COS 226 **Precept 6 Notes** September 3, 2021
Fall 2021
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In precept we explored two questions about percolation, questions raised by questions 4 and 5 in the first segment of the precept lesson, on percolation. We talked about question four first, but I'll reverse the order here, because I just got done drawing lots of figures related to question 5.

Question 5 asked whether there is a simple formula that will allow us to estimate the percolation threshold. The intended answer is yes: no one knows such a formula for the square grid, nor for many other tessellations for which one might ask the same question. But there are tessellations for which there is a simple formula, and knowing the answer for one kind of tessellation may give the answer for another. We explored this possibility. In the process, we discovered some interesting facts about the plane, planar graphs, and duality.

By the way, "tessellation" is just a big word for "an arrangement of shapes closely fitted together, especially of polygons in a repeated pattern without gaps or overlapping."

As motivation, consider the attached picture of black and white tiles, which cover the bathroom floor in my brother-in-law's house in Princeton. What does this have to do with percolation? It's a tessellation, and a relevant one, as we shall discover.

The kind of percolation in the programming assignment is 4-way percolation on a square grid: the cells are squares and we want to know if there is a path of connected squares with one in the top row and one in the bottom row, where "connected" means adjacent north, south, west, or east; that is, sharing a side. Sharing a corner does not suffice for connection. See Figure 1. This is 4-way connection. But we could allow squares sharing a corner to be connected: if there were holes in the corners of the squares, water could flow between squares sharing only a corner. An intermediate possibility is 6-way connection: two squares are connected if they share a northwest-southeast corner. Figure 2 shows these three different kinds of connection. I asked you to think about what the percolation thresholds might be for 4-way, 6-way, and 8-way connectivity, and how they might relate.

Given an arrangement of open and closed cells, if the arrangement percolates with 4-way connectivity, then it will percolate with 6-way connectivity, and if it percolates with 6-way connectivity, then it will percolate with 8-way connectivity, since adding connections can make an arrangement percolate, but if an arrangement already percolates, then adding connections cannot stop it from percolating. But we can say much more.

A tessellation in two dimensions is a planar graph: the cells are the faces, the corners are the vertices, the sides are the edges. An equivalent way to look at percolation is to take the *dual* of the graph, which is formed by putting a vertex in the middle of each face and connecting two vertices by an edge if their faces share a side. Since we are interested in north-south percolation we add two extra vertices, N and S, connect N by an edge to each vertex representing a face in the top row, and connect S by an edge to each vertex representing a face in the bottom row. See

Figure 3. An arrangement percolates if and only if there is a path of open vertices connecting *N* and *S*.
If we rotate Figure 3 ninety degrees and rename *N* and *S* *E* and *W*. We get Figure 4. Because of the ninety degree symmetry, asking whether there is a path of open vertices connecting *W* and *E* in Figure 4 is effectively the same as asking whether there is a path of open cells connecting *N* and *S* in Figure 3.

If we overlap these figures, we get Figure 4, which reveals something interesting: given any arrangement of open and closed vertices with *N* and *S* open and *W* and *S* closed, if there is an open path (a path of open vertices) connecting *N* and *S*, there *cannot* be a closed path (a path of closed vertices) connecting *W* and *E*. This is by the Jordan curve theorem, a fundamental theorem of topology that says that a simple closed curve divides the plane into an inside and an outside. Any path connecting a point on the inside to a point on the outside must contain a point on the curve. (Think of a circle made out of a rubber band laid on the plane that can be arbitrarily stretched and shrunk but not twisted, so it cannot cross over itself.). If we connect *N* and *S* with an edge around the outside of the grid as in Figure 5, then any open path connecting *N* and *S* becomes a simple closed curve with the addition of the new edge. Vertex *W* is inside this curve and vertex *E* is outside, so any closed path connecting *W* and *E* would have to contain a vertex on the closed curve. But this vertex would be both open and closed, a contradiction.

Because of the symmetry, the same argument shows that if there is a closed path connecting *W* and *E*, there cannot be an open path connecting *N* and *S*. That is, no arrangement can percolate both north-south with respect to open cells and west-east with respect to closed cells. This means that the percolation threshold p_t for the square grid with four-way connections must be at least ½: if p_t were less than ½, then choosing p = ½ and n large enough, it would be extremely likely that a random arrangement would percolate both north-south for open cells and west-east for closed cells, which is impossible.

Now let's consider the square grid with 8-way connectivity. To model 8-way connectivity, let's put a little diamond in the corner of each square of our square grid. This produces a tessellation of octagons and diamonds (which are just squares tilted forty-five degrees). See Figure 6. Given an arrangement of open and closed squares on the original square grid, we obtain a corresponding arrangement on the new grid by making each octagon open or closed depending on whether the corresponding square is open or closed, and making every diamond open. Then the arrangement on the square grid percolates north-south with 8-way connectivity if, and only if, the corresponding arrangement percolates on the new grid, where two cells are connected if they share a side. What's more, if instead of making all the diamonds open we make them all closed, then an arrangement of the square grid percolates north-south with 4-way connectivity if, and only if, the corresponding arrangement percolates on the new grid. See Figure 7, the bathroom tiling!

Now suppose we do the same thing to the octagon-diamond tessellation that we did to the square grid: construct the dual and add extra vertices *N*, *S*, *W*, and *E* with *N* and *S* open and *W* and *E* closed. We obtain Figure 8. Suppose we map an arrangement of open and closed squares on the original square grid to the corresponding arrangement of octagons on the new tessellation. Then the arrangement percolates north-south on the square grid with 4-way connectivity if and only if it percolates north-south on the tessellation with all diamonds closed, and the arrangement percolates north-south on the square grid with eight-way connectivity if and only if it percolates north-south on the tessellation with all diamonds open.

Now consider the octagon-diamond tessellation with all diamonds closed. By the Jordan curve argument we used on the square grid, if there is a path of open cells connecting N and S, there cannot be a path of closed cells connecting W and E, and if there is a path of closed cells connecting W and E, there cannot be a path of open cells connecting N and S. This is like what happened on the square grid. But now something stronger is true: for any arrangement, there is either a path of open cells connecting N and S, or a path of closed cells connecting W and E. In class you asked me to prove this, and I punted on the question, but I'll return to it below. First let's figure out what this tells us about 4-way and 8-way percolation on the square grid.

The square grid as well as the octagon-diamond tessellation are the same when rotated ninety degrees. Suppose we choose a probability p and generate an arrangement of open and closed cells in which the probability of a cell being open is p and the probability of a cell being closed is $1 - p$. In the octagon-diamond tessellation, either there is an open path connecting N and S, to the arrangement percolates north-south, or there is a closed path connecting W and E, but not both. We can view a closed path connecting W and E as the arrangement percolating west-east if we treat closed cells as open and vice-versa. That is, each arrangement open-percolates north-south, or closed percolates east-west, but not both. This means that if the open percolation threshold is p_t, the closed percolation threshold is $1 - p_t$. Translating back to the square grid, if the 4-way percolation threshold is p_t, the 8-way percolation threshold is $1 - p_t$. Knowing one gives us the other!

Now let's consider 6-way connectivity. If we modify the square grid to add sides between squares sharing a northwest-southeast corner, we obtain the hexagonal grid in Figure 9. As in the case of the octagon-diamond tessellation, any arrangement of open and closed cells has a north-south path of open cells, or an east-west path of closed cells, but not both. We have added one more degree of symmetry, since now open and closed are exactly the same! It follows that the percolation threshold for this grid is exactly ½ - no computation necessary!

The fact that in any arrangement of open and closed cells on the hexagonal grid there is a north-south of open cells or an east-west path of closed cells, but not both, is the basis of the game of Hex. I precept I mistakenly claimed that this game was invented by John Horton Conway, but this is not true: Conway invented many games, most notably the "game" (not really a game) of life, but not Hex. Hex was invented by Piet Hein and independently re-invented by John Nash, an even more famous Princetonian who won a Nobel Prize in economics for his work in game theory. Nash's colleagues called the game "Nash." At the following link you can find a nice paper on Hex, including a proof that one player always wins: https://web.mit.edu/sp.268/www/hex-notes.pdf. The same proof shows that there is either a north-south open path or an east-west closed path in the octagon-diamond tessellation, and indeed in any tessellation in which there are at most three faces at each corner. Rather than repeating the argument here, I'll let you read it in the paper. An amazing fact about this result is that it is equivalent to Brower's fixed point theorem, another classical result in mathematics. See

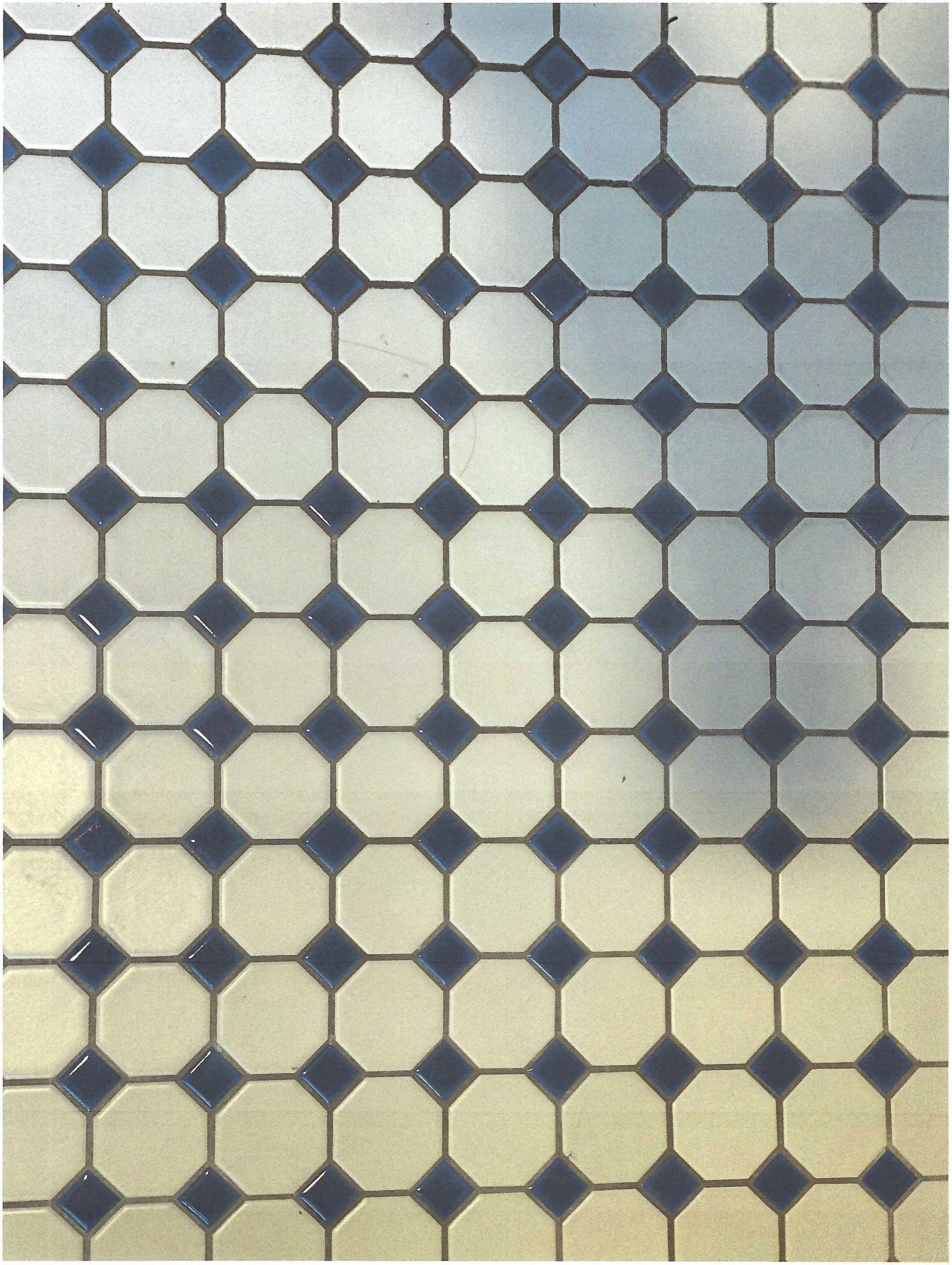
the paper on Hex. Who would have known? Our "simple" algorithmic problem has connections two deep results in mathematics.
Now let's turn to question 4 on the precept assignment, which we also discussed. The question asks, "Opening just one additional site can *drastically* increase the odds that a system percolates." True or false? The approved answer is true. But is this correct? Empirical evidence (some of which you will provide) suggests that as n, the number of rows and columns, grows, the probability that a system percolates has threshold behaviour: just below the critical threshold p_t, the probability that the system percolates is close to zero; just above, the probability of percolation is close to one. This means that if n is very large, a very small change in p, the probability that a cell is open, can drastically affect the probability that the system percolates. But as n grows, the number of cells grows. Opening one additional site corresponds to a *very* tiny change in p, not by a small constant but by something proportional to $1/n^2$. This much smaller change may well *not* drastically affect the odds of percolation. The threshold curve is steep, but it *never* becomes a step function, no matter hos big n is. Indeed it is an interesting question to estimate the variance in the threshold curve; that is, to estimate how big the range of p (as a function of n) over which the percolation probability changes from close to zero to close to one.

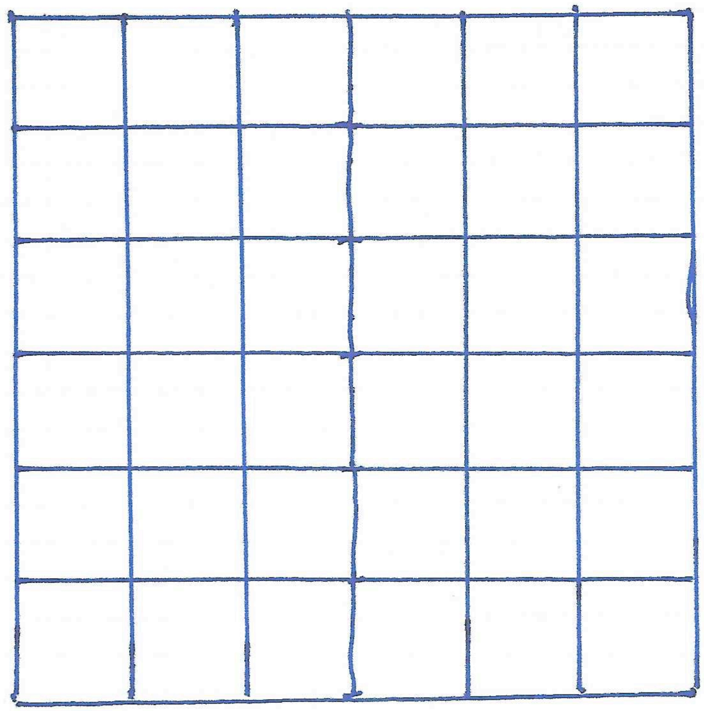
We won't pursue this issue further here, but I encourage the statistically minded among you to learn about the law of large numbers, properties of the normal distribution, and Chebyshev's inequalities, which come in handy in estimating the behavior of random events, including that of randomized algorithms.

Finally, I left you with the following question: Is the model we are using to estimate the percolation threshold probability correct? Recall the original problem definition: Cells are open with some fixed probability p, independent for each cell. (A given cell is open with probability p no matter what the state of the other cells is.). A sample generated according to this distribution may or may not percolate. We want to estimate the value p_t such that the percolation probability is ½. (If the threshold curve is steep, the system will almost certainly percolate if $p > p_t$, and almost certainly *not* percolate if $p < p_t$.). But we are generating samples in a different way, by choosing a uniformly random permutation of the cells, opening the cells in permutation order, and counting the number of cells open when the system first percolates. Does this estimate give the correct answer? Why or why not?

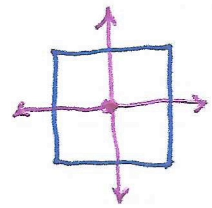
In case the answer is no, let's consider an alternative way of estimating p_t. We repeat the following experiment: For each cell, generate a random real number uniformly in the interval from zero to one. Given the sample, compute the largest p such that if all cells with number at least p are open, the system percolates. Use this p as an estimate of the percolation threshold.

Does this method work? If it does, can you devise a fast algorithm to compute the percolation value for a given sample? If you can, and you have time and ambition, you might try doing the programming assignment (or at least parts of it) using this method to see how it compares.

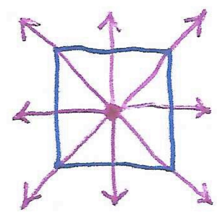




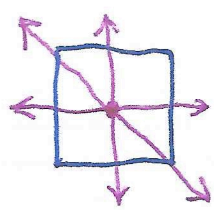
A square grid



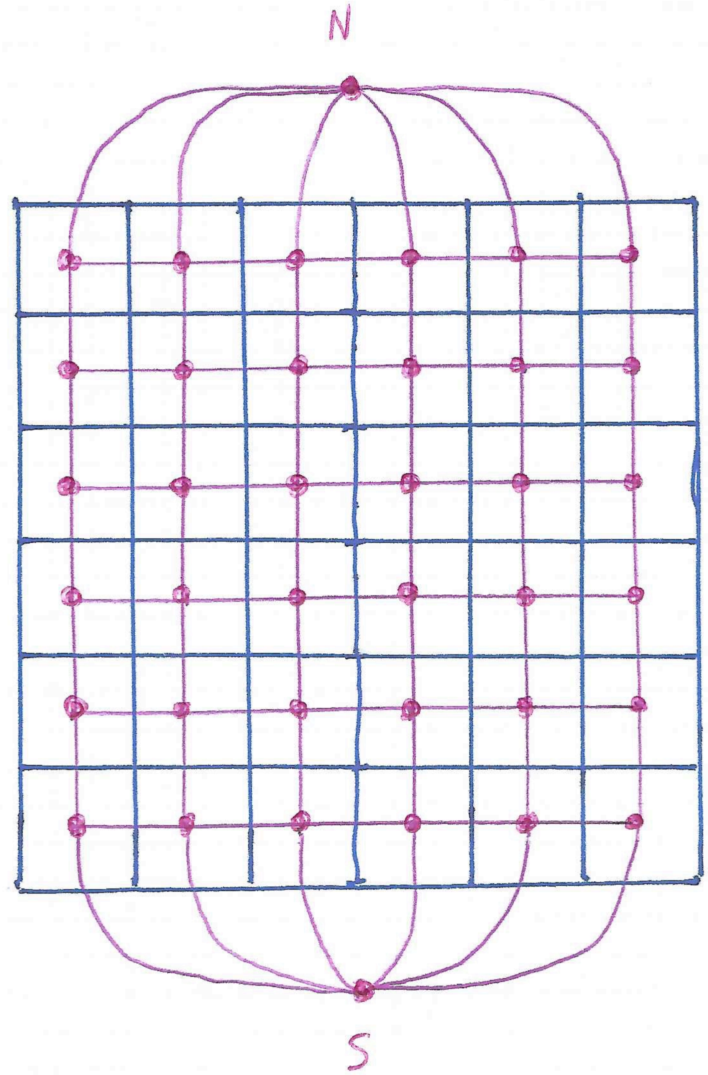
4-way flow



8-way flow

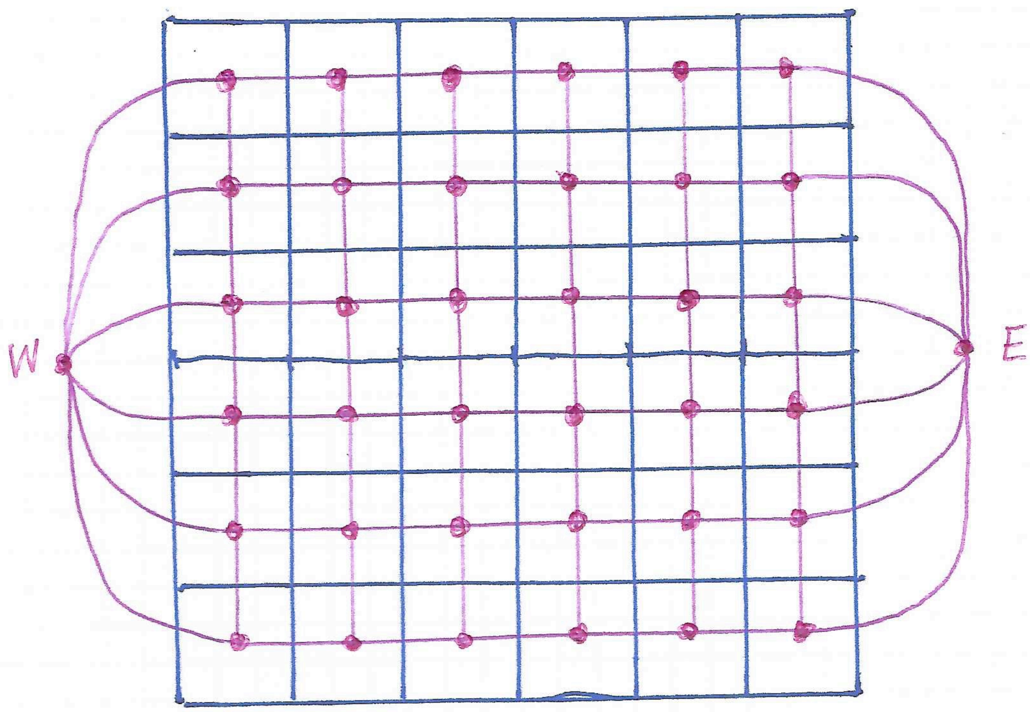


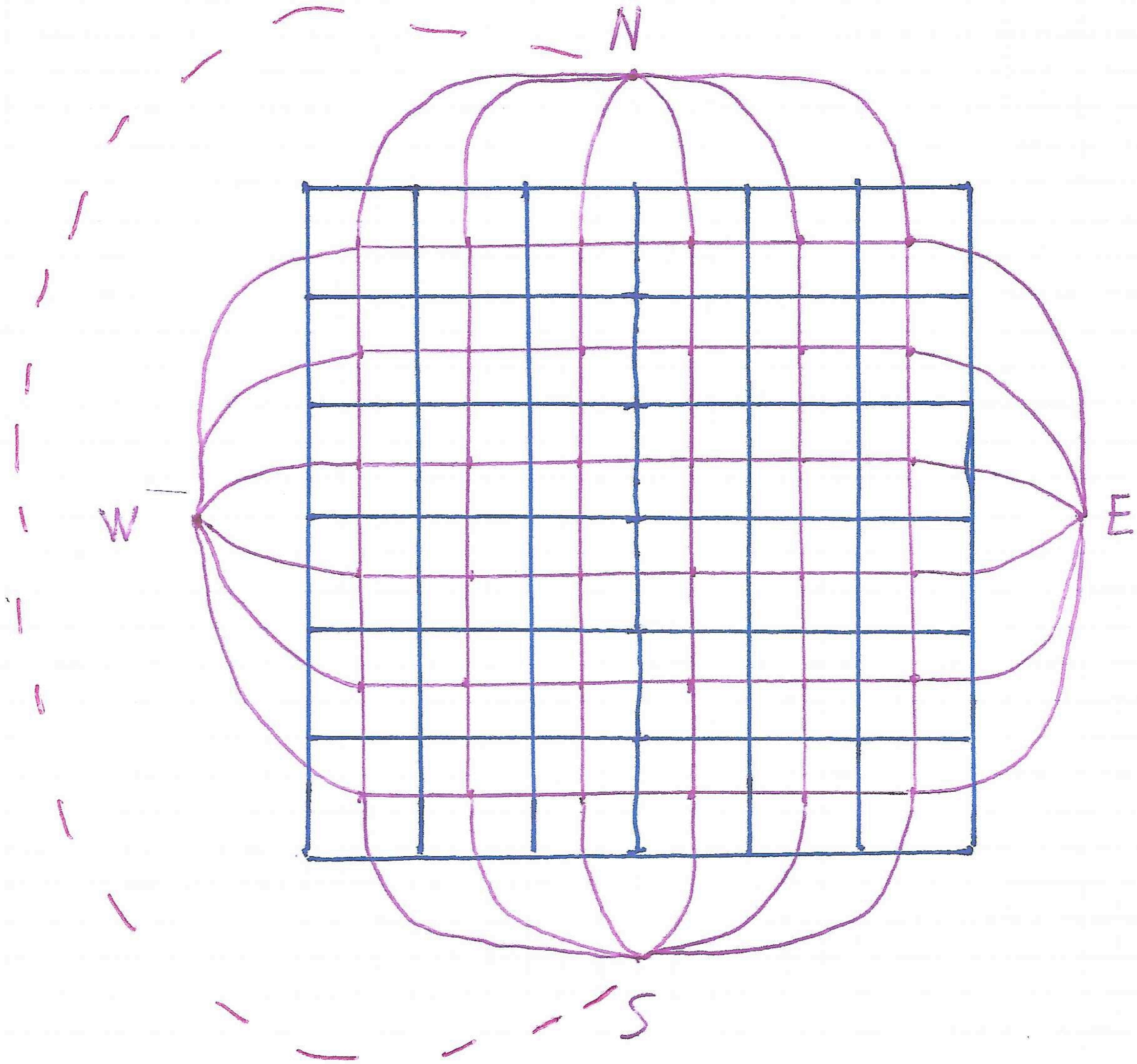
6-way flow



Dual graph with N, S

N-S path = percolation


W
E



NS open path $\Rightarrow$
no WE closed path

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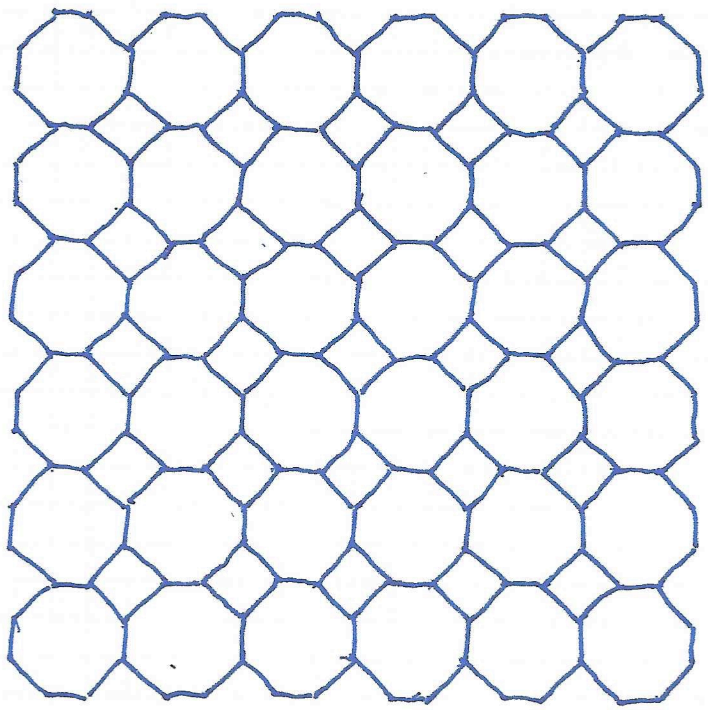
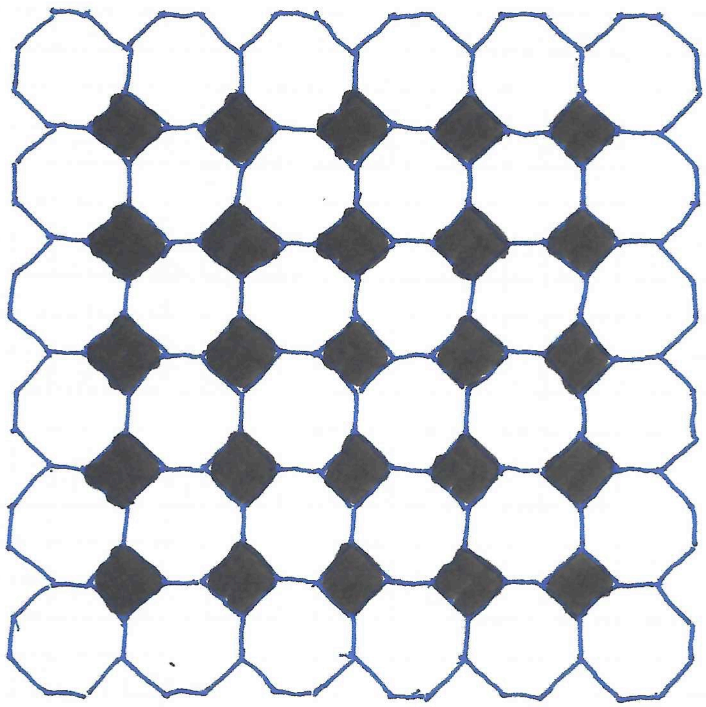
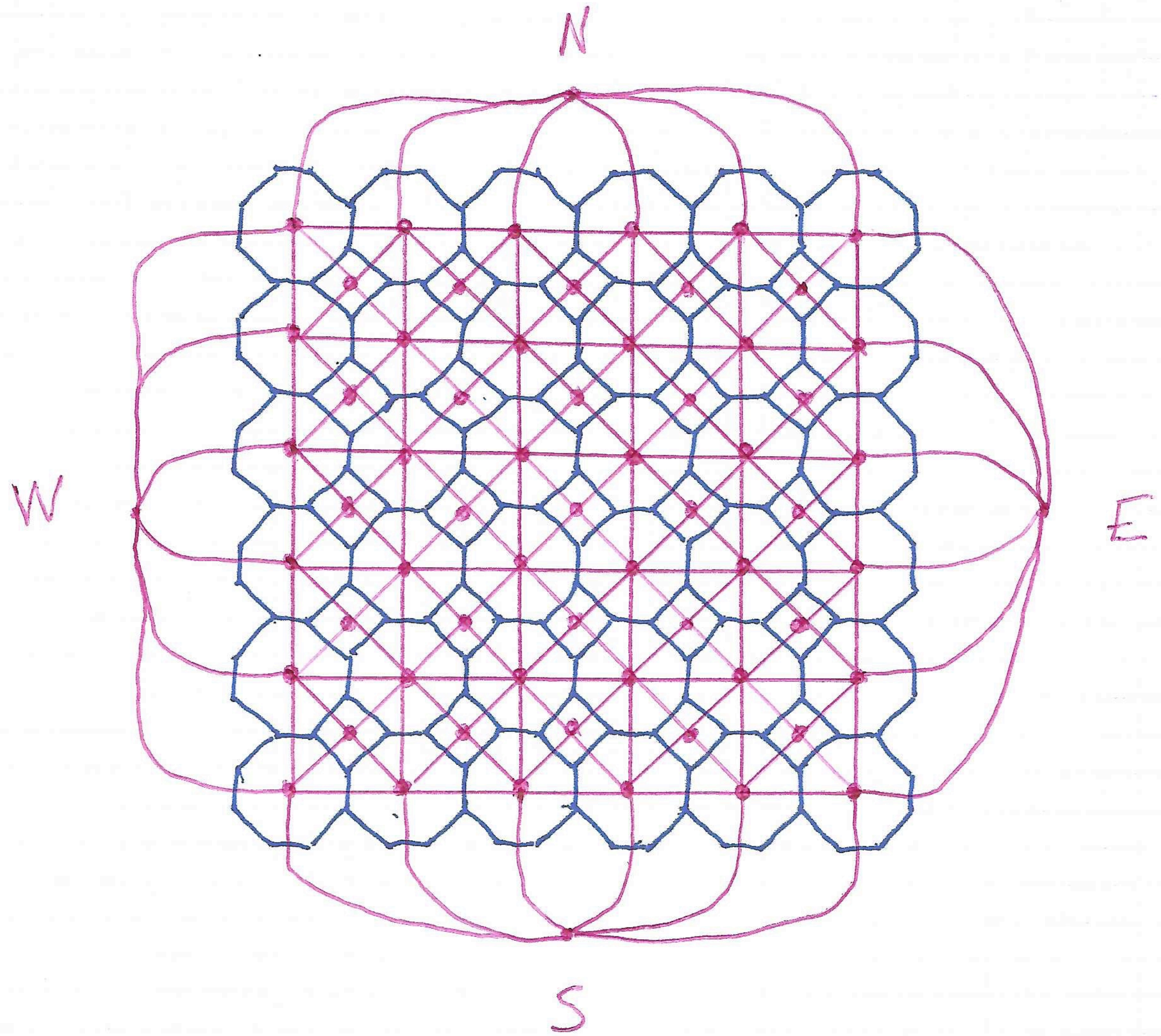
no NS open path



8-way perdation:
all diamonds open



4- way percolation:
all diamonds closed

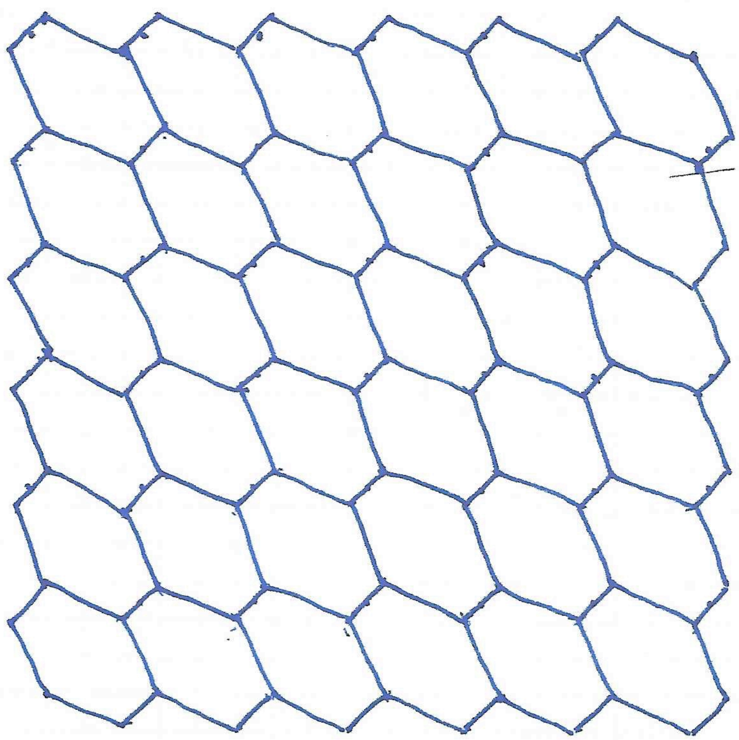


All diamonds closed

NS 4-way open path

if and only if

no WE 8-way closed path



6-way percolation

NS open path iff

no WE closed path