Polymorphism

COS 326
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Princeton University

polymorphic,
higher-order
programming

POLY-HO!

Here’s an annoying thing

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

What if I want to increment a list of floats?

Alas, I can’t just call this map. It works on ints!

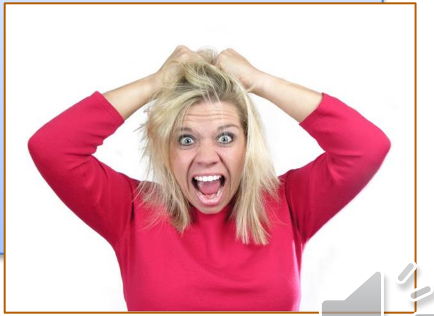
Here's an annoying thing

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

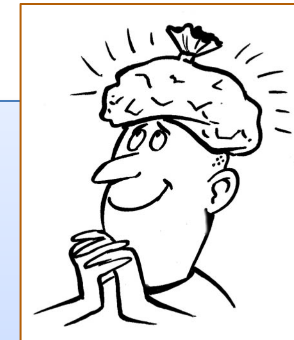
What if I want to increment a list of floats?

Alas, I can't just call this map. It works on ints!

```
let rec mapfloat (f:float->float) (xs:float list) :
           float list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(mapfloat f tl);;
```



Turns out



```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

let ints = map (fun x -> x + 1) [1; 2; 3; 4]

let floats = map (fun x -> x +. 2.0) [3.1415; 2.718]

let strings = map String.uppercase [“sarah”; “joe”]
```

Type of the undecorated map?

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

map : ('a -> 'b) -> 'a list -> 'b list
```

Type of the undecorated map?

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```

We often use greek letters like α or β to represent type variables.

Read as:

- for any types 'a and 'b,
- if you give map a function from 'a to 'b,
- it will return a function
  - which when given a list of 'a values
  - returns a list of 'b values.

We can say this explicitly

```
let rec map (f:'a -> 'b) (xs:'a list) : 'b list =
  match xs with
   | [] -> []
   | hd::tl -> (f hd)::(map f tl)

map : (‘a -> ‘b) -> ‘a list -> ‘b list
```

The OCaml compiler is smart enough to figure out that this is the *most general* type that you can assign to the code.

(technical term: *principal type*)

We say map is *polymorphic* in the types 'a and 'b – just a fancy way to say map can be used on any types 'a and 'b.

Java generics derived from ML-style polymorphism (but added after the fact and more complicated due to subtyping)

More realistic polymorphic functions

```
let rec merge (lt:'a->'a->bool) (xs:'a list) (ys:'a list) : 'a list =
  match (xs,ys) with
  | ([],_) -> ys
  | (_,[]) -> xs
  | (x::xst, y::yst) ->
      if lt x y then x::(merge lt xst ys)
      else y::(merge lt xs yst)

let rec split (xs:'a list)(ys:'a list)(zs:'a list) : 'a list * 'a list =
  match xs with
  | [] -> (ys, zs)
  | x::rest -> split rest zs (x::ys)

let rec mergesort (lt:'a->'a->bool) (xs:'a list) : 'a list =
  match xs with
  | ([] | _::[]) -> xs
  | _ -> let (first,second) = split xs [] [] in
         merge lt (mergesort lt first) (mergesort lt second)
```

More realistic polymorphic functions

```
mergesort : ('a->'a->bool) -> 'a list -> 'a list

mergesort (<) [3;2;7;1]
  == [1;2;3;7]

mergesort (>) [2; 3; 42]
  == [42 ; 3; 2]

mergesort (fun x y -> String.compare x y < 0) [“Hi”; “Bi”]
  == [“Bi”; “Hi”]

let int_sort = mergesort (<)
let int_sort_down = mergesort (>)
let str_sort = mergesort (fun x y -> String.compare x y < 0)
```

Another Interesting Function

```
let comp f g x = f (g x)

let mystery = comp (add 1) square
```

```
let comp = fun f -> (fun g -> (fun x -> f (g x)))

let mystery = comp (add 1) square
```

```
let mystery =
  (fun f -> (fun g -> (fun x -> f (g x)))) (add 1) square
```

```
let mystery =               fun x -> (add 1) (square x)
```

```
let mystery x = add 1 (square x)
```

Function composition!

```
let comp f g x = f (g x)

let mystery = comp (add 1) square
```

$(f \circ g)(x) = f(g(x))$

mystery = (add 1) $\circ$ square

mystery(x) = (add 1) (square (x))

What is the type of comp?

```
let comp f g x = f (g x)
```

```
let comp (f: 'b->'c) (g: 'a->'b) (x: 'a) : 'c
   = f (g x)
```

```
comp : ('b -> 'c) ->
       ('a -> 'b) ->
       ('a -> 'c)
```

Optimization

What does this program do?

```
map f (map g [x1; x2; …; xn])
```

For each element of the list x1, x2, x3 ... xn, it executes g, creating:

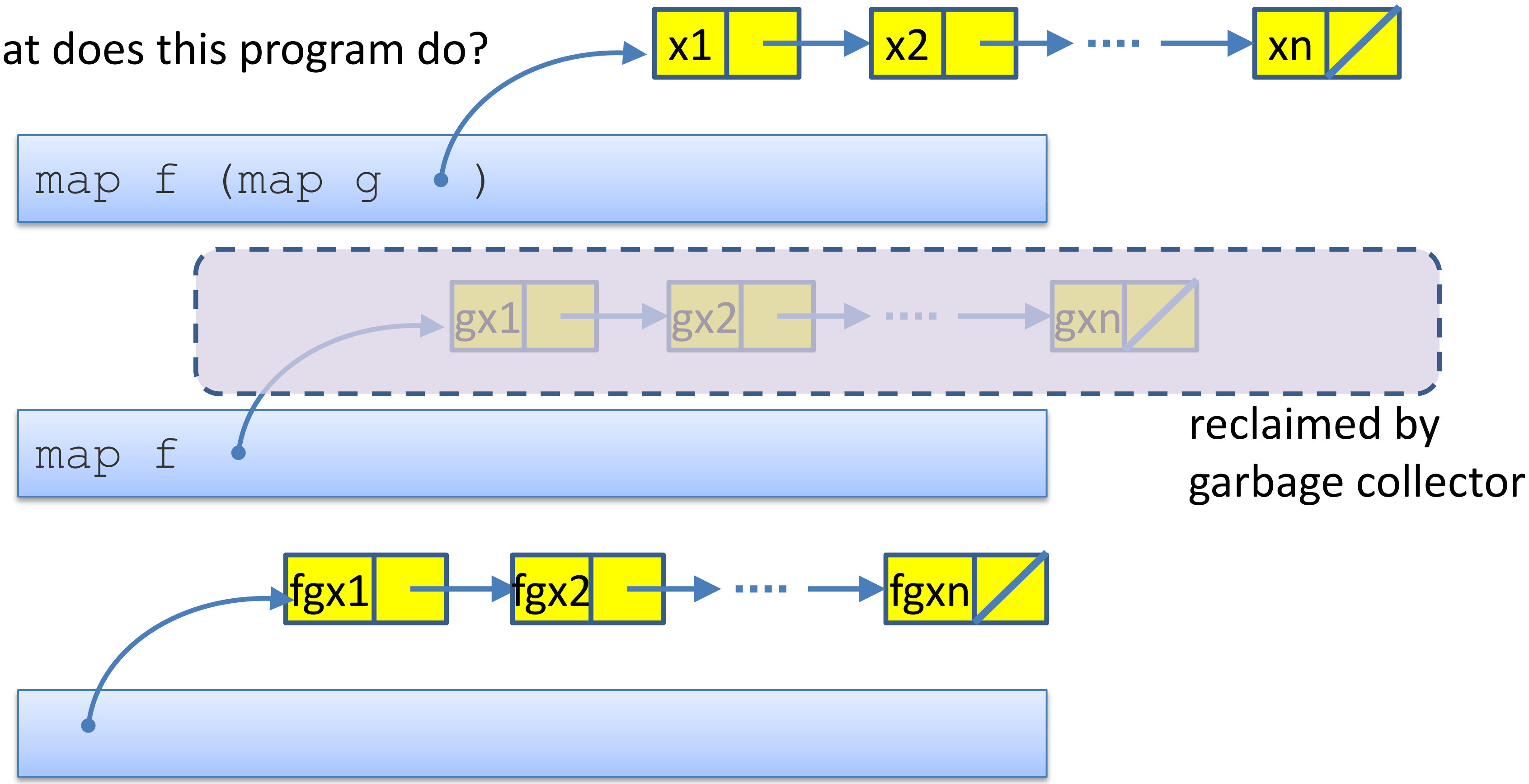
```
map f ([g x1; g x2; …; g xn])
```

Then for each element of the list [g x1, g x2, g x3 ... g xn], it executes f, creating:

```
[f (g x1); f (g x2); …; f (g xn)]
```

Optimization

What does this program do?



Optimization

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map f ([g x1; g x2; …; g xn])
```

Then for each element of the list [g x1, g x2, g x3 ... g xn], it executes f, creating:

```
[f (g x1); f (g x2); …; f (g xn)]
```

Is there a faster way? Yes! (And query optimizers for SQL do it for you.)

```
map (comp f g) [x1; x2; ...; xn]
```

Deforestation

```
map f (map g [x1; x2; …; xn])
```

This kind of optimization has a name:

deforestation

(because it eliminates intermediate lists and, um, trees…)

```
map (comp f g) [x1; x2; ...; xn]
```

How about reduce?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general ty

Based on the patterns, we know xs must be a ('a list) for some type 'a.

How about reduce?

```
let rec reduce f u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

How about reduce?

```
let rec reduce f u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

f is called so it must be a function of two arguments.

How about reduce?

```
let rec reduce (f:? -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:? -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

Furthermore, hd came from xs, so f must take an 'a value as its first argument.

How about reduce?

```
let rec reduce (f:'a -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> ? -> ?) u (xs: 'a list)  =
   match xs with
   | [] -> u
   | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

The second argument to f must have the same type as the result of reduce. Let's call it 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

The result of f must have the same type as the result of reduce overall: 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

If xs is empty, then reduce returns u. So u's type must be 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

reduce returns the result of f. So f's result type must be 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

```
('a -> 'b -> 'b) -> 'b -> 'a list -> 'b
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
     (fun x y -> 1+y) hd (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
      (fun x y -> 1+y) hd (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
     (fun y -> 1+y) (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + reduce (fun ...) 0 tl
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + mystery0 tl
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + mystery0 tl   List Length!
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery1 = reduce (fun x y -> x::y) []
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery1 = reduce (fun x y -> x::y) []

let rec mystery1 xs =
  match xs with
  | [] -> []
  | hd::tl -> hd::(mystery1 tl)   Copy!
```

And this one?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery2 g =
   reduce (fun a b -> (g a)::b) []
```

And this one?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery2 g =
   reduce (fun a b -> (g a)::b) []

let rec mystery2 g xs =
  match xs with
  | [] -> []
  | hd::tl -> (g hd)::(mystery2 g tl) map!
```

Map and Reduce

```
val map : ('a -> 'b) -> 'a list -> 'b list
```

```
val reduce : ('a -> 'b -> 'b) -> 'b ->  'a list -> 'b
```

We coded map in terms of reduce:

- ie: we showed we can compute map f xs using a call to reduce ? ? ? just by passing the right arguments in place of ? ? ?

Can we code reduce in terms of map?

Map and Reduce

```
val map : ('a -> 'b) -> 'a list -> 'b list
```

```
val reduce : ('a -> 'b -> 'b) -> 'b ->  'a list -> 'b
```

let reduce f u xs =   … map (…) (…) …

(use only: map, f, u, xs; don't use **rec** *)*

reduce (+) 0 [1;2;3] = … map (…) (…) …

Some Other Combinators: List Module

https://caml.inria.fr/pub/docs/manual-ocaml/libref/List.html

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a

val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b
```

```
val mapi : (int -> 'a -> 'b) -> 'a list -> 'b list

List.mapi f [a0; ...; an] == [f 0 a0; … ; f n an]
```

```
val map2 : ('a -> 'b -> 'c) -> 'a list -> 'b list -> 'c list

List.map2 f [a0; ...; an] [b0; ...; bn] == [f a0 b0 ; … ; f an bn]
```

```
val iter : ('a -> unit) -> 'a list -> unit

List.iter f [a0; ...; an] == f a0; … ; f an
```

Summary

- Map and reduce are two *higher-order functions* that capture very, very common *recursion patterns*

- Reduce is especially powerful:
  - related to the "visitor pattern" of OO languages like Java.
  - can implement most list-processing functions using it, including things like copy, append, filter, reverse, map, etc.

- We can write clear, terse, reusable code by exploiting:
  - higher-order functions
  - anonymous functions
  - first-class functions
  - polymorphism

Practice Problems

Using map, write a function that takes a list of pairs of integers, and produces a list of the sums of the pairs.

- e.g., list_add [(1,3); (4,2); (3,0)] = [4; 6; 3]
- Write list_add directly using reduce.

Using map, write a function that takes a list of pairs of integers, and produces their quotient if it exists.

- e.g., list_div [(1,3); (4,2); (3,0)] = [Some 0; Some 2; None]
- Write list_div directly using reduce.

Using reduce, write a function that takes a list of optional integers, and filters out all of the None's.

- e.g., filter_none [Some 0; Some 2; None; Some 1] = [0;2;1]
- Why can't we directly use filter? How would you generalize filter so that you can compute filter_none? Alternatively, rig up a solution using filter + map.

Using reduce, write a function to compute the sum of squares of a list of numbers.

- e.g., sum_squares = [3,5,2] = 38