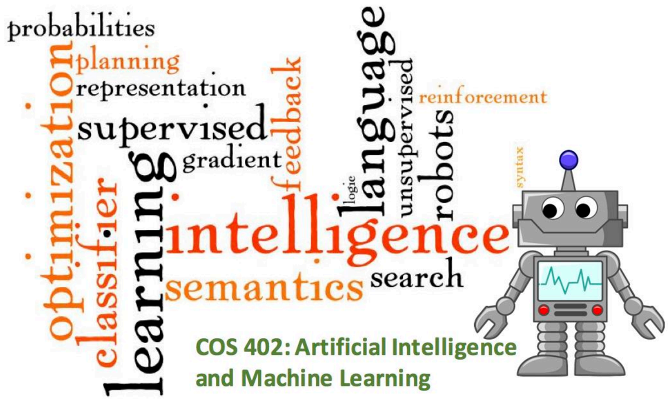

COS 402 – Machine Learning and Artificial Intelligence
Fall 2016

Lecture 16: Hidden Markov Models

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Course progress

- Learning from examples
  - Definition + fundamental theorem of statistical learning, motivated efficient algorithms/optimization
  - Convexity, greedy optimization – gradient descent
  - Neural networks
- Knowledge Representation
  - NLP
  - Logic
  - Bayes nets
  - Optimization: MCMC
  - HMM (today) (a special case of Bayes nets)
- Next: reinforcement learning

Admin

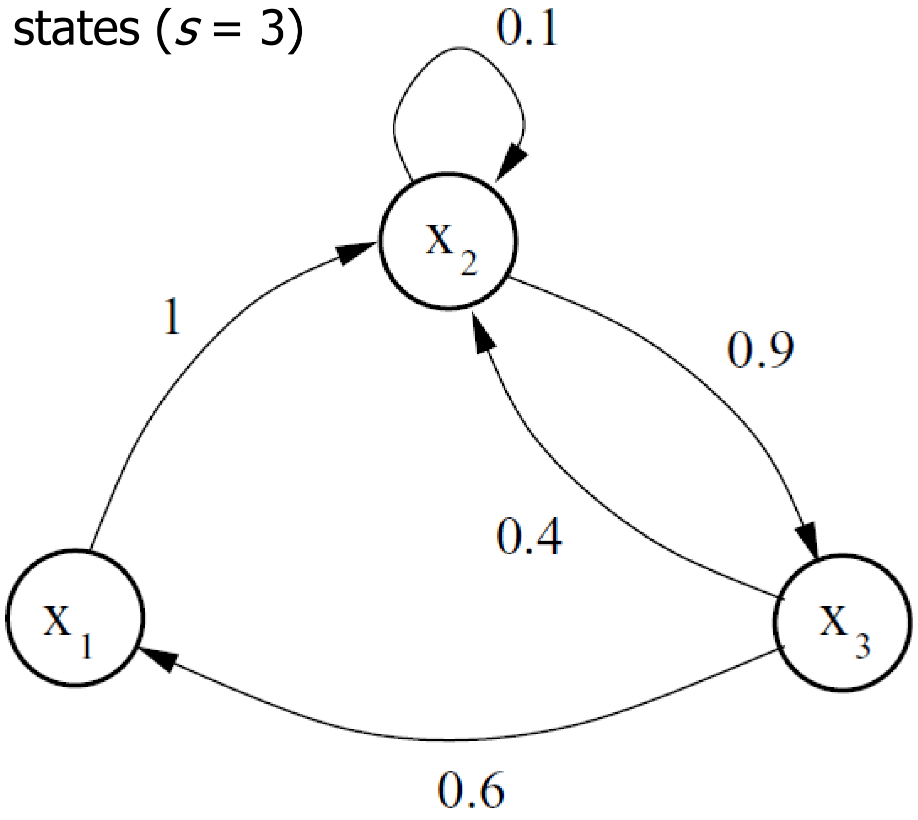
- (written) ex4 – announced today
- Due after Thanksgiving (Thu)

Markov Chain

Markov chain with three states (s = 3)

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.1 & 0.9 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

Transition matrix



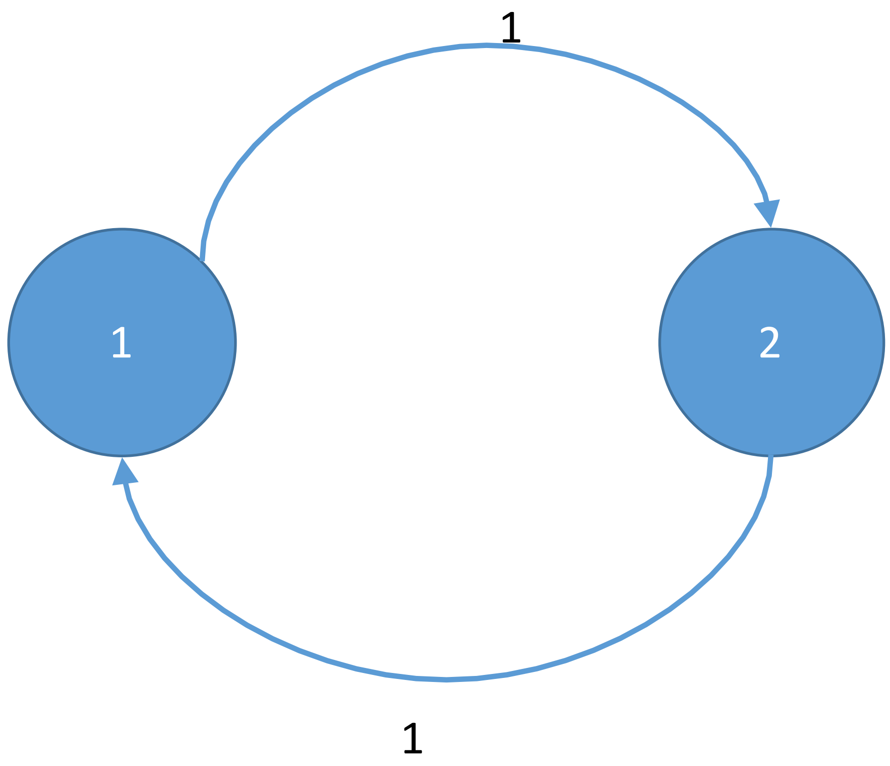
Transition graph

Directed graph, and a transitition matrix giving, for each i, j   the probability of stepping to j when at i.

Ergodic theorem

Every irreducible and a-periodic Markov chain has a unique stationary distribution, and every random walk starting from any node converges to it!

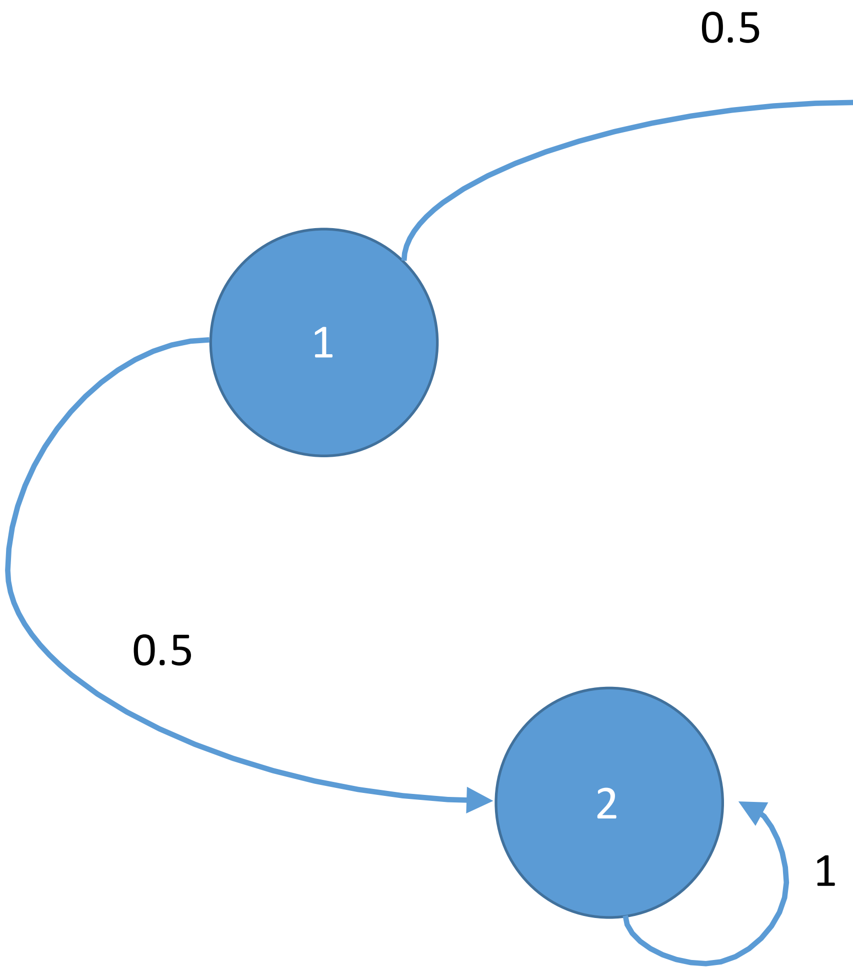
Non-stationary Markov chains



"periodic"

Notice: self-loop → not periodic anymore

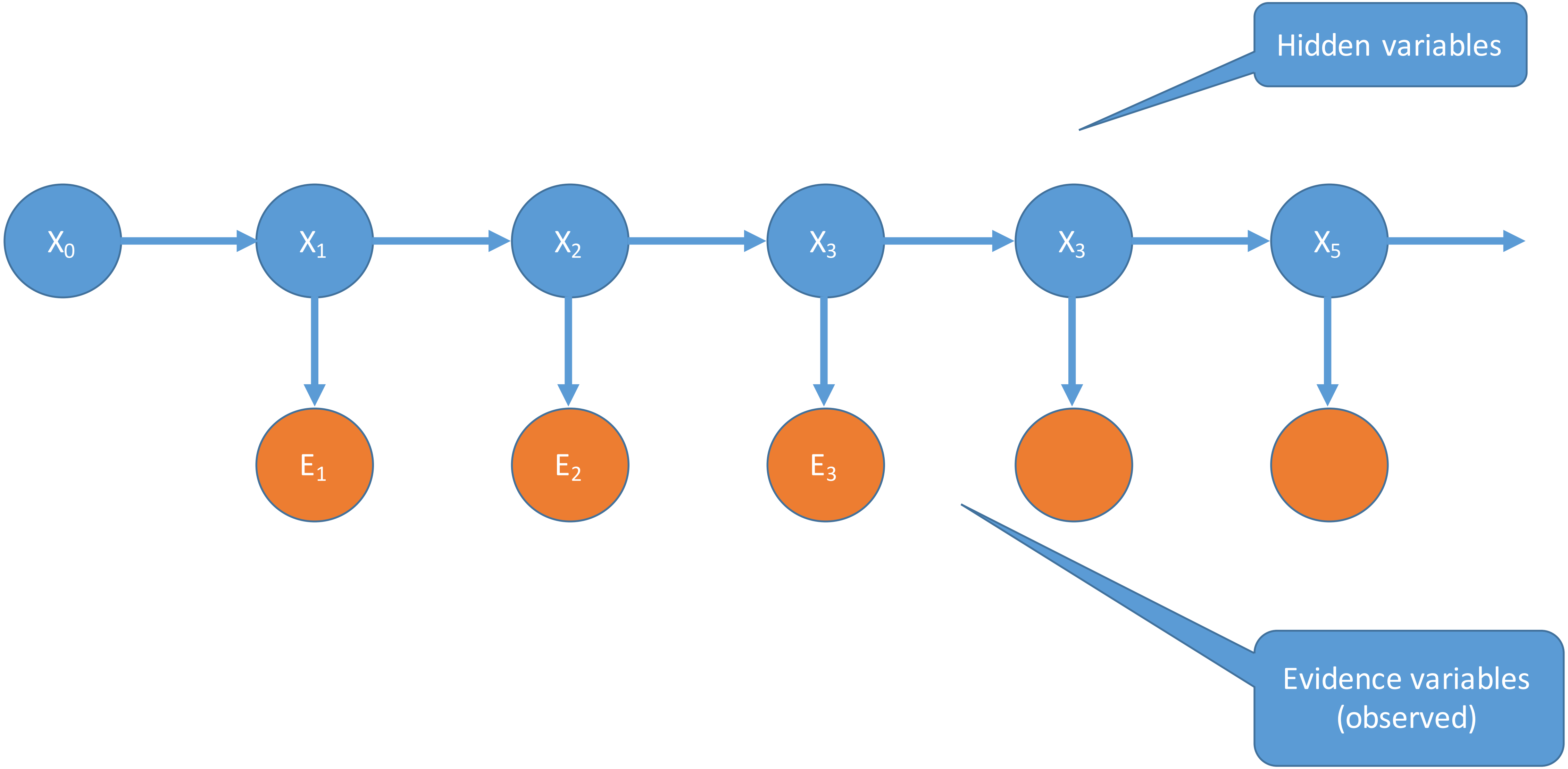
Non-stationary Markov chains



"reducible"

Irreducible → for any pair of vertices, $T_{ij} > 0$ after finitely many iterations.

This lecture: temporal models
Hidden Markov Models



Applications

- Time-dependent variables / problems (e.g. treating patients with changing biometrics over time)

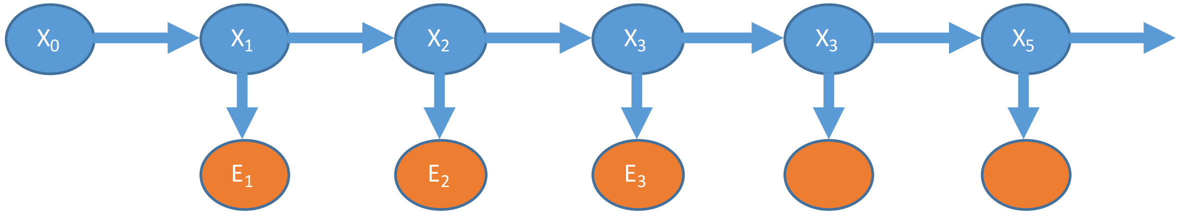
- Natural sequential data (speech, text, etc.).

- Example - text tagging:

the dog saw a cat

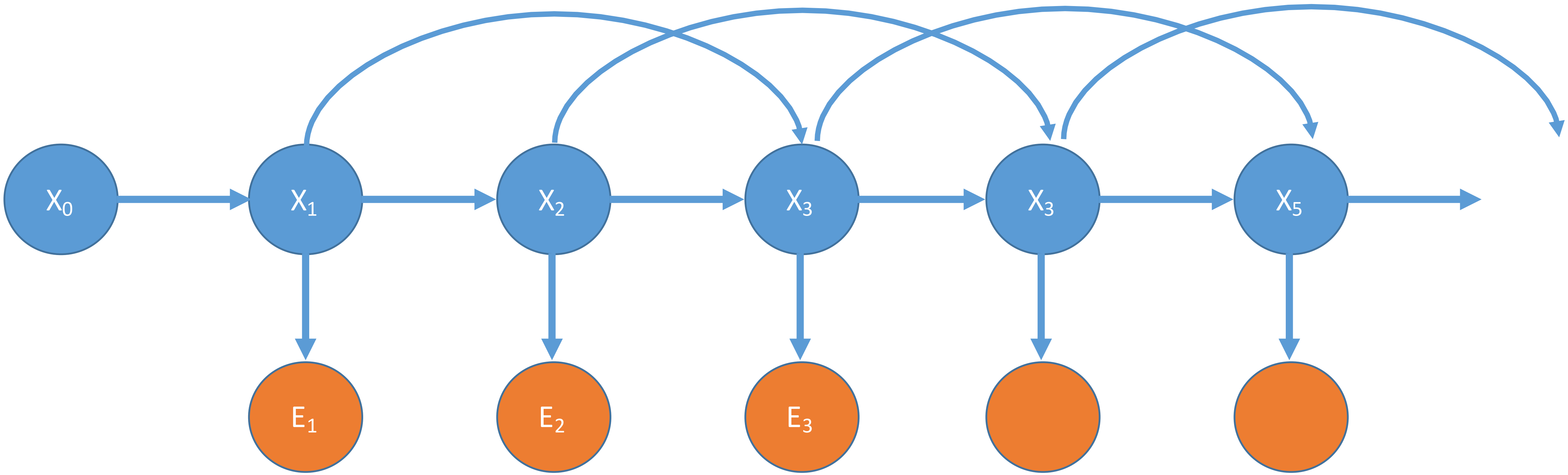
D N V D N

Hidden Markov Models: definitions



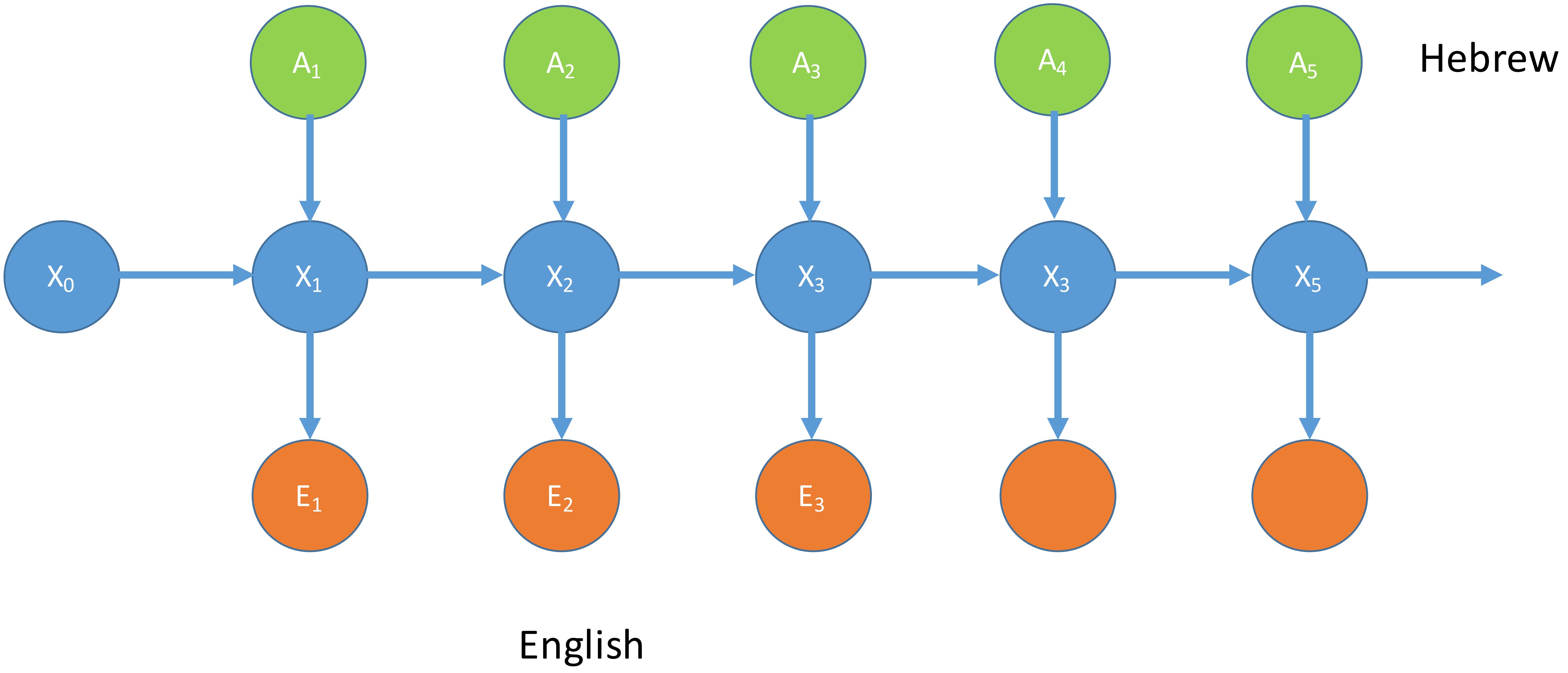
- X_t = state at time t
- E_t = evidence at time t
- $P(X_0)$ = initial state
- $P(X_t|X_{t-1})$ – transition model = Markov chain
- $P(E_t|X_t)$ – sensor/observation model, random
- Assumptions:
  - Future is independent of past given present (1st order)
    $P(X_t|X_{0:t-1}) = P(X_t|X_{t-1})$
  - Current evidence only depends on current state
    $P(E_t|X_{0:t}, E_{1:t-1}) = P(E_t|X_t)$

Hidden Markov Models – 2^{nd} order dependencies natural extension



$$P(X_t | X_{0:t-1}) = P(X_t | X_{t-1}, X_{t-2})$$

Hidden Markov Models – translation



HMMs – questions we want to solve

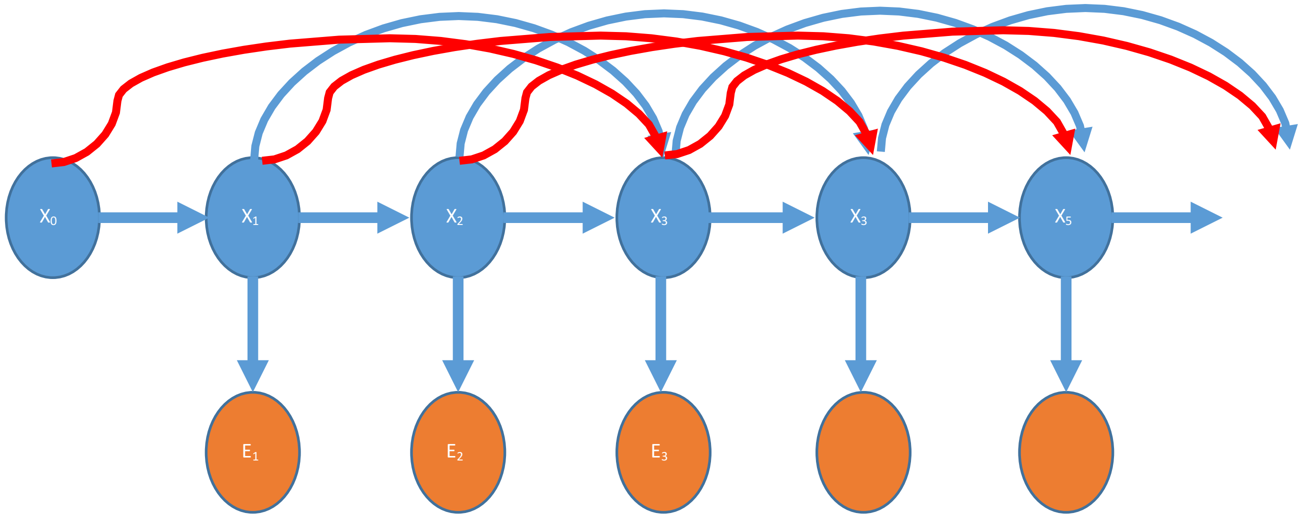
1. Filtering: what's the current state?
   $P(X_t|E_t) = ?$

2. Prediction: where will I be in k steps?
   $P(X_{t+k}|E_{1:t}) = ?$

3. Smoothing: where was I in the past?
   $P(X_k|E_{1:t}) = ?$

4. Most likely sequence to the data
   $\arg\max_{X_{0:t}} P(X_{0:t}|E_{1:t}) = ?$

Example – word tagging by trigram HMM



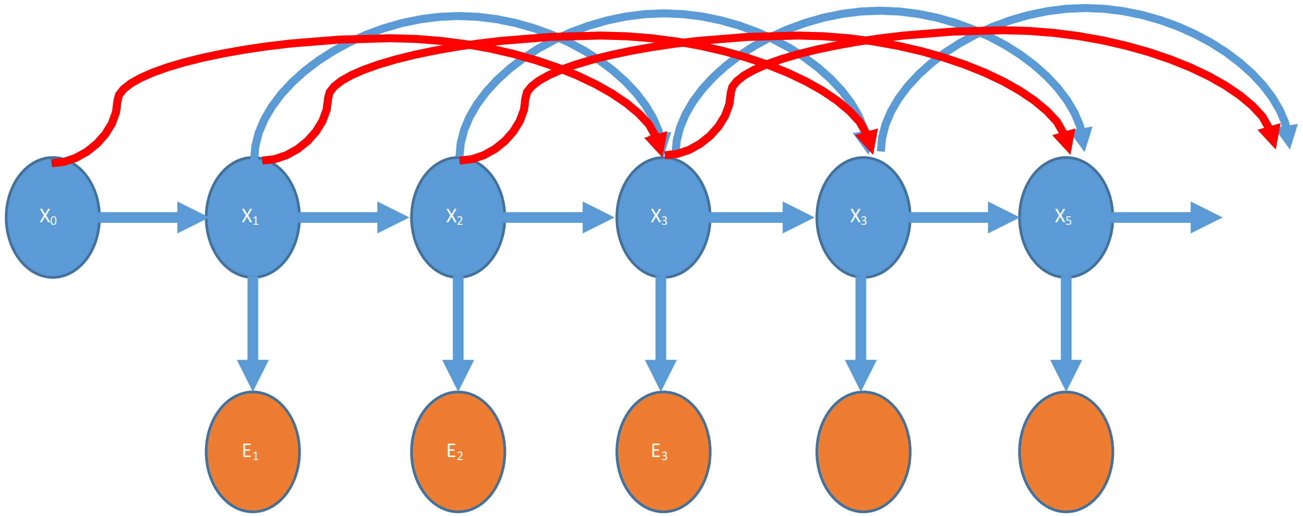
the dog saw a cat

D N V D N

- Let K = {V,N,D,Adv,...,*,STOP} be a set of labels. These are going to be our *states*
- V = dictionary words, these are the *observations*
- Model = HMM with 3 arcs back. Trigram assumption:

$$\mathrm{P}(\mathrm{X_t}|\mathrm{X_{0:t-1}}) = \mathrm{P}(\mathrm{X_t}|\mathrm{X_{t-1}},\mathrm{X_{t-2}}), \qquad P(E_t|X_{0:t},E_{1:t-1}) = P(E_t|X_t)$$

Example – word tagging by trigram HMM



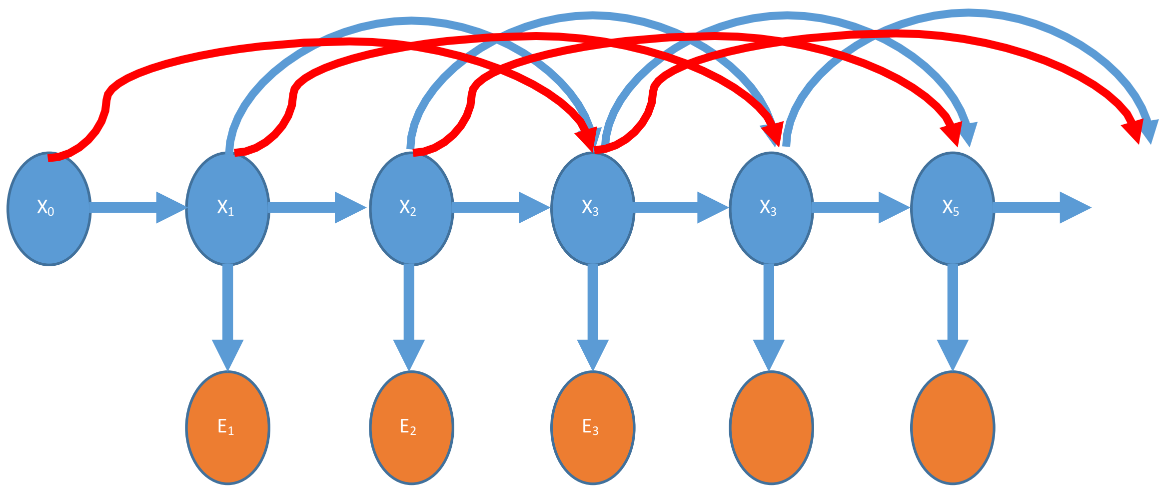
the dog saw a cat

D N V D N

- We will see,

$$P(\text{the dog laughs}, \text{D N V STOP}) =$$
$$P(D|*,*)\times P(N|*,D)\times P(V|D\ N)\times P(STOP|N,V)\times$$
$$\times P(the\ |D)\times P(dog\ |N)\times P(laughs|V)$$

Example – word tagging by trigram HMM



- Assume we know transition probabilities $P(X_t|X_{t-1}, X_{t-2})$
- Assume we know observation frequencies $P(E_t|X_t)$
- (how can we estimate these from labelled data?)

Decoding HMMs

Input: sentence $E_1, E_2, \ldots, E_t$
Output: tagging according to labels in K (N,V,…), i.e. the states $X_1, \ldots, X_t$

i.e. $\arg\max_{x_{0:t}} P(X_{0:t} = x_{0:t} | E_{1:t} = e_{1:t})$

$= \arg\max_{x_{0:t}} P(X_{0:t} = x_{0:t}, E_{1:t} = e_{1:t}) \times \frac{1}{P(E_{1:t} = e_{1:t})}$

By the trigram Markov assumption, we have:

$$P(x_{0:t}, e_{1:t}) = \prod_{i=1 \; to \; t} P(x_i | x_{i-1}, x_{i-2}) \prod_{i=1 \; to \; t} P(e_i | x_i)$$

Why?

Decoding HMMs

$P(x_{0:t}, e_{1:t}) =$

$= P(x_{1:t}) \times P(e_{0:t}|x_{1:t})$ (complete probability)

$= \prod_{i=1\ to\ t} P(x_i|x_{1:i-1}) \times \prod_{i=1\ to\ t} P(e_i|x_{1:i-1}, e_{1:t})$ (chain rule)

$= \prod_{i=1\ to\ t} P(x_i|x_{i-1}, x_{i-2}) \times \prod_{i=1\ to\ t} P(e_i|x_{1:i-1}, e_{1:t})$ (2nd order MC)

$= \prod_{i=1\ to\ t} P(x_i|x_{i-1}, x_{i-2}) \times \prod_{i=1\ to\ t} P(e_i|x_i)$ (cond. independence)

Decoding HMMs – Viterbi algorithm

Let

$$f(X_{0:k}) = \prod_{i=1\ to\ k} P(X_i|X_{i-1},X_{i-2}) \prod_{i=1\ to\ k} P(e_i|X_i)$$

And define

$$\pi_k(u,v) = \max_{X_{0:k-2}} f(X_{0:k-2},u,v)$$

Recall: we want to compute:

$$\arg\max_{x_{0:t}} P(x_{0:t},e_{1:t}) = \arg\max f(x_{0:t})$$

Decoding HMMs – Viterbi algorithm

Let

$$f(X_{0:k}) = \prod_{i=1\ to\ k} P(X_i | X_{i-1}, X_{i-2}) \prod_{i=1\ to\ k} P(e_i | X_i)$$

And define

$$\pi_k(u, v) = \max_{X_{0:k-2}} f(X_{0:k-2}, u, v)$$

Main lemma:

$$\pi_k(u, v) = \max_{w} \{ \pi_{k-1}(w, u) \times P(v | w, u) \times P(e_k | v) \}$$

Now the algorithm is straightforward: compute this recursively! (a.k.a. dynamic programming)

Viterbi: explicit pseudo code

Input: observations $e_1,\ldots,e_t$

Output: most likely variable assignments $x_0,\ldots,x_t$

Initialize: set x_0,x_{-1} to be "*"

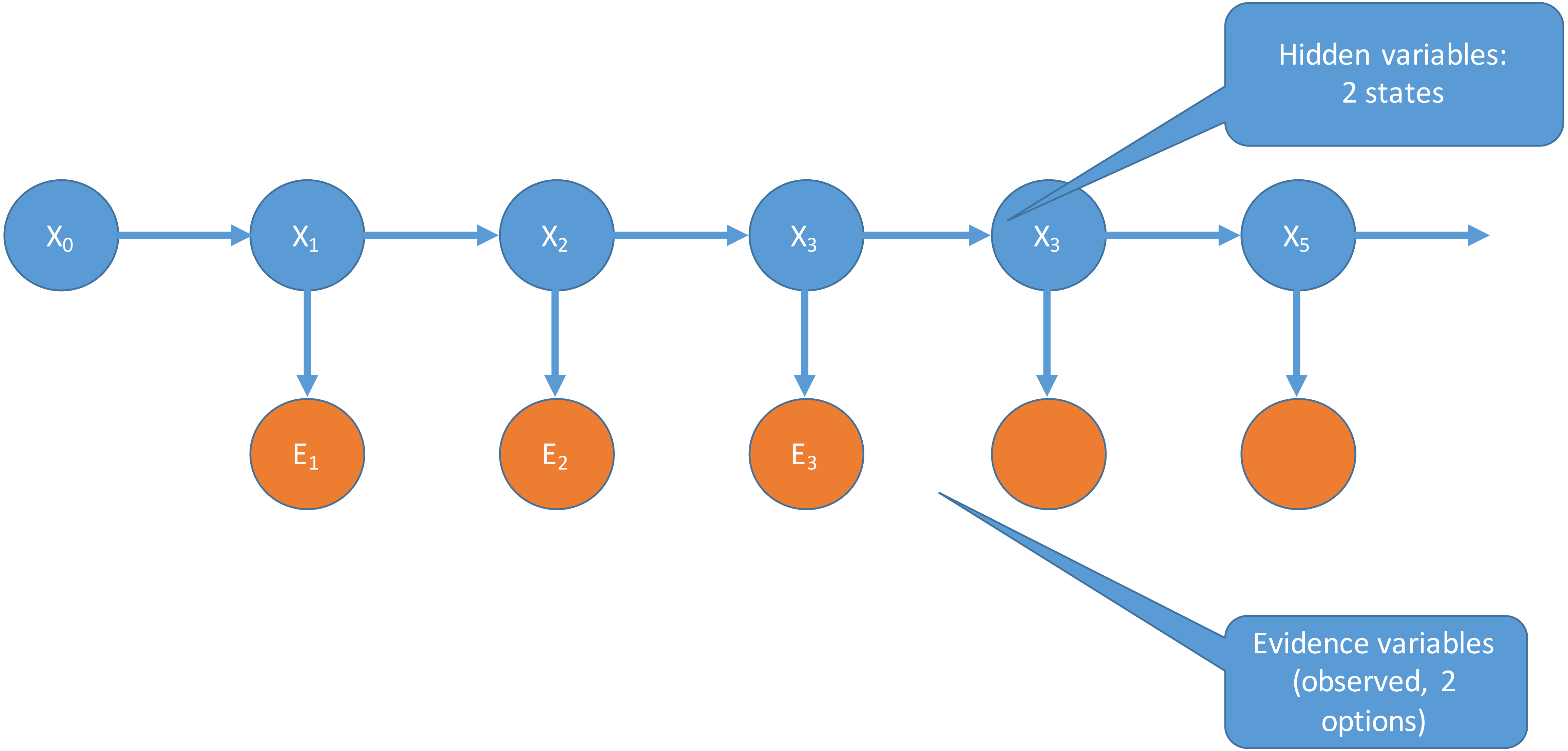
For k=1,2,…,t do:

- For u,v in K do:
  1. $\pi_k(u,v) = \max_w \{ \pi_{k-1}(w,u) \times P(v|w,u) \times P(e_k|v) \}$
  2. Save the $\pi_k(u,v)$ value and the assignments which meets it
- end

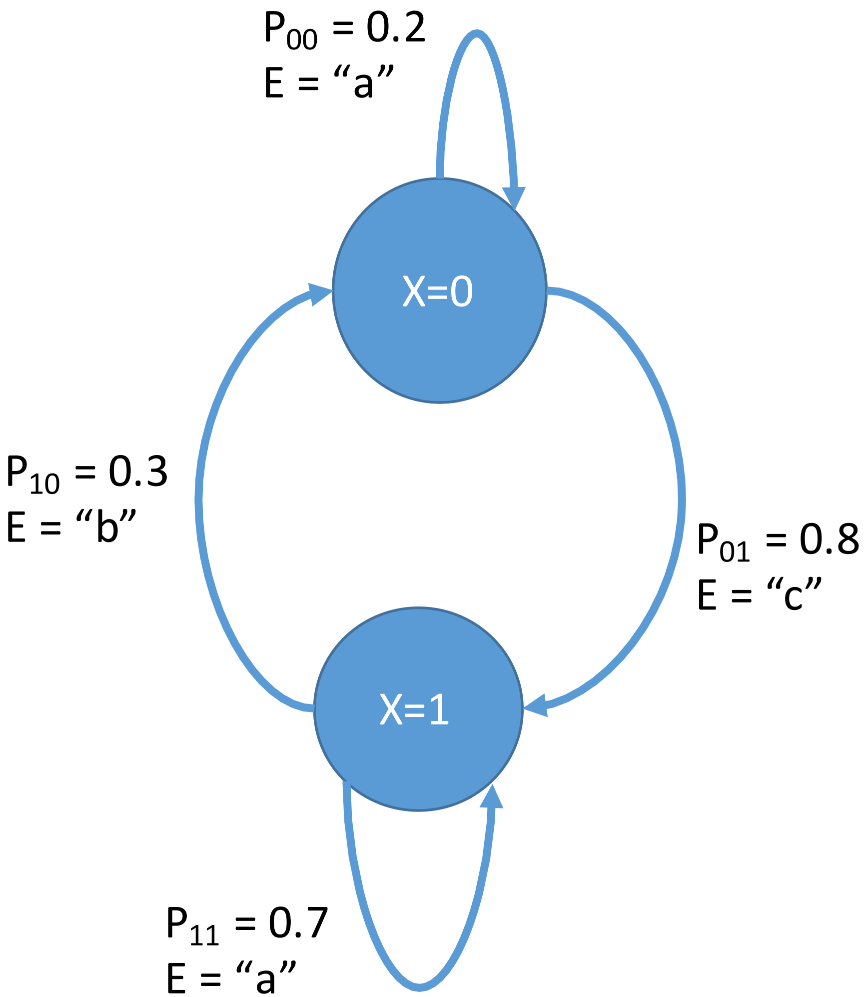
Return $\max_{u,v} \{ \pi_t(u,v) \times P(STOP\ |u,v) \}$ and assignments which meets it

Computational complexity?

Hidden Markov Models – another view



Hidden Markov Models – another view



Markov chain with:

1. Transition probabilities that govern state change
2. Distribution over signals/observations from each state

Transition matrix:

0.2	**0.8**
0.3	0.7

Observation matrices:

P(“a” | x_t)

0.2	**0**
0	0.7

P(“b” | x_t)

0	**0**
0	0.3

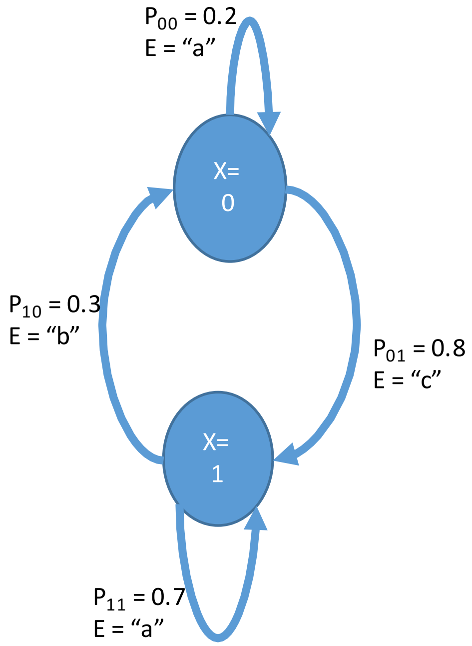
P(“c” | x_t)

0.8	**0**
0	0

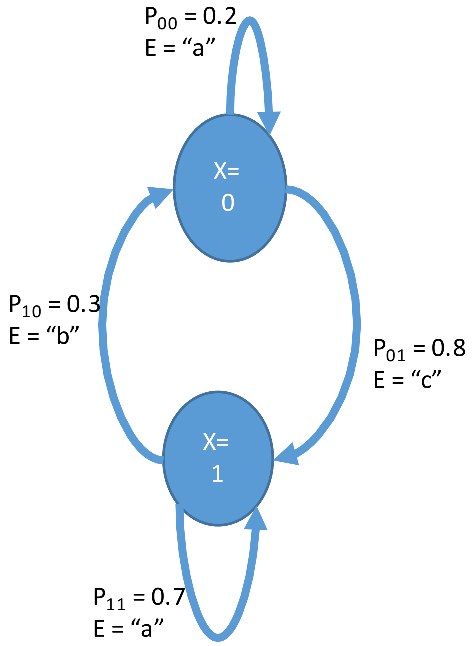
“forward algorithm”

To compute $P(X_{t+1}|e_{1:t+1})$, recursive formula (similar to what we did)

$$P(X_{t+1}|e_{1:t+1}) = \alpha\, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t)\, P(x_t|e_{1:t})$$



“forward algorithm”



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$$P(X_{t+1}|e_{1:t+1}) = \alpha\, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t)\, P(x_t|e_{1:t})$$

Derivation

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}|e_{1:t}, e_{t+1})$$

$$= \frac{1}{P(e_{t+1}|e_{1:t})}\, P(e_{t+1}|X_{t+1}, e_{1:t}) P(X_{t+1}|e_{1:t}) \qquad \text{(Bayes)}$$

$$= \alpha\;\; P(e_{t+1}|X_{t+1}) P(X_{t+1}|e_{1:t}) \qquad \text{(Markov assumption)}$$

$$= \alpha\;\; P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t)\, P(x_t|e_{1:t})$$

"forward algorithm"

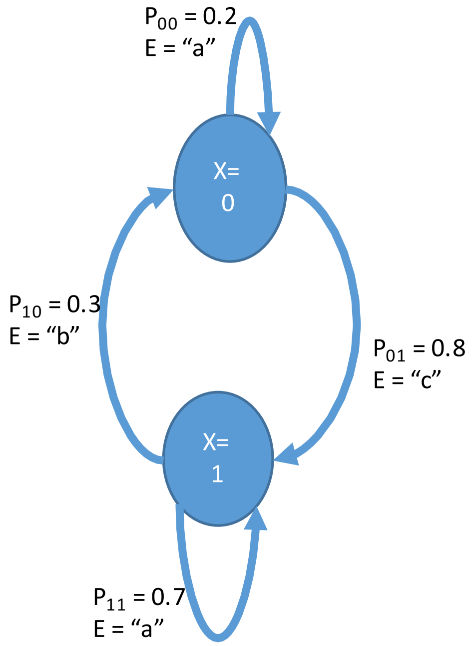
To compute $P(X_{t+1}|e_{1:t+1})$, recursive formula (similar to what we did)

$$P(X_{t+1}|e_{1:t+1}) = \alpha\, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t)\, P(x_t|e_{1:t})$$

Or in matrix form, if f_t is the vector of $\mathrm{f_t(x)} = P(X_t = x, e_{1:t})$:

$$f_{t+1} = \alpha\, O_{t+1} T^{\top} f_t$$

O_t - observation matrix corresponding to E_t.
α - normalizing constant to 1 (equal to $\frac{1}{P(e_{1:t})}$).



0.2	**0.8**
0.3	0.7

P("a" | x_t)

0.2	**0**
0	0.7

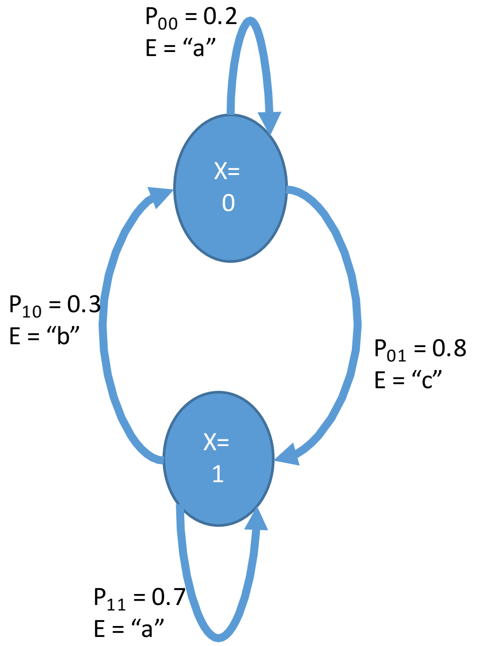
P("b" | x_t)

0	**0**
0	0.3

P("c" | x_t)

0.8	**0**
0	0

“backward algorithm”



Let b_t is the vector of $b_{k:t}(x) = P(e_{k:t}, X_{k-1})$:

$$b_{k+1:t} = T\, O_{k+1} b_{k+2:t}$$

O_t - observation matrix corresponding to E_t.

0.2	0.8
0.3	0.7

P(“a” | x_t)

0.2	0
0	0.7

P(“b” | x_t)

0	0
0	0.3

P(“c” | x_t)

0.8	0
0	0

Summary

- HMMs - useful to model time-dependent variables / problems (e.g. treating patients with changing biometrics over time)

- Example - text tagging

- Viterbi algorithm (dynamic programming) to find the most likely assignment to the hidden variables.
  (assuming the transition probabilities are known)

- Independence assumptions allow “forward” + “backward” computations of conditional probabilities