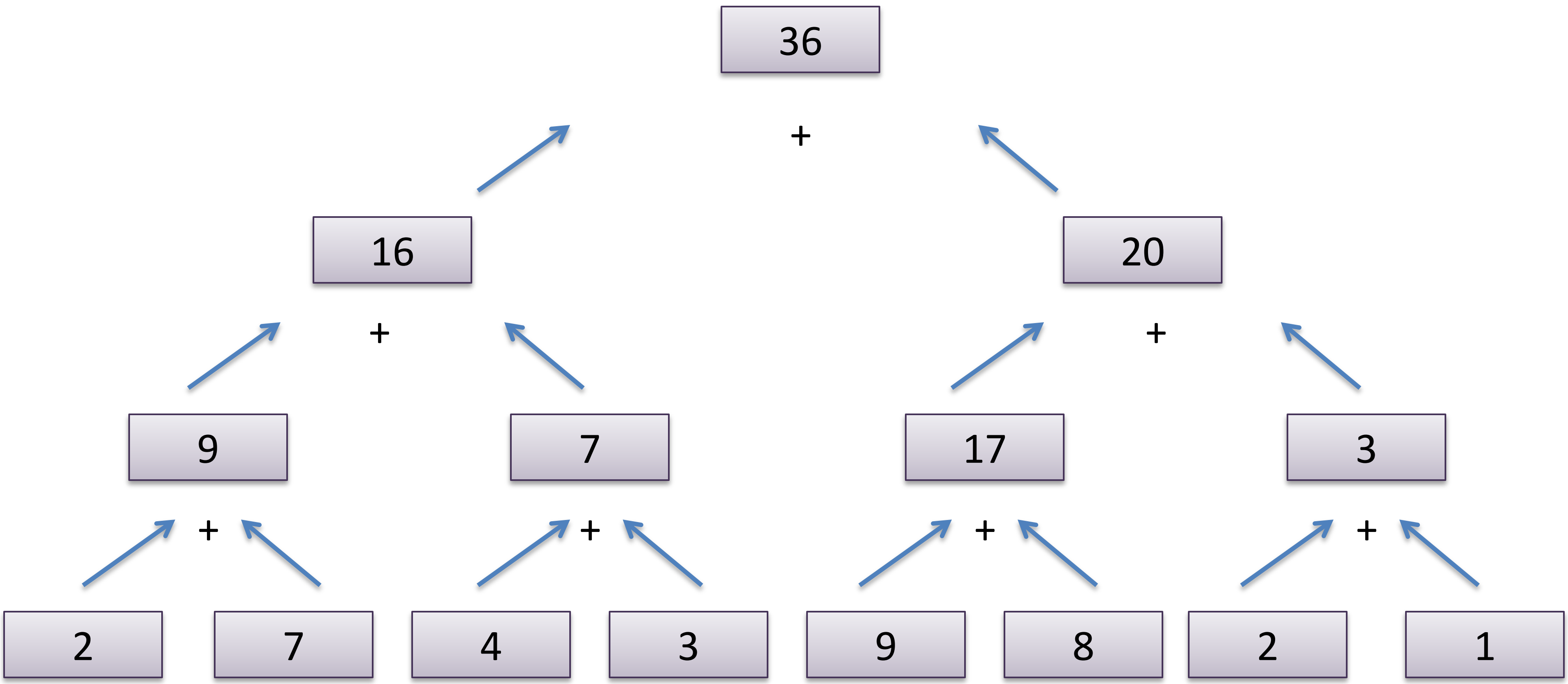
v1 + v2 == v2 + v1

But we don't have to reorder elements to obtain a significant speedup; we just have to reorder the execution of the operations.

Parallel Sum



Parallel Sum

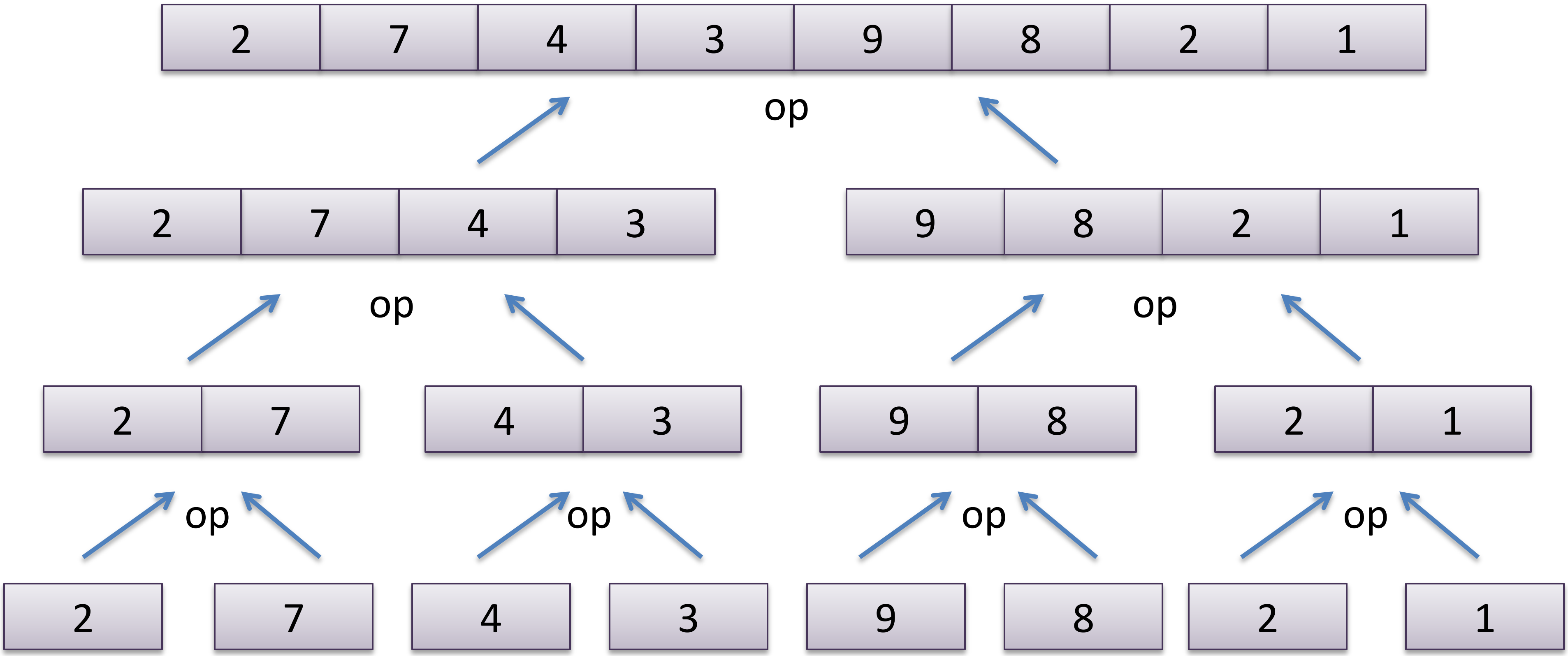


Parallel Sum

```
let both f x g y =
  let ff = future f x in
  let gv = g y in
  (force ff, gv)
```

```
let rec psum (s : int seq) : int =
  match length s with
    0 -> 0
  | 1 -> nth s 0
  | n ->
      let (s1,s2) = split (n/2) s in
      let (a1, a2) = both psum s1
                          psum s2 in
      a1 + a2
```

Parallel Reduce



If op is associative and the base case has the properties:

op base X == X   op X base == X

then the parallel reduce is equivalent to the sequential left-to-right fold.

Parallel Reduce

```
let rec reduce (f:'a -> 'a -> 'a) (base:'a) (s:'a seq) =
  match length s with
    0 -> base
  | 1 -> nth s 0
  | n ->
      let (s1,s2) = split (n/2) s in
      let (n1, n2) = both (reduce f base) s1
                          (reduce f base) s2 in
      f n1 n2
```

Parallel Reduce

```
let rec reduce (f:'a -> 'a -> 'a) (base:'a) (s:'a seq) =
  match length s with
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      let (s1,s2) = split (n/2) s in
      let (n1, n2) = both (reduce f base) s1
                          (reduce f base) s2 in
      f n1 n2
```

```
let sum s = reduce (+) 0 s
```

A little more general

```
let rec mapreduce (inject: 'a -> 'b)
                  (combine:'b -> 'b -> 'b)
                  (base:'b)
                  (s:'a seq) =
  match length s with
    0 -> base
  | 1 -> inject (nth s 0)
  | n ->
      let (s1,s2) = split (n/2) s in
      let (n1, n2) = both
                     (mapreduce inject combine base) s1
                     (mapreduce inject combine base) s2 in
      combine n1 n2
```

A little more general

```
let rec mapreduce (inject: 'a -> 'b)
                  (combine:'b -> 'b -> 'b)
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    0 -> base
  | 1 -> inject (nth s 0)
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      let (s1,s2) = split (n/2) s in
      let (n1, n2) = both
                     (mapreduce inject combine base) s1
                     (mapreduce inject combine base) s2 in
      combine n1 n2
```

```
let average s =
  let (count, total) =
    mapreduce (fun x -> (1,x))
              (fun (c1,t1) (c2,t2) -> (c1+c2, t1 + t2))
              (0,0) s in
  if count = 0 then 0 else total / count
```

DON'T PARALLELIZE AT TOO FINE A GRAIN

Parallel Reduce with Sequential Cut-off

When data is small, the overhead of parallelization isn't worth it.
Revert to the sequential version!

```
let sequential_reduce f base (s:'a seq) =
   let rec g i x =
       if i<0 then x else g (i-1) (f (nth a i) x)
    in g (length s - 1)
```

```
let SHORT = 1000

let rec reduce (f:'a -> 'a -> 'a) (base:'a) (s:'a seq) =
  if length s < SHORT
  then sequential_reduce f base s
  else let (s1,s2) = split ((length s)/2) s in
      let (n1, n2) = both (reduce f base) s1
                          (reduce f base) s2 in
      f n1 n2
```

BALANCED PARENTHESES

The Balanced Parentheses Problem

Consider the problem of determining whether a sequence of parentheses is balanced or not. For example:

- balanced: ()()(())
- not balanced: (
- not balanced:)(
- not balanced: ()))

We will try formulating a divide-and-conquer parallel algorithm to solve this problem efficiently:

```
type paren = L | R      (* L(eft) or R(ight) paren *)

let balanced (ps : paren seq) : bool = ...
```

First, a sequential approach

fold from left to right, keep track of
# of unmatched left parens

(	(	)	)	)	(	)	(

0

Warning! This solution does not generalize to a parallel map/reduce!

First, a sequential approach

fold from left to right, keep track of
# of unmatched left parens

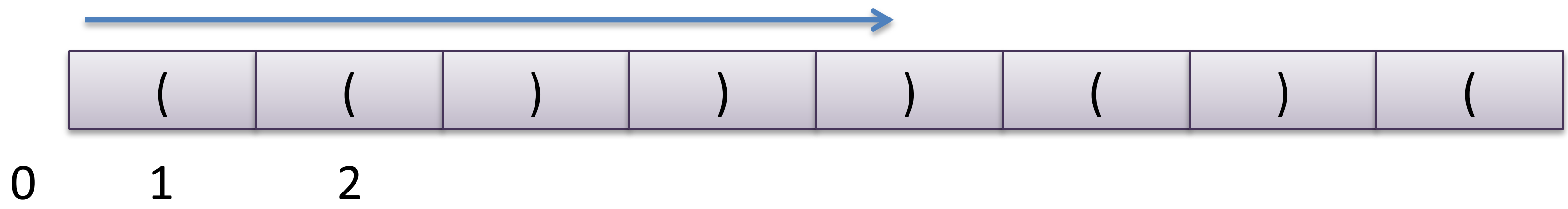
(	(	)	)	)	(	)	(

0 1

Warning! This solution does not generalize to a parallel map/reduce!

First, a sequential approach

fold from left to right, keep track of
# of unmatched left parens



Warning! This solution does not generalize to a parallel map/reduce!

First, a sequential approach

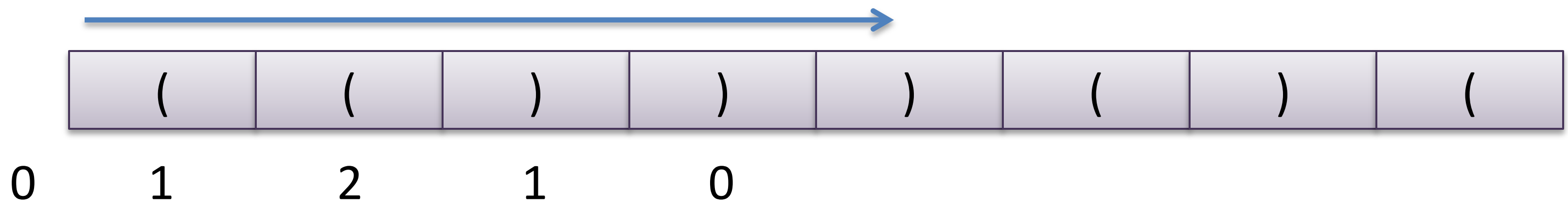
fold from left to right, keep track of
# of unmatched left parens

	(	(	)	)	)	(	)	(
0	1	2	1					

Warning! This solution does not generalize to a parallel map/reduce!

First, a sequential approach

fold from left to right, keep track of
# of unmatched left parens



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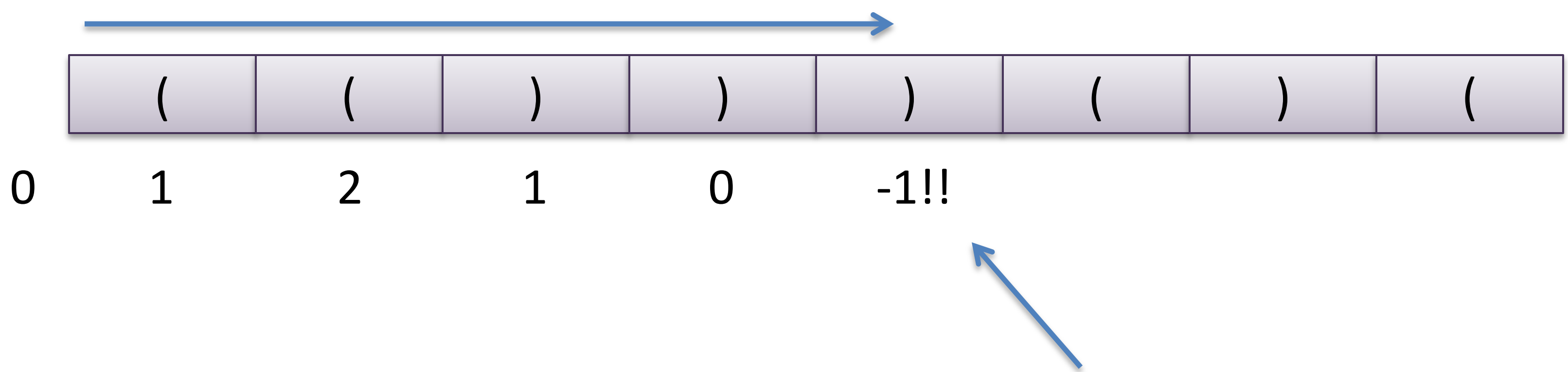
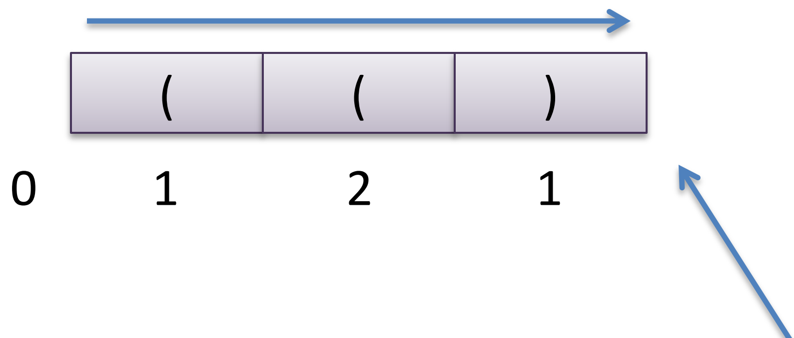


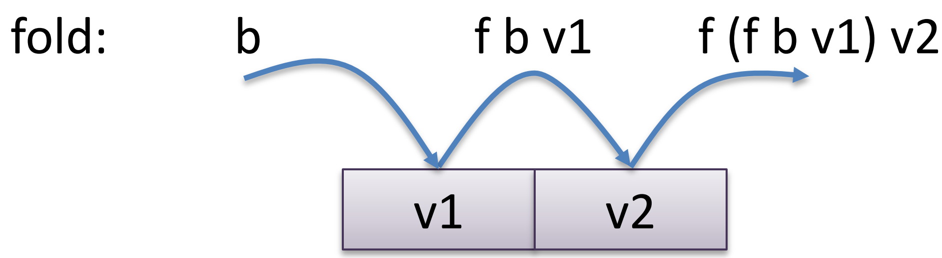
too many right parens
indicates no match

First, a sequential approach



if you reach the end of the end of the sequence, you should have no unmatched left parens

Easily Coded Using a Fold




```
let rec fold f b s =
  let rec aux n accum =
    if n >= length s then
      accum
    else
      aux (n+1) (f (nth s n) accum)
  in
  aux 0 b
```

Easily Coded Using a Fold

```
(* check to see if we have too many unmatched R parens

   so_far : number of unmatched parens so far
            or None if we have seen too many R parens

 *)

let check (p:paren) (so_far:int option) : int option =
  match (p, so_far) with
    (_, None) -> None
  | (L, Some c) -> Some (c+1)
  | (R, Some 0) -> None          (* violation detected *)
  | (R, Some c) -> Some (c-1)
```

Easily Coded Using a Fold

```
let fold f base s = ...

let check so_far s = ...

let balanced (s: paren seq) : bool =
  match fold check (Some 0) s with
      Some 0 -> true
    | (None | Some n) -> false
```

That was easy enough. But the "check" function is not associative, that means it can't be used in a parallel "reduce".

That's what I was warning about!

Parallel Version

Key insights

- if you find () in a sequence, you can delete it without changing the balance

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- if you have deleted all of the pairs (), you are left with:
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Combining two sequences where we have deleted all ():

-))) … j …))) (((… k … ((( ))) … x …))) (((… y … (((

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- if you find () in a sequence, you can delete it without changing the balance
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Combining two sequences where we have deleted all ():

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- if x ≥ k then))) … j …)))))) … x – k …))) (((… y … (((

Parallel Version

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For divide-and-conquer, splitting a sequence of parens is easy

Combining two sequences where we have deleted all ():

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- if x ≥ k then))) … j …)))))) … x – k …))) (((… y … (((
- if x ≤ k then))) … j …))) (((… k – x … ((((((… y … (((

Parallel Matcher

```
(* delete all () and return the (j, k) corresponding to:

   ))) ... j ... ))) ((( ... k ... (((
 *)

let rec matcher s =
    match length s with
      0 -> (0, 0)
    | 1 -> (match nth s 0 with
              | L -> (0, 1)
              | R -> (1, 0))
    | n ->
       let (left, right) = split (n/2) s in
       let ((j, k), (x, y)) = both matcher left
                                   matcher right  in
       if x > k
       then (j + (x - k), y)
       else (j, (k - x) + y)
```

))) … j …))) (((… k … (((

))) … x …))) (((… y … (((

Parallel Balance

```
(*   *)
let matcher s = ...

(* true if s is a sequence of balanced parens *)
let balanced s =
    match matcher s with
    | (0, 0) -> true
    | (j,k) -> false
```

Parallel Matcher

```
(* delete all () and return the (j, k) corresponding to:

   ))) ... j ... ))) ((( ... k ... (((
 *)

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    match length s with
      0 -> (0, 0)
    | 1 -> (match nth s 0 with
            | L -> (0, 1)
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    | n ->
       let (left, right) = split (n/2) s in
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                                   matcher right  in
       if x > k
       then (j + (x - k), y)
       else (j, (k - x) + y)
```

This looks just like mapreduce!

Using a Parallel Fold

```
let rec mapreduce(inject: 'a -> 'b)
                 (combine:'b -> 'b -> 'b)
                 (base:'b)
                 (s:'a seq) = ...
```

```
let inject paren =
  match paren with
    L -> (0, 1)
  | R -> (1, 0)

let combine (j,k) (x,y) =
      if x > k then (j + (x - k), y)
      else          (j, (k - x) + y)

let balanced s =
    match mapreduce inject combine (0,0) s with
    | (0, 0) -> true
    | (i,j) -> false
```

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```

Work: O(N)
Span: O(log N)

Using a Parallel Fold

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```

For correctness, check the associativity of combine

also check: combine base (i,j) == (i, j)

Summary

Parallel complexity can be described in terms of work and span

Folds and reduces are easily coded as parallel divide-and-conquer algorithms with O(n) work and O(log n) span

The map-reduce paradigm, inspired by functional programming, is a winner when it comes to big-data processing (more about that in the next lecture).

Sanity checks

```
let combine (j,k) (x,y) =
      if x > k then (j + (x - k), y)
      else          (j, (k - x) + y)

base = (0,0)
```

check the associativity of combine

also check:
combine base (i,j) == (i, j)

Prove for yourself:

combine (combine (j,k) (x,y)) (a,b) = combine (j,k) (combine (x,y)(a,b))

combine (j,k) (0,0) = (j,k)

combine (0,0) (j,k) = (j,k)