Simple Functional Data

COS 326
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What is the single most important mathematical concept ever developed in human history?

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An answer: The mathematical variable

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(runner up: natural numbers/induction)

Why is the mathematical variable so important?

The mathematician says:

“Let x be some integer, we define a polynomial over x ...”

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The mathematician says:

"Let x be some integer, we define a polynomial over x ..."

What is going on here? The mathematician has separated a *definition* (of x) from its *use* (in the polynomial).

This is the most primitive kind of *abstraction* (x is *some* integer)

Abstraction is the key to controlling complexity and without it, modern mathematics, science, and computation would not exist.

It allows *reuse* of ideas, theorems ... functions and programs!

OCAML BASICS:
LET DECLARATIONS

Abstraction & Abbreviation

In OCaml, the most basic technique for factoring your code is to use let expressions

Instead of writing this expression:

```
(2 + 3) * (2 + 3)
```

We write this one:

```
let x = 2 + 3 in
x * x
```

A Few More Let Expressions

```
let x = 2 in
let squared = x * x in
let cubed = x * squared in
squared * cubed
```

```
let a = "a" in
let b = "b" in
let as = a ^ a ^ a in
let bs = b ^ b ^ b in
as ^ bs
```

A Technical Note: The Structure of a .ml File

Foo.ml

```
<declaration>

<declaration>

      …
```

Every .ml file is a sequence of *declarations*

These “declarations” are a little different than “expressions”

A Technical Note: The Structure of a .ml File

Bar.ml

```
let x = 17 + 5

let y = x + 22
```

Bar.ml contains two *let declarations*

Let declarations do not end with “in”

Let declarations have the form:

let *<var>* = *<expression>*

A Technical Note: The Structure of a .ml File

Baz.ml

```
let x =
  let z = 22 in
  z + z

let y =
  if x < 17 then
    let w = x + 1 in
    2 * w
  else
    26
```

Because let declarations have this form:

let *<var>* = *<expression>*

they contain expressions

... including “let expressions” which have the form:

let *<var>* = *<expression>* in *<expression>*

OCaml Variables are Immutable

Once *bound* to a value, a variable is never modified or changed.

```
let x = 3

let add_three (y:int) : int = y + x
```

given a *use* of a variable, like this one for *x*, work outwards and upwards through a program to find the closest enclosing *definition*. That is the value of this use *forever and always*.

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OCaml Variables are Immutable

Once *bound* to a value, a variable is never modified or changed.

```
let x = 3

let add_three (y:int) : int = y + x
```

It does not matter what I write next. add_three will always add 3!

OCaml Variables are Immutable

Once *bound* to a value, a variable is never modified or changed.

a distinct variable that "happens to be spelled the same"

```
let x = 3

let add_three (y:int) : int = y + x

let x = 4

let add_four (y:int) : int = y + x
```

OCaml Variables are Immutable

A use of a variable always refers to it's *closest* (in terms of syntactic distance) enclosing declaration. Hence, we say OCaml is a *statically scoped* (or *lexically scoped*) language

we can use add_three without worrying about the second definition of x

```
let x = 3

let add_three (y:int) : int = y + x

let x = 4

let add_four (y:int) : int = y + x

let add_seven (y:int) : int =
  add_three (add_four y)
```

OCaml Variables are Immutable

Since the two variables (both happened to be named x) are actually different, unconnected things, we can rename them.

This is known as *alpha-conversion*.

you can rename
x to zzz
by replacing
the definition
and all its uses with
the new name

```
let x = 3

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let x = 4

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let add_three (y:int) : int = y + x

let zzz = 4

let add_four (y:int) : int = y + zzz

let add_seven (y:int) : int =
  add_three (add_four y)
```

How does OCaml execute a let expression?

```
let x = <expression1> in
<expression2>
```

In a nutshell:

- execute <expression1>, until you get a value v1
- substitute that value v1 for x in <expression2>
- execute <expression2>, until you get a value v2
- the result of the whole execution is v2

How does OCaml execute a let expression?

```
let x = 2 + 1 in x * x
```

-->

```
let x = 3 in x * x
```

How does OCaml execute a let expression?

```
let x = 2 + 1 in x * x
```

-->

```
let x = 3 in x * x
```

substitute
3 for x

-->

```
        3 * 3
```

How does OCaml execute a let expression?

```
let x = 2 + 1 in x * x
```

-->

```
let x = 3 in x * x
```

substitute
3 for x

-->

```
3 * 3
```

-->

```
9
```

How does OCaml execute a let expression?

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let x = 2 + 1 in x * x
```

-->

```
let x = 3 in x * x
```

substitute
3 for x

-->

```
3 * 3
```

-->

```
9
```

Note: I write
e1 --> e2
when e1 evaluates
to e2 in one step

Meta-comment

OCaml expression

OCaml expression

```
let x = 2 in x + 3      -->      2 + 3
```

I defined the language in terms of itself:
By reduction of one OCaml expression to another

I'm trying to train you to think at a high level of abstraction.

I didn't have to mention low-level abstractions like assembly code or registers or memory layout to tell you how OCaml works.

Another Example

```
let x = 2 in
let y = x + x in
y * x
```

Another Example

```
let x = 2 in
let y = x + x in
y * x
```

substitute
2 for x

-->

```
let y = 2 + 2 in
y * 2
```

Another Example

```
let x = 2 in
let y = x + x in
y * x
```

substitute
2 for x

-->

```
let y = 2 + 2 in
y * 2
```

-->

```
let y = 4      in
y * 2
```

Another Example

```
let x = 2 in
let y = x + x in
y * x
```

substitute
2 for x

-->

```
let y = 2 + 2 in
y * 2
```

-->

```
let y = 4       in
y * 2
```

substitute
4 for y

-->

```
4 * 2
```

Another Example

```
let x = 2 in
let y = x + x in
y * x
```

substitute
2 for x

-->

```
let y = 2 + 2 in
y * 2
```

-->

```
let y = 4       in
y * 2
```

substitute
4 for y

-->

```
4 * 2
```

-->

```
8
```

Moral: Let operates by *substituting* computed values for variables

OCAML BASICS: TYPE CHECKING AGAIN

Back to Let Expressions ... Typing

x granted type of e1 for use in e2

```
let x = e1 in

e2
```

overall expression
takes on the type of e2

Back to Let Expressions ... Typing

x granted type of e1 for use in e2

```
let x = e1 in

e2
```

overall expression
takes on the type of e2

x has type int
for use inside the
let body

```
let x = 3 + 4 in

string_of_int x
```

overall expression
has type string

Let Expressions Really Are Expressions

$$\boxed{2 + 3}$$

an expression

Let Expressions Really Are Expressions

```
2 + 3
```

an expression

```
let x = 2 + 3 in
x + x
```

an expression

Let Expressions Really Are Expressions

```
2 + 3
```

an expression

```
let x = 2 + 3 in
x + x
```

an expression

an expression

```
let x = let y = 2 + 3 in y + 5 in
1 + x
```

let expressions can appear anywhere other expressions can appear. they can be *nested*

Exercise

(a)

```
let x =
  let y = 2 + 3 in y
in
  let x = "1" in
x + x
```

(b)

```
let x =
  let y = "2" ^ "3" in y
in
  let x = 1 in
x + x
```

Which of (a) or (b) type check? Explain why.

On a piece of paper (or in your favorite editor), show the step-by-step evaluation of the example that type checks.

Critique the *programming style* used in these examples.

OCAML BASICS: FUNCTIONS

Defining functions

```
let add_one (x:int) : int = 1 + x
```

Defining functions

let keyword

```
let add_one (x:int) : int = 1 + x
```

function name

argument name

type of argument

type of result

expression that computes value produced by function

Note: recursive functions with begin with "**let rec**"

Defining functions

Nonrecursive functions:

```
let add_one (x:int) : int = 1 + x

let add_two (x:int) : int = add_one (add_one x)
```

definition of add_one
must come before use

Defining functions

Nonrecursive functions:

```
let add_one (x:int) : int = 1 + x

let add_two (x:int) : int = add_one (add_one x)
```

With a local definition:

local function definition hidden from clients

```
let add_two' (x:int) : int =
  let add_one x = 1 + x in
  add_one (add_one x)
```

I left off the types. OCaml figures them out

Good style: types on top-level definitions

Types for Functions

Some functions:

```
let add_one (x:int) : int = 1 + x

let add_two (x:int) : int = add_one (add_one x)

let add (x:int) (y:int) : int = x + y
```

function with two arguments

Types for functions:

```
add_one : int -> int

add_two : int -> int

add : int -> int -> int
```

Rule for type-checking functions

General Rule:

If a function f : T1 → T2
and an argument e : T1
then f e : T2

Example:

```
add_one : int -> int

3 + 4 : int

add_one (3 + 4) : int
```

Rule for type-checking functions

Recall the type of add:

Definition:

```
let add (x:int) (y:int) : int =
  x + y
```

Type:

```
add : int -> int -> int
```

Rule for type-checking functions

Recall the type of add:

Definition:

```
let add (x:int) (y:int) : int =
  x + y
```

Type:

```
add : int -> int -> int
```

Same as:

```
add : int -> (int -> int)
```

Rule for type-checking functions

General Rule:

If a function $f : T1 \rightarrow T2$
and an argument $e : T1$
then $f\ e : T2$

$A \rightarrow B \rightarrow C$

same as:

$A \rightarrow (B \rightarrow C)$

Example:

```
add : int -> int -> int

3 + 4 : int

add (3 + 4) : ???
```

Rule for type-checking functions

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If a function f : T1 -> T2
and an argument e : T1
then f e : T2

A -> B -> C

same as:

A -> (B -> C)

Example:

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add : int -> (int -> int)

3 + 4 : int

add (3 + 4) :
```

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same as:

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Example:

```
add : int -> int -> int

3 + 4 : int

add (3 + 4) : int -> int

(add (3 + 4)) 7 : int
```

Rule for type-checking functions

General Rule:

If a function f : T1 -> T2
and an argument e : T1
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A -> B -> C

same as:

A -> (B -> C)

Example:

```
add : int -> int -> int

3 + 4 : int

add (3 + 4) : int -> int

add (3 + 4) 7 : int
```

extra parens
not necessary

One key thing to remember

- If you have a function f with a type like this:

$$A \to B \to C \to D \to E \to F$$

- Then each time you add an argument, you can get the type of the result by knocking off the first type in the series

f a1 : $B \to C \to D \to E \to F$ (if a1 : A)

f a1 a2 : $C \to D \to E \to F$ (if a2 : B)

f a1 a2 a3 : $D \to E \to F$ (if a3 : C)

f a1 a2 a3 a4 a5 : F (if a4 : D and a5 : E)

TYPE ERRORS

Type Checking Rules

Type errors for if statements can be confusing sometimes. Recall:

```
let rec concatn s n =
  if n <= 0 then
    ...
  else
    s ^ (concatn s (n-1))
```

Type Checking Rules

Type errors for if statements can be confusing sometimes. Recall:

```
let rec concatn s n =
  if n <= 0 then
    ...
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```

ocaml might point to (concatn s (n-1)) and says:

```
Error: This expression has type int but an
expression was expected of type string
```

Type Checking Rules

Type errors for if statements can be confusing sometimes. Recall:

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let rec concatn s n =
  if n <= 0 then
    ...
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    s ^ (concatn s (n-1))
```

ocaml might say:

```
Error: This expression has type int but an
expression was expected of type string
```

or ocaml might point to the expression (s ^ (concatn ...)) and say:

```
Error: This expression has type string but an
expression was expected of type int
```

Type Checking Rules

Type errors for if statements can be confusing sometimes.

Example. We create a string from s, concatenating it n times:

```
let rec concatn s n =
  if n <= 0 then
    ...
  else
    s ^ (concatn s (n-1))
```

???

```
Error: This expression has type int but an
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Type Checking Rules

Type errors for if statements can be confusing sometimes.
Example. We create a string from s, concatenating it n times:

they don't *agree*!

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let rec concatn s n =
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```

???

```
Error: This expression has type int but an
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Type Checking Rules

Type errors for if statements can be confusing sometimes.
Example. We create a string from s, concatenating it n times:

they don't *agree*!

```
let rec concatn s n =
  if n <= 0 then
    0
  else
    s ^ (concatn s (n-1))
```

???

The type checker points to *some* place where there is *disagreement*.

Moral: *Sometimes you need to look in an earlier branch for the error* even though the type checker points to a later branch.
The type checker doesn't know what the user wants.

A Tactic: Add Typing Annotations

```
let rec concatn (s:string) (n:int) : string =
  if n <= 0 then
    0
  else
    s ^ (concatn s (n-1))
```

```
Error: This expression has type int but an
expression was expected of type string
```

Exercise

Given the following code:

```
let munge b x =
  if not b then
    string_of_int x
  else
    "hello"

let y = 17
```

What are the types of the following expressions?
(And what must the types of f and g be?)

```
munge : ??

munge (y > 17) : ??

munge true (f (munge false 3)) : ??

munge true (g munge) : ??
```

DATA STRUCTURES: THE TUPLE

* it is really our second complex data structure since functions are data structures too!

Tuples

A tuple is a fixed, finite, ordered collection of values

Some examples with their types:

```
(1, 2)                          : int * int
("hello", 7 + 3, true)          : string * int * bool
('a', ("hello", "goodbye"))     : char * (string * string)
```

Tuples

To use a tuple, we extract its components

General case:

```
let (id1, id2, …, idn) = e1 in e2
```

An example:

```
let (x,y) = (2,4) in x + x + y
```

Tuples

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```

An example:

```
let (x,y) = (2,4) in x + x + y
-->  2 + 2 + 4
```

substitute!

Tuples

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General case:

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let (id1, id2, …, idn) = e1 in e2
```

An example:

```
let (x,y) = (2,4) in x + x + y
-->  2 + 2 + 4
-->  8
```

Rules for Typing Tuples

if e1 : t1 and e2 : t2
then (e1, e2) : t1 * t2

Rules for Typing Tuples

if e1 : t1 and e2 : t2
then (e1, e2) : t1 * t2

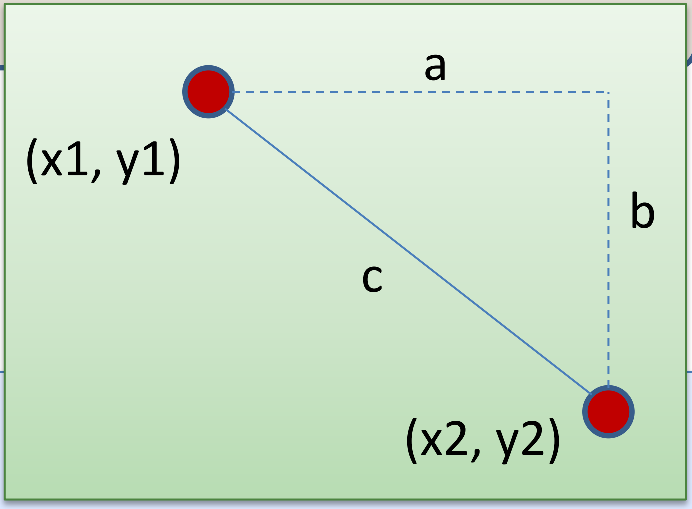
if e1 : t1 * t2 then
x1 : t1 and x2 : t2
inside the expression e2

```
let (x1,x2) = e1 in

e2
```

overall expression
takes on the type of e2

Distance between two points



$c^2 = a^2 + b^2$

Problem:

- A point is represented as a pair of floating point values.
- Write a function that takes in two points as arguments and returns the distance between them as a floating point number

Writing Functions Over Typed Data

Steps to writing functions over typed data:

1. Write down the function and argument names
2. Write down argument and result types
3. Write down some examples (in a comment)

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   - *the argument types suggests how to do it*
5. Build new output values
   - *the result type suggests how you do it*

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   - define and reuse helper functions
   - your code should be elegant and easy to read

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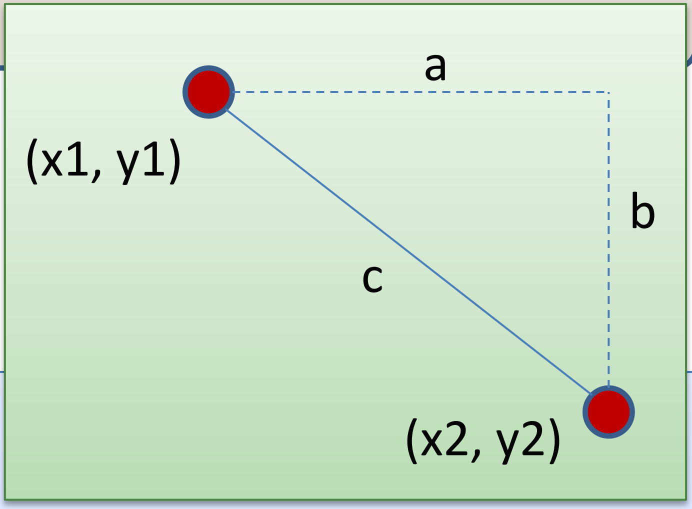
   - define and reuse helper functions
   - your code should be elegant and easy to read

Types help structure your thinking about how to write programs.

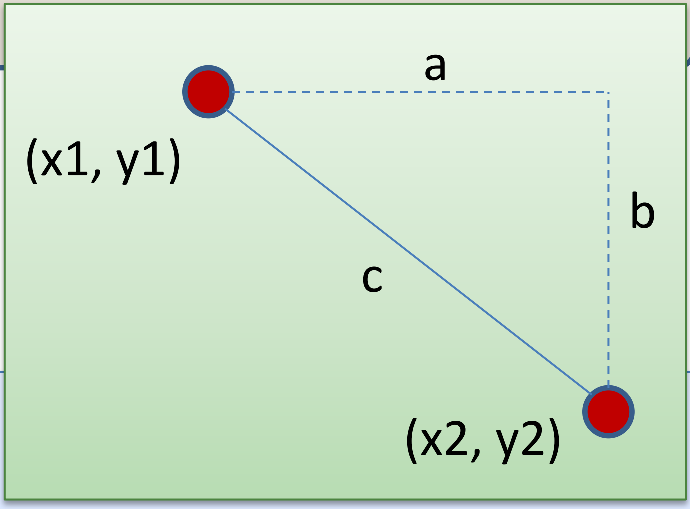
Distance between two points



a type abbreviation

```
type point = float * float
```

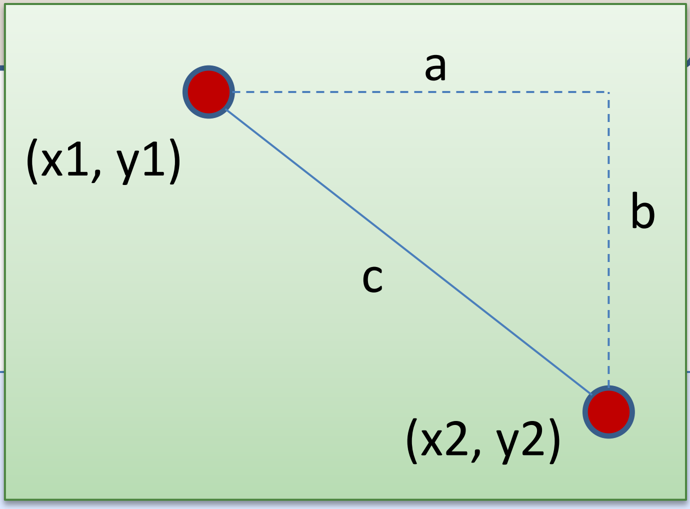
Distance between two points




```
type point = float * float

let distance (p1:point) (p2:point) : float =
```

write down function name
argument names and types

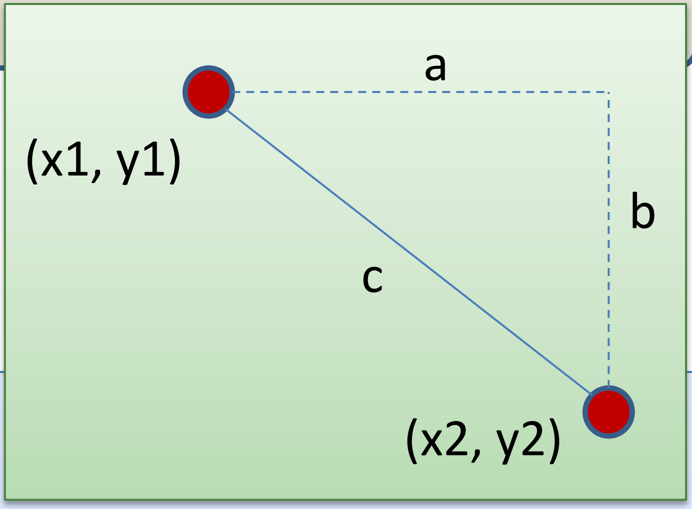
Distance between two points



examples

```
type point = float * float


(* distance (0.0,0.0) (0.0,1.0) == 1.0
 * distance (0.0,0.0) (1.0,1.0) == sqrt(1.0 + 1.0)
 *
 * from the picture:
 * distance (x1,y1) (x2,y2) == sqrt(a^2 + b^2)
 *)


let distance (p1:point) (p2:point) : float =
```

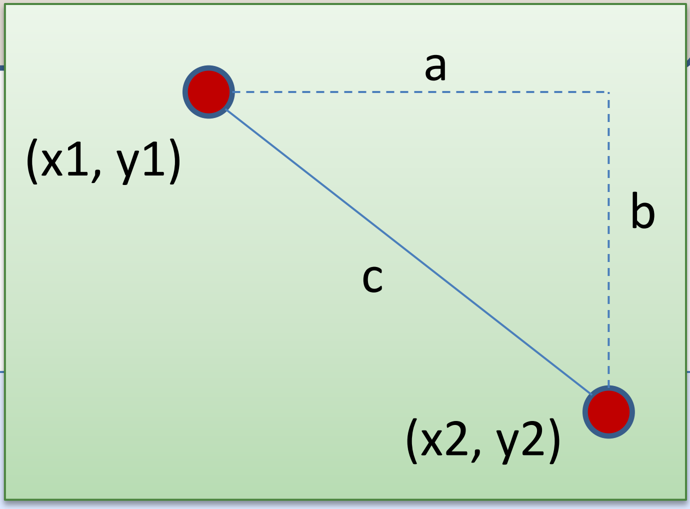
Distance between two points




```
type point = float * float

let distance (p1:point) (p2:point) : float =

  let (x1,y1) = p1 in
  let (x2,y2) = p2 in
  ...
```

deconstruct
function inputs

Distance between two points

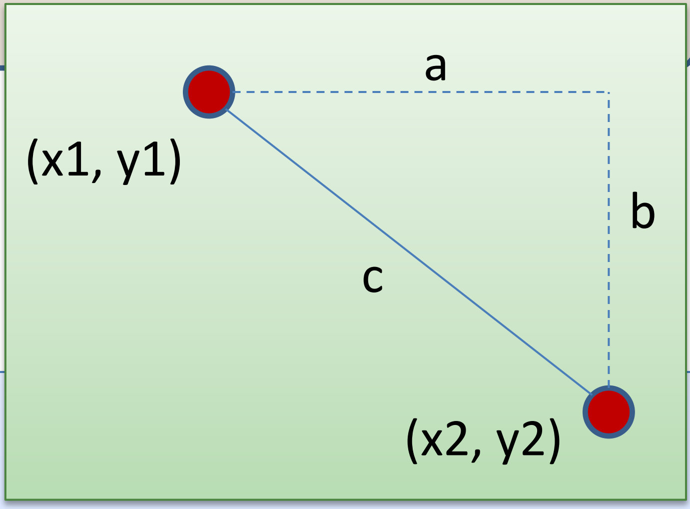



```
type point = float * float

let distance (p1:point) (p2:point) : float =

  let (x1,y1) = p1 in
  let (x2,y2) = p2 in
  sqrt ((x2 -. x1) *. (x2 -. x1) +.
        (y2 -. y1) *. (y2 -. y1))
```

compute function results

notice operators on floats have a "." in them

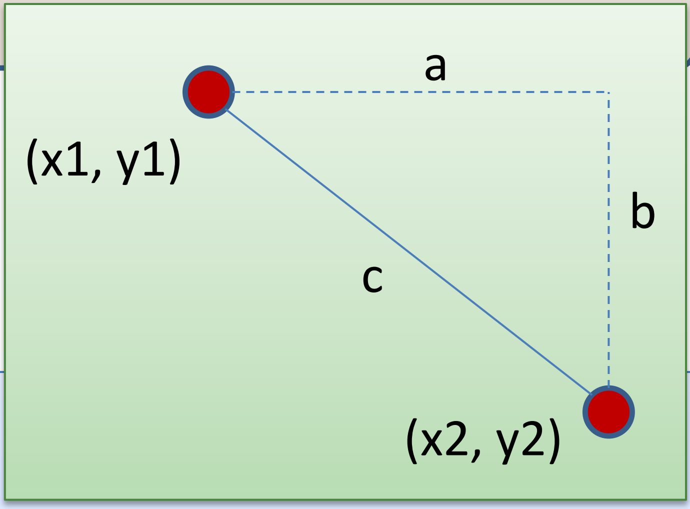
Distance between two points




```
type point = float * float

let distance (p1:point) (p2:point) : float =
  let square x = x *. x in
  let (x1,y1) = p1 in
  let (x2,y2) = p2 in
  sqrt (square (x2 -. x1)) +.
        square (y2 -. y1))
```

define helper functions to avoid repeated code

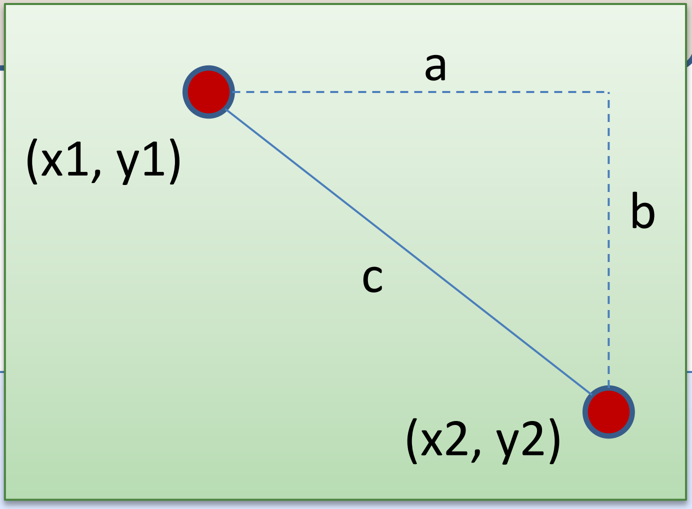
Distance between two points




```
type point = float * float

let distance (x1,y1) (x2,y2) =
  let square x = x *. x in
  sqrt (square (x2 -. x1) +. square (y2 -. y1))
```

use tuple patterns
in function arguments
if you'd like

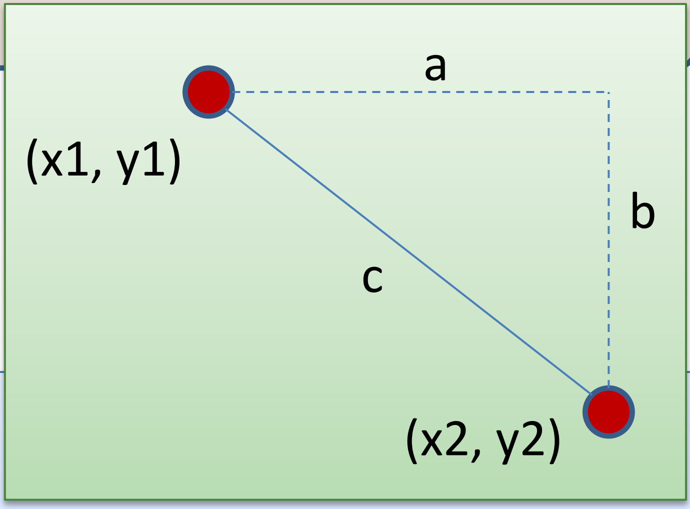
Distance between two points




```
type point = float * float

let distance ((x1,y1):point) ((x2,y2):point) : float =
  let square x = x *. x in
  sqrt (square (x2 -. x1) +. square (y2 -. y1))
```

type annotations
can be included

Distance between two points




```
type point = float * float

let distance (p1:point) (p2:point) : float =
  let square x = x *. x in
  let (x1,y1) = p1 in
  let (x2,y2) = p2 in
  sqrt (square (x2 -. x1) +. square (y2 -. y1))


let pt1 = (2.0,3.0)
let pt2 = (0.0,1.0)
let dist12 = distance pt1 pt2
```

implement some tests

MORE TUPLES

Tuples

Here's a tuple with 2 fields:

(4.0, 5.0) : float * float

Tuples

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```

Tuples

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(4.0, 5, "hello", 55) : float * int * string * int

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Here's a tuple with 4 fields:

(4.0, 5, "hello", 55) : float * int * string * int

Here's a tuple with 0 fields:

() : unit

SUMMARY:
BASIC FUNCTIONAL PROGRAMMING

Writing Functions Over Typed Data

Steps to writing functions over typed data:

1. Write down the function and argument names
2. Write down argument and result types
3. Write down some examples (in a comment)
4. Deconstruct input data structures
5. Build new output values
6. Clean up by identifying repeated patterns

For tuple types:

- when the input has type t1 * t2
  - use let (x,y) = … to deconstruct
- when the output has type t1 * t2
  - use (e1, e2) to construct

We will see this paradigm repeat itself over and over

Records

Records are a lot like tuples. It's just that they have named fields.

Having named fields (records rather than tuples) often makes it easier to understand a program, especially when there are more than just 2 or 3 fields in a structure.

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An example:

```
type name = {first:string; last:string;}

let my_name = {first="David"; last="Walker";}

let to_string (n:name) = n.last ^ ", " ^ n.first
```

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type name = {first:string; last:string;}

let my_name = {first="David"; last="Walker";}

let to_string (n:name) = n.last ^ ", " ^ n.first
```

Note: Records come with several other useful features, like functional updates via "with expressions."

See *Real World OCaml* for more info.

WRAP-UP

Writing Functions Over Typed Data

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Exercise

What error do you get when you try to compile this file? (Type it in.) Why?

```
type item = {
   number: int;
   name: string;
}

type contact = {
   name: string*string;   (* first and last name *)
   phone: item;
}

let get_name x = x.name

let myphone = {number=122; name="iphone";}

let _ = print_endline (get_name myphone)
```