Managing Multiple Mutable Data Structures

COS 326
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Last Time

We explored two programming disciplines that help us manage parallelism and concurrency:

- Futures:
  - future : ('a -> 'b) -> 'a -> 'b future
  - force : 'a future -> 'a
  - create a future to run a function in the background
  - useful in divide-and-conquer parallel programming
- Mutexes:
  - with_lock : mutex -> (unit -> 'b) -> 'b
  - associate each mutable data structure with a lock m
  - protect all accesses to a mutable data structure with with_lock m

This Time: Sometimes a computation depends upon several mutable data structures.

- eg: to transfer a balance from one bank account to another
- our existing techniques break down

Another Example

```
type 'a stack = { mutable contents : 'a list;
                  lock : Mutex.t
                };;

let empty () = {contents=[]; lock=Mutex.create()};;

let push (s:'a stack) (x:'a) : unit =
    with_lock s.lock (fun _ ->
      s.contents <- x::s.contents)
;;

let pop (s:'a stack) : 'a option =
    with_lock s.lock (fun _ ->
      match s.contents with
      | [] -> None
      | h::t -> (s.contents <- t ; Some h))
;;
```

Another Example

```
type 'a stack = { mutable contents : 'a list;
                  lock : Mutex.t }

val empty : () -> 'a stack
val push  : 'a stack -> a -> unit
val pop   : 'a stack -> 'a option

let transfer_one (s1:'a stack) (s2: 'a stack) =
  with_lock s1.lock (fun _ ->
    match pop s1 with
   | None -> ()
   | Some x -> push s2 x)
```

Another Example

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val push  : 'a stack -> a -> unit
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Unfortunately, we already hold `s1.lock` when we invoke `pop s1` which tries to acquire the lock.

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```

Unfortunately, we already hold `s1.lock` when we invoke `pop s1` which tries to acquire the lock.

So we end up *dead-locked*.

Another Example

```
type 'a stack = { mutable contents : 'a list;
                  lock : Mutex.t }

val empty : () -> 'a stack
val push  : 'a stack -> a -> unit
val pop   : 'a stack -> 'a option

let transfer_one (s1:'a stack) (s2: 'a stack) =
  with_lock s1.lock (fun _ ->
    match pop s1 with
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    | Some x -> push s2 x)
```

Avoid deadlock by deleting the line that aquires s1.lock initially

A trickier problem

```
type 'a stack = { mutable contents : 'a list;
                  lock : Mutex.t }

val empty : () -> 'a stack
val push  : 'a stack -> a -> u
val pop   : 'a stack ->

let pop_two (s1:'a stack)
            (s2:'a stack) : ('a * 'a) option =
  match pop s1, pop s2 with
     | Some x, Some y -> Some (x,y)
     | Some x, None -> push s1 x ; None
     | None, Some y -> push s2 y ; None
```

Either:

(1) pop one from each if both nonempty, or

(2) have no effect at all

A trickier problem

```
type 'a stack = { mutable contents : 'a list;
                  lock : Mutex.t }

val empty : () -> 'a stack
val push  : 'a stack -> a -> unit
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  match pop s1, pop s2 with
     | Some x, Some y -> Some (x,y)
     | Some x, None -> push s1 x ; None
     | None, Some y -> push s2 y ; None
```

But some other thread could sneak in here and try to perform an operation on our contents before we've managed to push the value back on.

Yet another broken solution

```
let no_lock_pop (s1:'a stack) : 'a option =
  match s1.contents with
  | [] -> None
  | h::t -> (s1.contents <- t ; Some h)

let no_lock_push (s1:'a stack) (x :'a) : unit =
  contents <- x::contents

let pop_two (s1:'a stack)
            (s2:'a stack) : ('a * 'a) option =
  with_lock s1.lock (fun _ ->
  with_lock s2.lock (fun _ ->
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     | None, Some y -> no_lock_push s2 y ; None))
```

Yet another broken solution

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```

Problems?

Yet another broken solution

```
let no_lock_pop (s1:'a stack) : 'a option =
  match s1.contents with
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  contents <- x::contents

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```

What happens if we call pop_two x x?

Yet another broken solution

```
let no_lock_pop (s1:
  match s1.contents w
  | [] -> None
  | h::t -> (s1.contents <- t ;

let no_lock_push (s1:'a stack)
  contents <- x::contents


let pop_two (s1:'a stack)
            (s2:'a stack) : ('a * 'a) option =
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     | Some x, None -> no_lock_push s1 x ; None
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```

In particular, consider:

```
Thread.create (fun _ -> pop_two x y)
Thread.create (fun _ -> pop_two y x)
```

What happens if two threads are trying to call pop_two at the same time?

Yet another broken solution

```
let no_lock_pop (s1:
  match s1.contents w
  | [] -> None
  | h::t -> (s1.contents <- t ; So    h)

let no_lock_push (s1:'a stack)
  contents <- x::contents

let pop_two (s1:'a stack)
            (s2:'a stack) : (
  with_lock
  with_lock
  match no_                            with
     | Some x, Some y -> Some (x,y)
     | Some x, None -> no_lock_push s1 x ; None
     | None, Some y -> no_lock_push s2 y ; None))
```

More general problem:

```
Thread.create (fun _ -> pop_two x y)
Thread.create (fun _ -> pop_two y x)
```

One possible interleaving:
T1 acquires x's lock.
T2 acquires y's lock.
T1 tries to acquire y's lock and blocks.
T2 tries to acquire x's lock and blocks.

DEADLOCK

A fix

```
type 'a stack = { mutable contents : 'a list; lock : Mutex.t; id : int }

let new_id : unit -> int =
  let c = ref 0 in (fun _ -> c := (!c) + 1 ; !c)

let empty () = {contents=[]; lock=Mutex.create(); id=new_id()};;

let no_lock_pop_two (s1:'a stack) (s2:'a stack) : ('a * 'a) option =
      match no_lock_pop s1, no_lock_pop s2 with
       | Some x, Some y -> Some (x,y)
       | Some x, None -> no_lock_push s1 x; None
       | None, Some y -> no_lock_push s2 y; None

let pop_two (s1:'a stack) (s2:'a stack) : ('a * 'a) option =
  if s1.id < s2.id then
    with_lock s1.lock (fun _ ->
    with_lock s2.lock (fun _ ->
      no_lock_pop_two s1 s2))
  else if s1.id > s2.id then
    with_lock s2.lock (fun _ ->
    with_lock s1.lock (fun _ ->
      no_lock_pop_two s1 s2))
  else with_lock s1.lock (fun _ -> no_lock_pop_two s1 s2)
```

sigh ...

```
type 'a stack = { mutable contents : 'a list; lock : Mutex.t; id : int }

let new_id : unit -> int =
  let c = ref 0 in let l = Mutex.create() in
 (fun _ -> with_lock l (fun _ -> (c := (!c) + 1 ; !c)))

let empty () = {contents=[]; lock=Mutex.create(); id=new_id()};;

let no_lock_pop_two (s1:'a stack) (s2:'a stack) : ('a * 'a) option =
      match no_lock_pop s1, no_lock_pop s2 with
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       | Some x, None -> no_lock_push s1 x; None
       | None, Some y -> no_lock_push s2 y; None

let pop_two (s1:'a stack) (s2:'a stack) : ('a * 'a) option =
  …
;;
```

Refined Design Pattern

- *Associate a lock with each shared, mutable object.*
- *Choose some ordering on shared mutable objects.*
  - doesn't matter what the order is, as long as it is total.
  - in C/C++, often use the address of the object as a unique number.
  - Our solution: *add a unique ID number to each object*
- *To perform actions on a set of objects S atomically:*
  - acquire the locks for the objects in S *in order*.
  - perform the actions.
  - release the locks.

Refined Design Pattern

- *Associate a lock with each shared, mutable object.*
- *Choose some ordering on shared mutable objects.*
  - doesn't matter what the order
  - in C/C++, often use the address … number.
  - Our solution: *add a unique ID …*
- *To perform actions on a set of ob…*
  - acquire the locks for the objects in S *in order*.
  - perform the actions.
  - release the locks.

Important!

Acquire all the locks you will need
BEFORE
performing any irreversible actions!

BUT: IN A BIG PROGRAM, IT IS REALLY HARD TO GET THIS RIGHT

A HUGE COMPONENT OF PL RESEARCH INVOLVES TRYING TO FIND THE MISTAKES PEOPLE MAKE WHEN DOING THIS. AVOID WHENEVER POSSIBLE! USE FUNCTIONAL ABSTRACTIONS!

SUMMARY

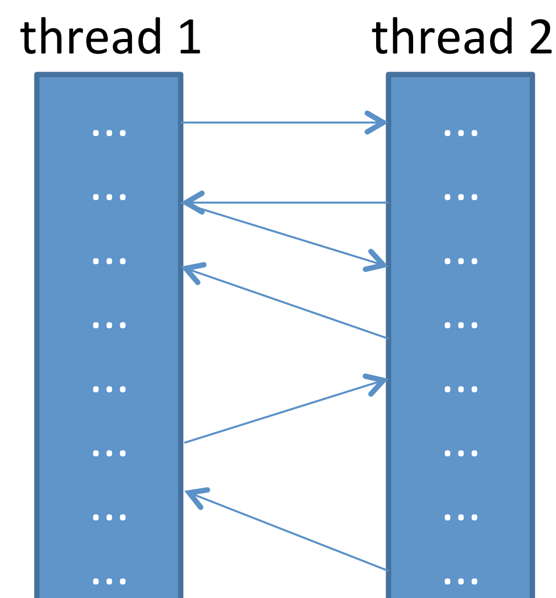
Programming with mutation, threads and locks

Reasoning about the correctness of pure parallel programs that include futures is easy -- no harder than ordinary, sequential programs. (Reasoning about their performance may be harder.)

Reasoning about shared variables
and semaphores is *hard* in general,
but *futures* are a *discipline*
for getting it right.

Much of programming-language design
is the art of finding good disciplines
in which it's harder* to write bad programs.

Even aside from PL design, the same is true of
software engineering with Abstract Data Types:
engineer *disciplines* in your interfaces,
harder for the user to get it wrong.



*but somebody will always find a way…

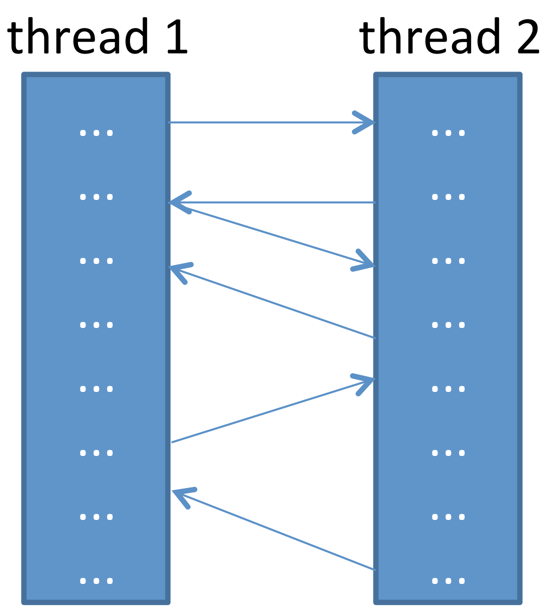
Programming with mutation, threads and locks

Reasoning about the correctness of pure parallel programs that include futures is easy -- no harder than ordinary, sequential programs. (Reasoning about their performance may be harder.)

Reasoning about concurrent programs with *effects* requires considering *all interleavings* of instructions of concurrently executing threads.*

- often too many interleavings for normal humans to keep track of
- nonmodular: you often have to look at the details of each thread to figure out what is going on
- locks cut down interleavings
- but knowing you have done it right still requires deep analysis

**and worse…*



Scheduling
Parallel Computations

Visualizing Computational Costs

```
let x = 1 + 2 in
3 + x
```

Visualizing Computational Costs

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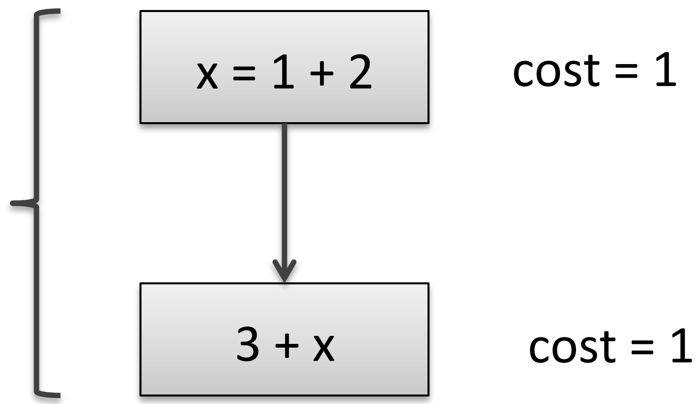
3 + x



Visualizing Computational Costs

let x = 1 + 2 in
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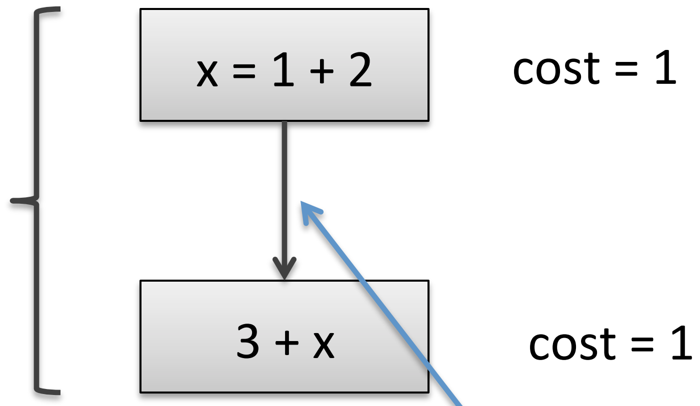
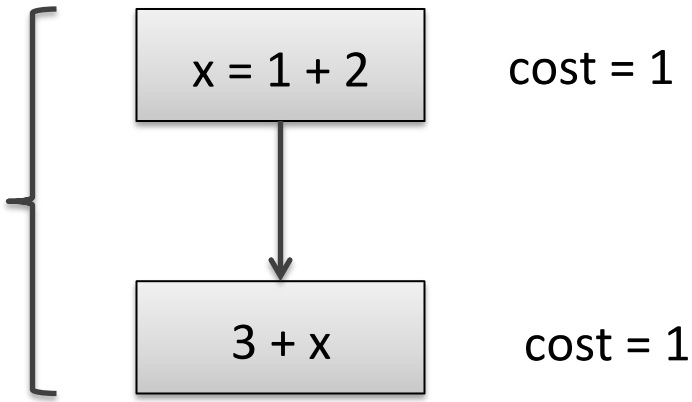


dependence:
x = 1 + 2 *happens before* 3 + x

Visualizing Computational Costs

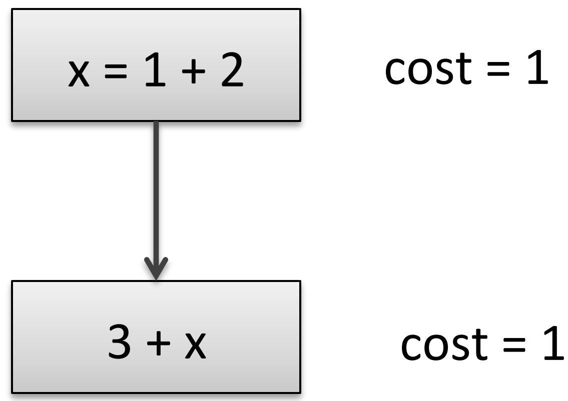
```
let x = 1 + 2 in
3 + x
```



<u>**Execution of dependency diagrams:**</u> A processor can only begin executing the computation associated with a block when the computations of all of its predecessor blocks have been completed.

Visualizing Computational Costs

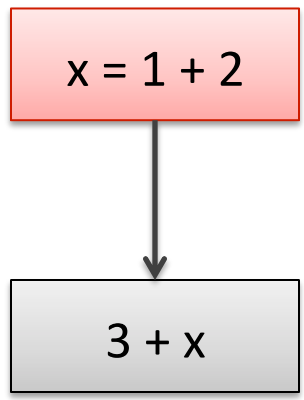
step 1:
execute first block



Cost so far: 0

Visualizing Computational Costs

step 1:
execute first block

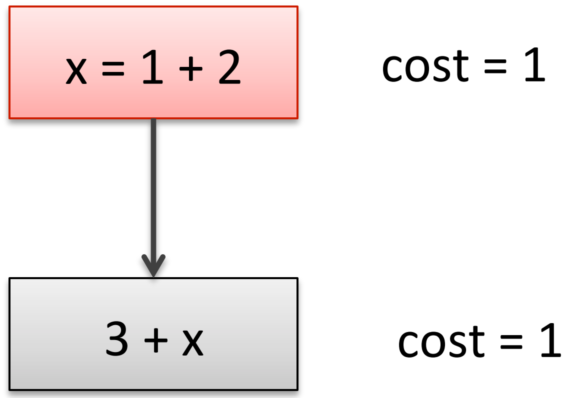


cost = 1

cost = 1

Cost so far: 1

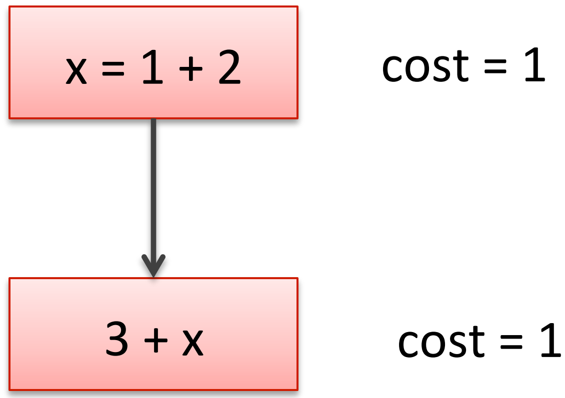
Visualizing Computational Costs



step 2:
execute second block
because all of its
predecessors have
been completed

Cost so far: 1

Visualizing Computational Costs

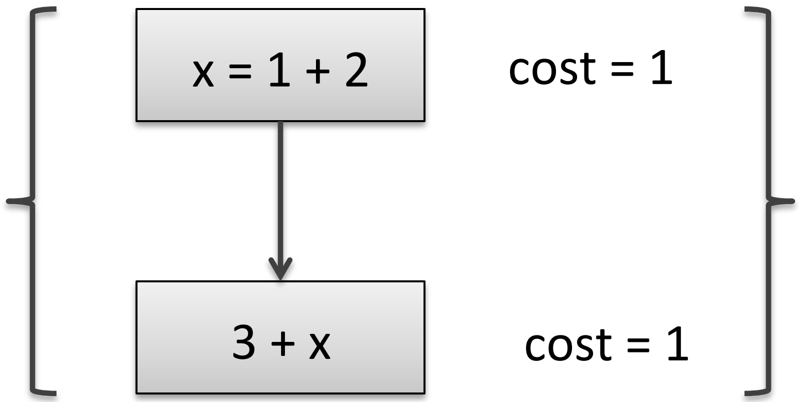


step 2:
execute second block because all of its predecessors have been completed

Cost so far: 1 + 1

Visualizing Computational Costs

let x = 1 + 2 in
3 + x



total cost
= 1 + 1
= 2

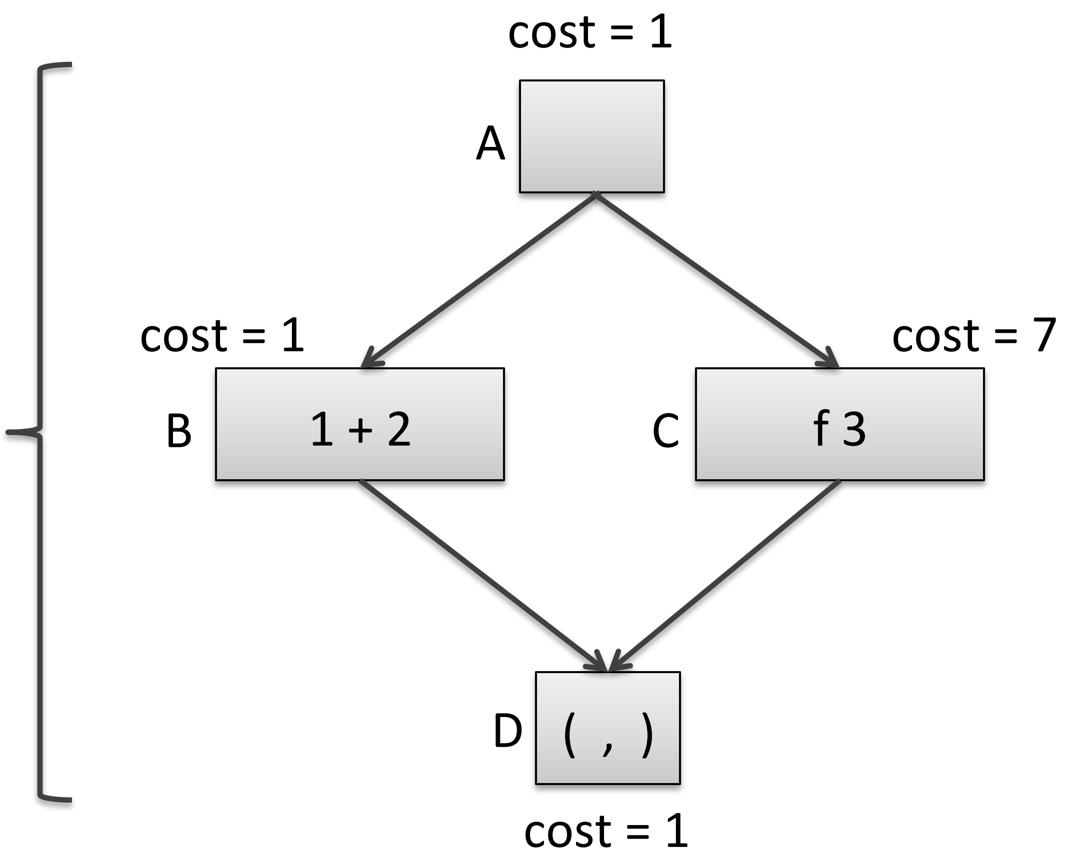
Visualizing Computational Costs

(1 + 2 || f 3)

parallel pair:
compute both left and right-hand sides independently
return pair of values
(easy to implement using futures)

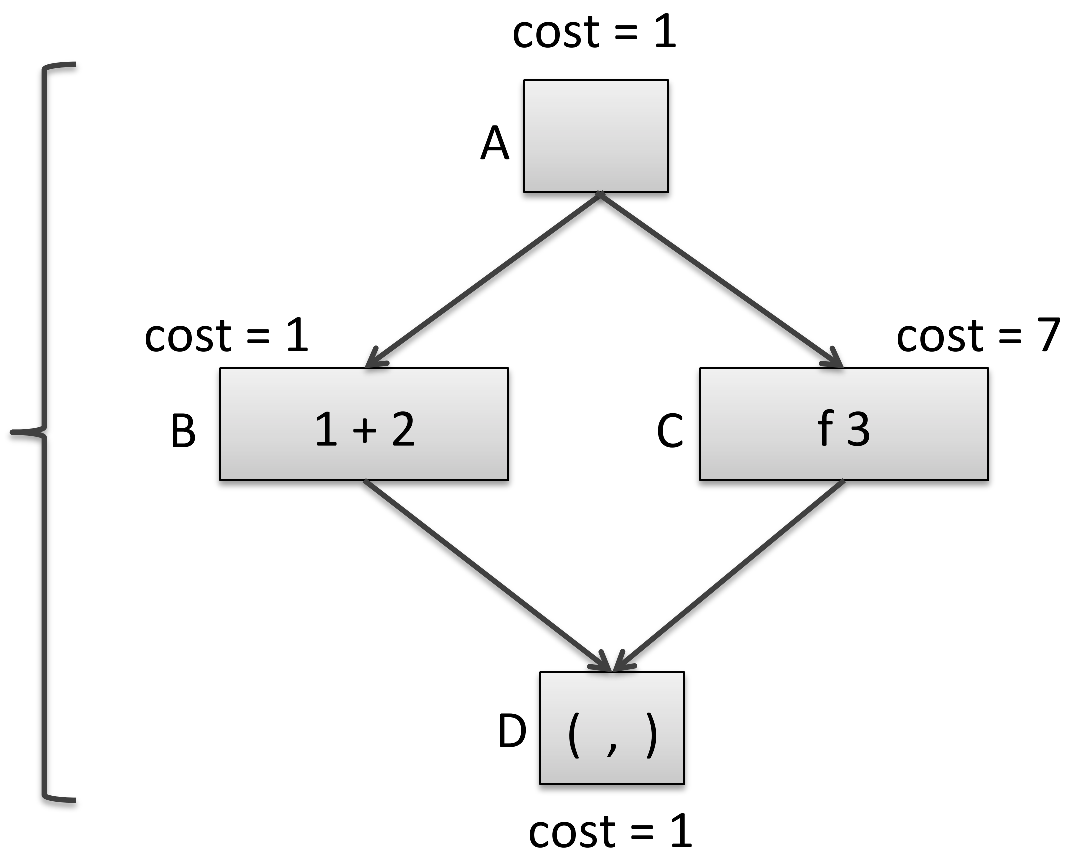
Visualizing Computational Costs

(1 + 2 || f 3)



Visualizing Computational Costs

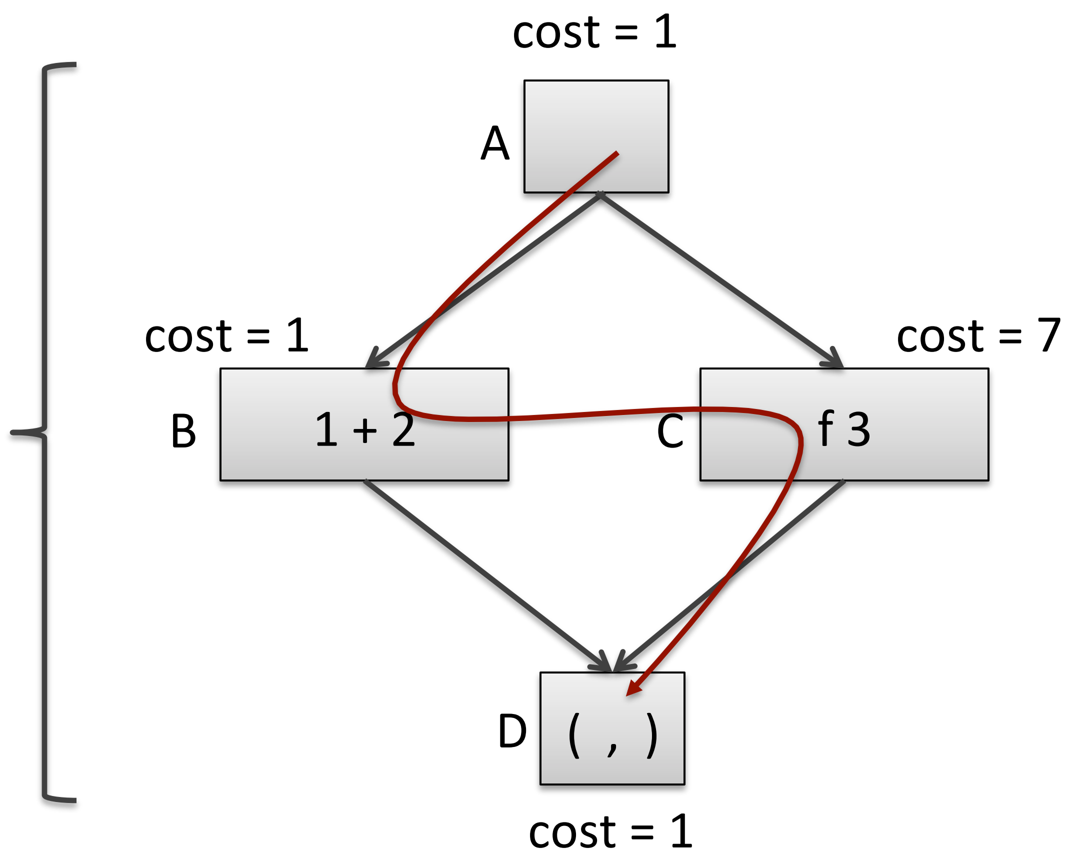
(1 + 2 || f 3)



Suppose we have 1 processor. How much time does this computation take?

Visualizing Computational Costs

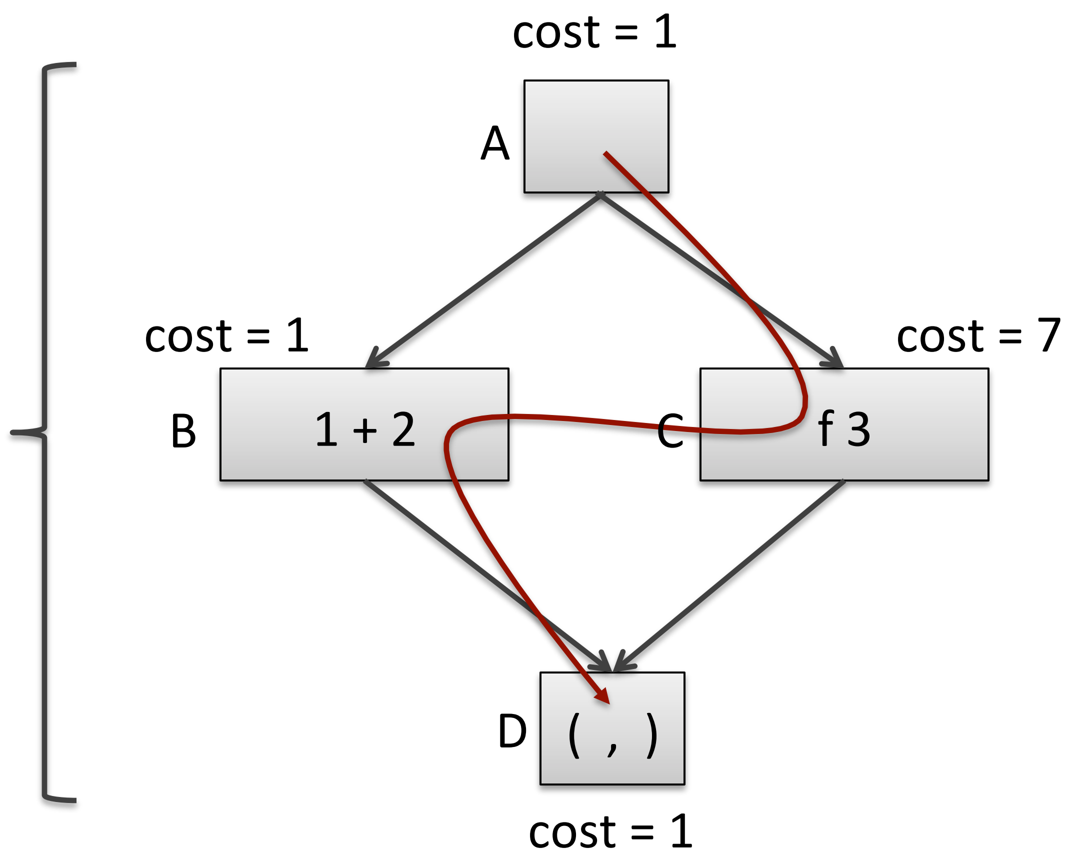
(1 + 2 || f 3)



Suppose we have 1 processor. How much time does this computation take?
Scheduld A-B-C-D: 1 + 1 + 7 + 1

Visualizing Computational Costs

(1 + 2 || f 3)



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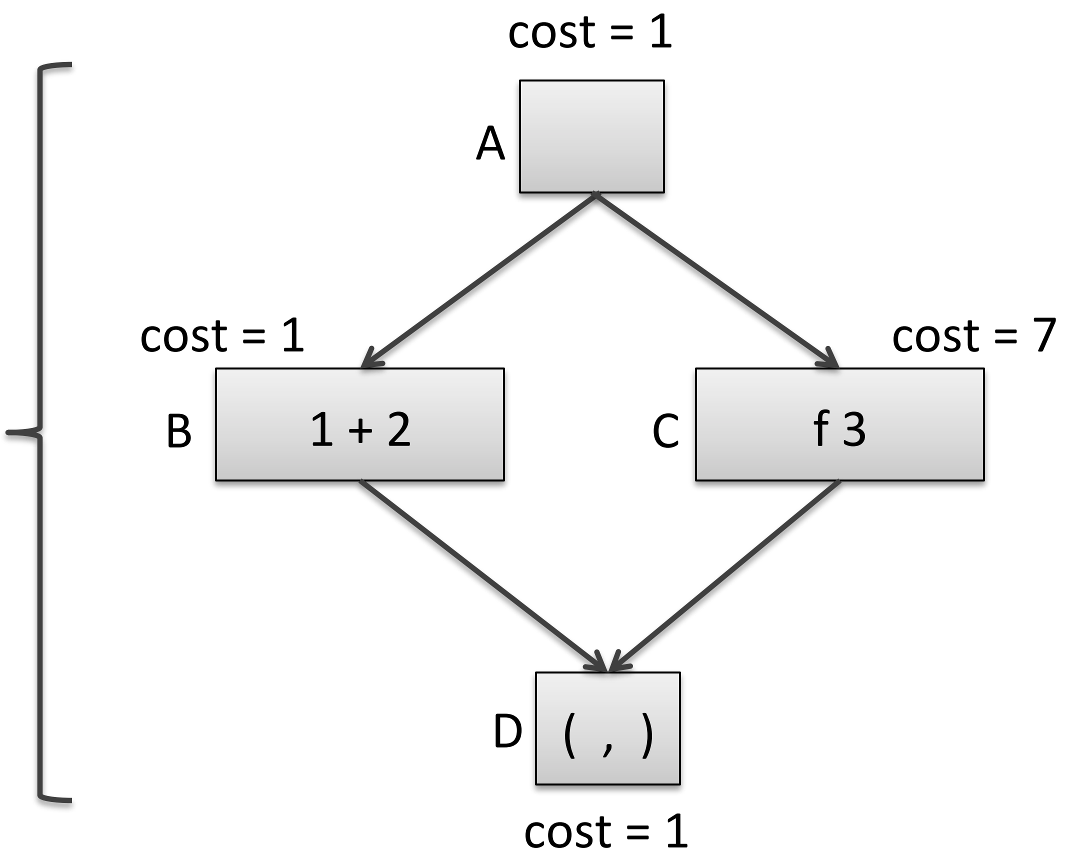
Schedule A-C-B-D: 1 + 1 + 7 + 1

Visualizing Computational Costs

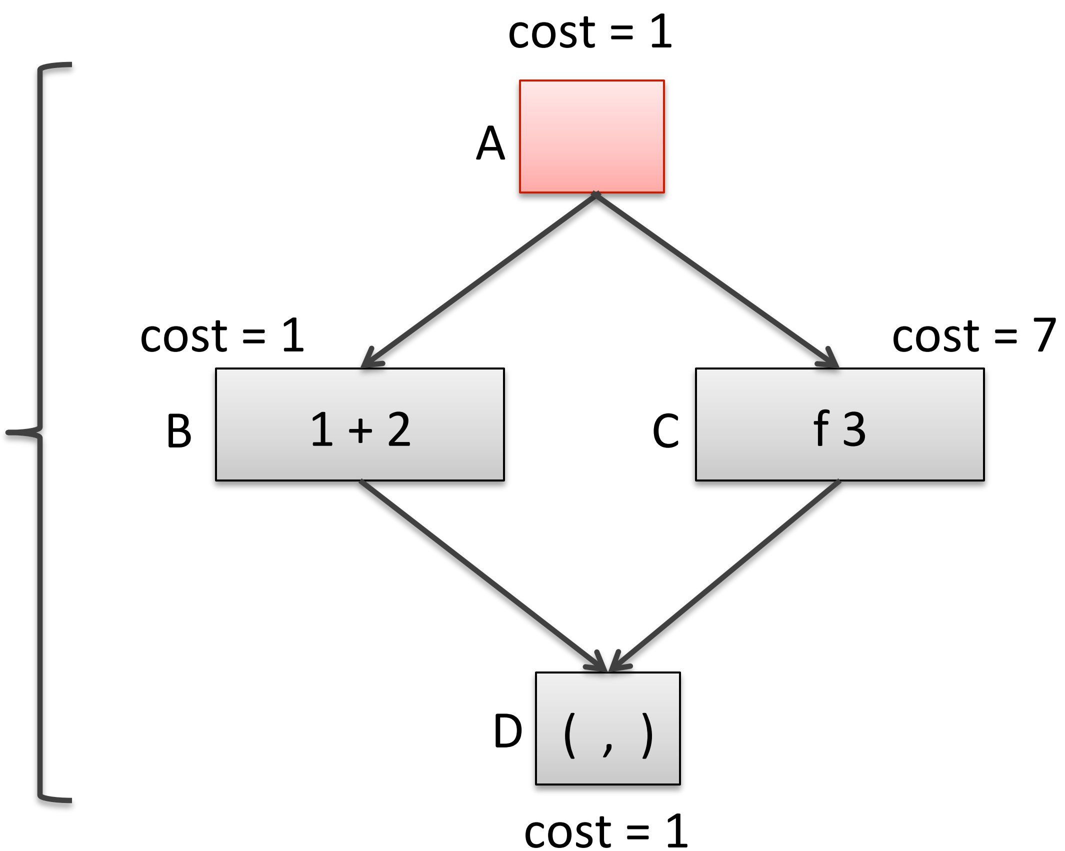
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?

Visualizing Computational Costs

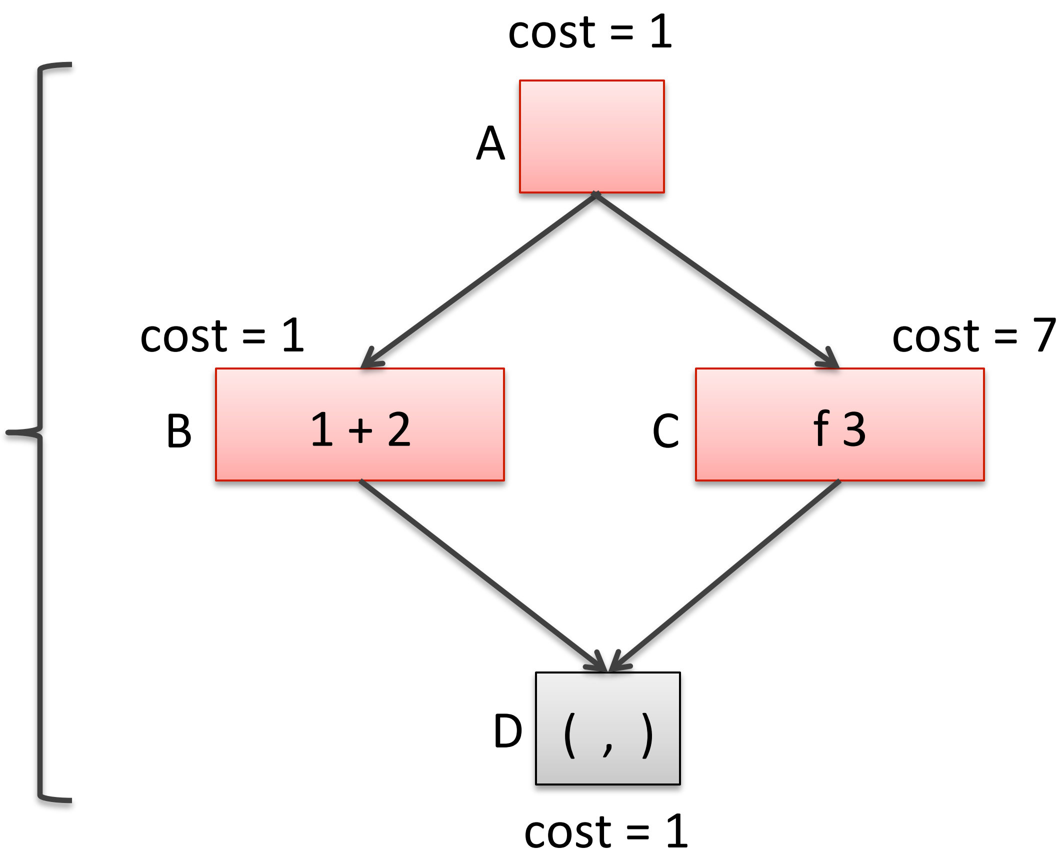
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1

Visualizing Computational Costs

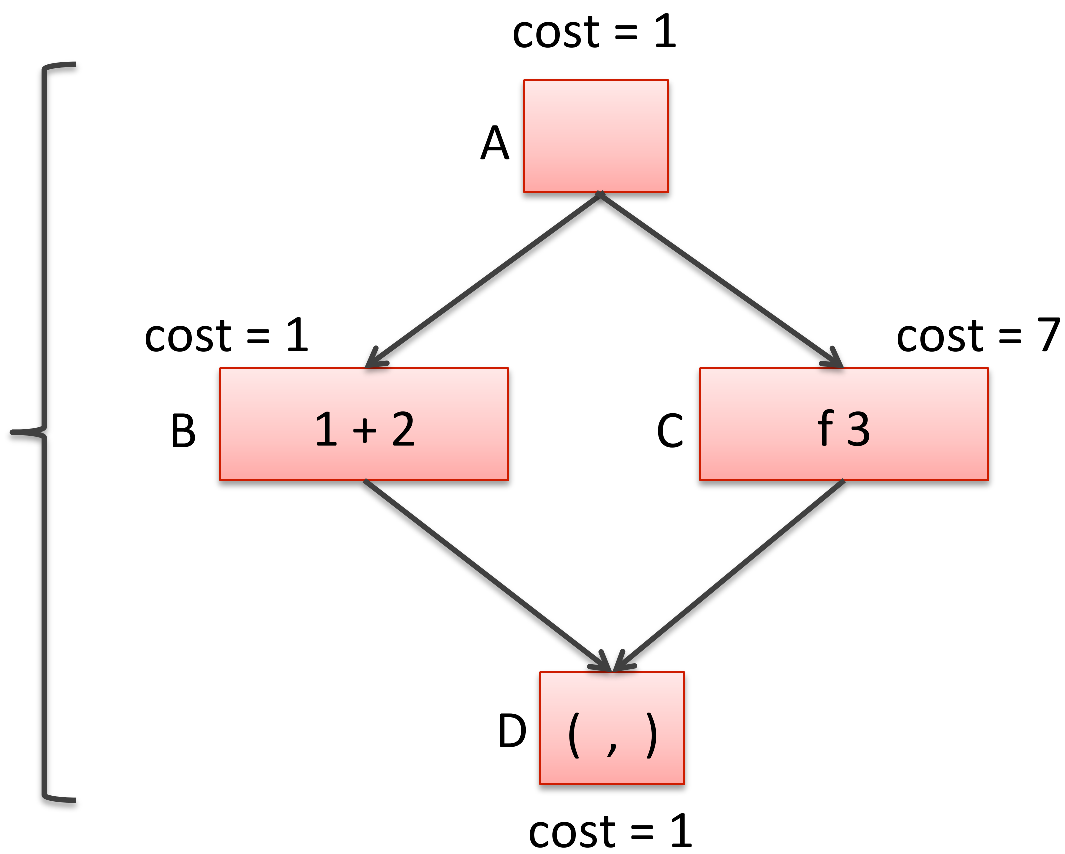
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1 + max(1,7)

Visualizing Computational Costs

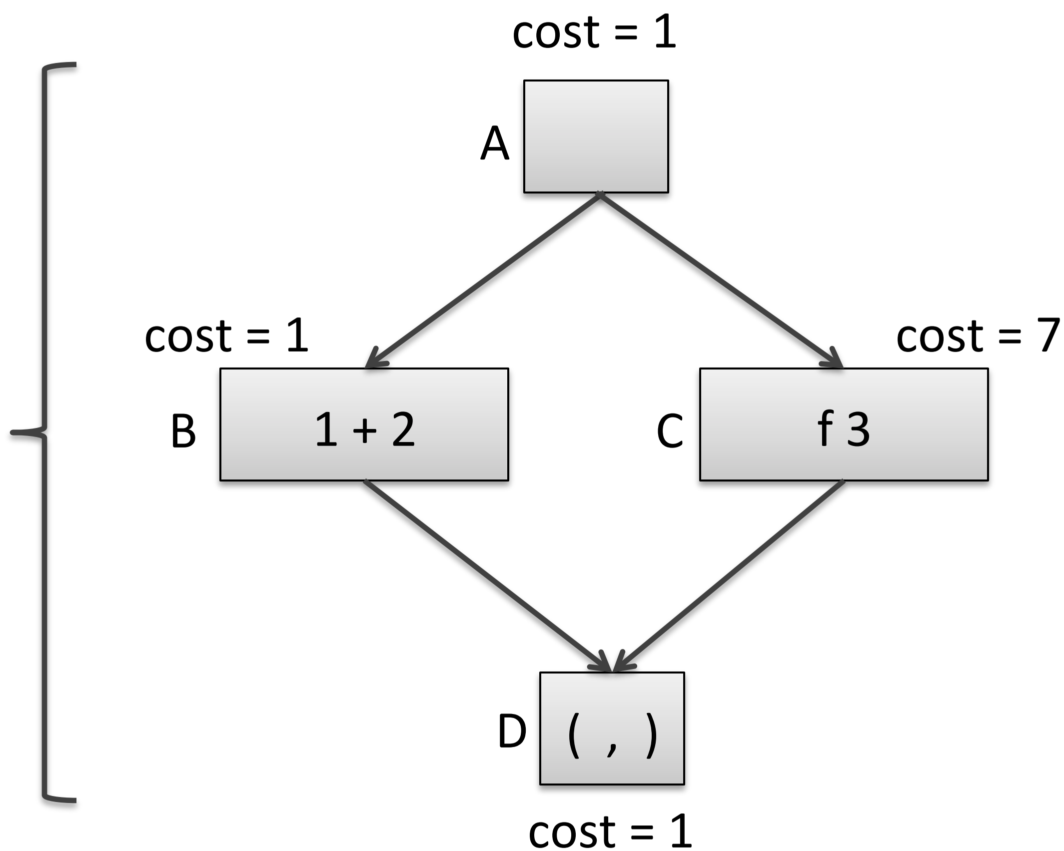
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Cost so far: 1 + max(1,7) + 1

Visualizing Computational Costs

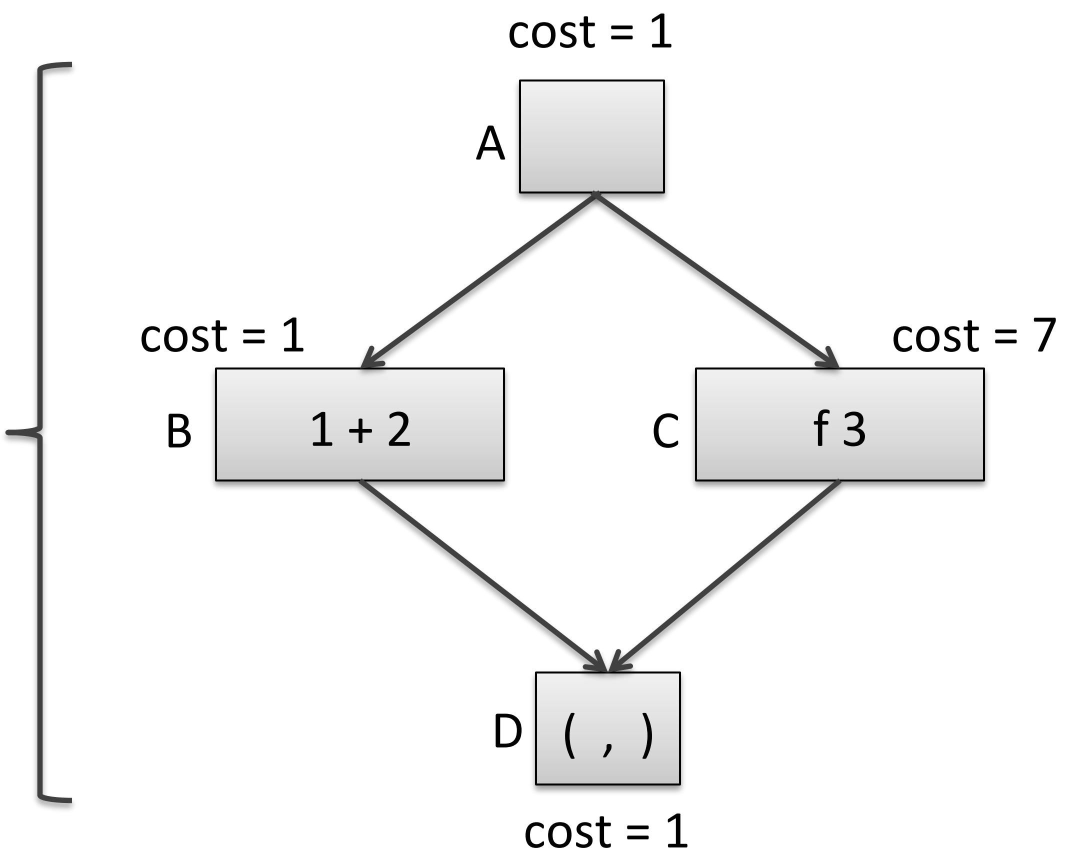
(1 + 2 || f 3)



Suppose we have 2 processors. How much time does this computation take?
Total cost: 1 + max(1,7) + 1. We say the *schedule* we used was: A-CB-D

Visualizing Computational Costs

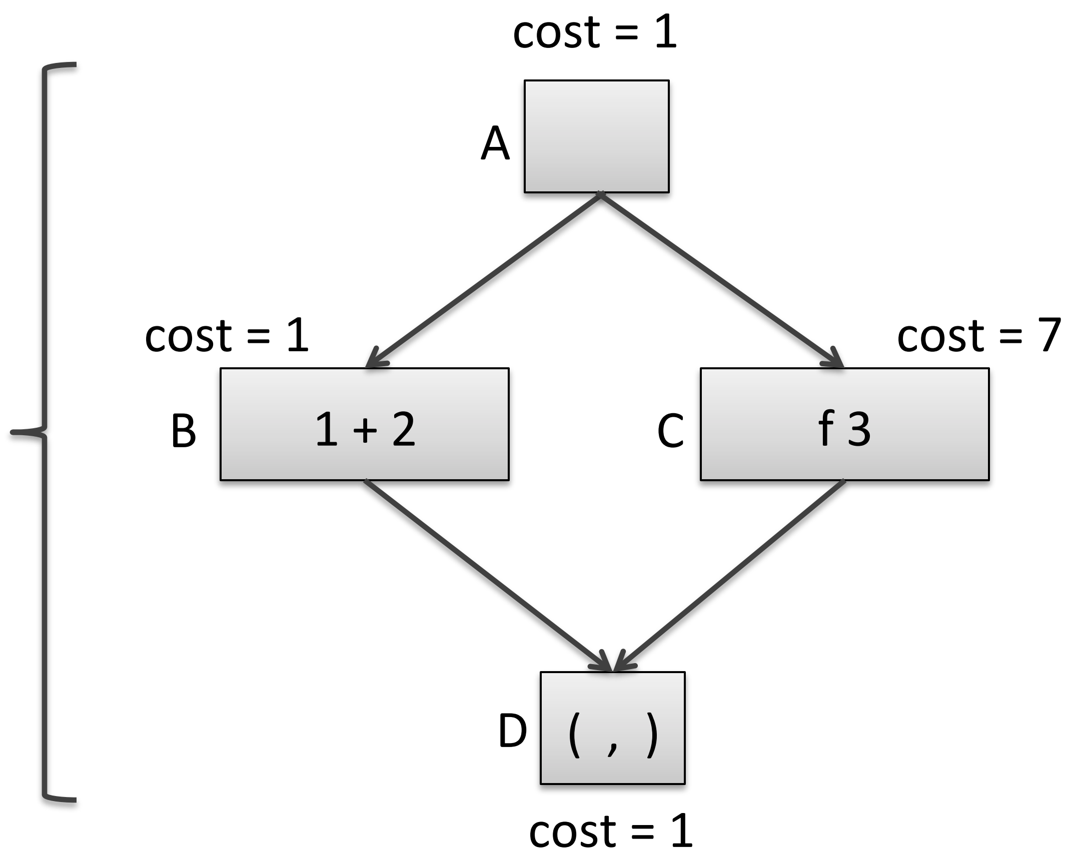
(1 + 2 || f 3)



Suppose we have 3 processors. How much time does this computation take?

Visualizing Computational Costs

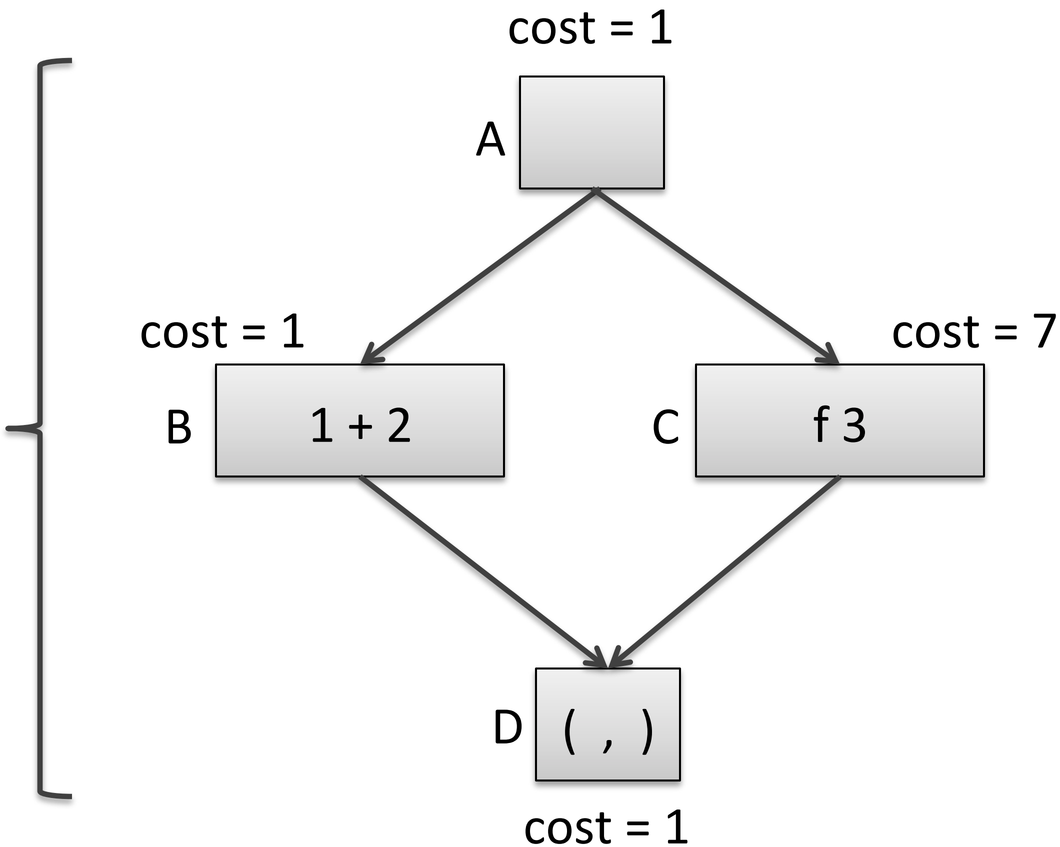
(1 + 2 || f 3)



Suppose we have 3 processors. How much time does this computation take?
Schedule A-BC-D: 1 + max(1,7) + 1 = 9

Visualizing Computational Costs

(1 + 2 || f 3)



Suppose we have infinite processors. How much time does this computation take?

Schedule A-BC-D: 1 + max(1,7) + 1 = 9

Work and Span

- Understanding the complexity of a parallel program is a little more complex than a sequential program
  - the number of processors has a significant effect

- One way to *approximate* the cost is to consider a parallel algorithm independently of the machine it runs on is to consider *two* metrics:
  - Work: The cost of executing a program with just 1 processor.
  - Span: The cost of executing a program with an infinite number of processors

- Always good to minimize work
  - Every instruction executed consumes energy
  - Minimize span as a second consideration
  - Communication costs are also crucial (we are ignoring them)

Parallelism

The parallelism of an algorithm is an estimate of the maximum number of processors an algorithm can profit from.

- parallelism = work / span

If work = span then parallelism = 1.

- We can only use 1 processor
- It's a sequential algorithm

If span = ½ work then parallelism = 2

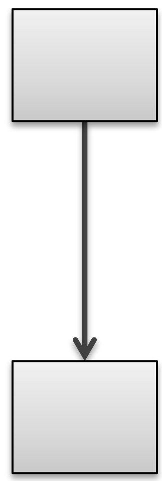
- We can use up to 2 processors

If work = 100, span = 1

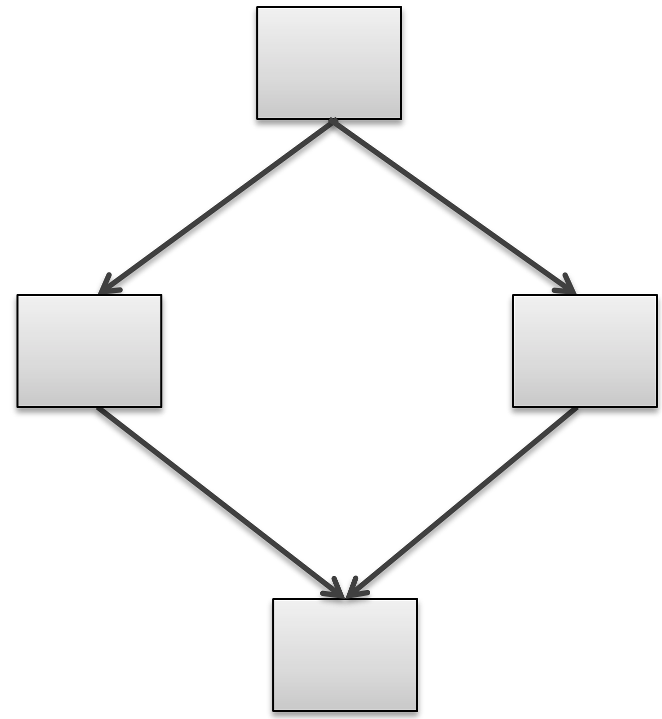
- All operations are independent & can be executed in parallel
- We can use up to 100 processors

Series-Parallel Graphs

one operation



two operations
in sequence
e1; e2

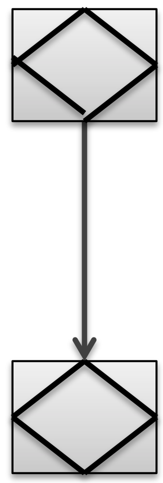


two operations
in parallel
(e1 || e2)

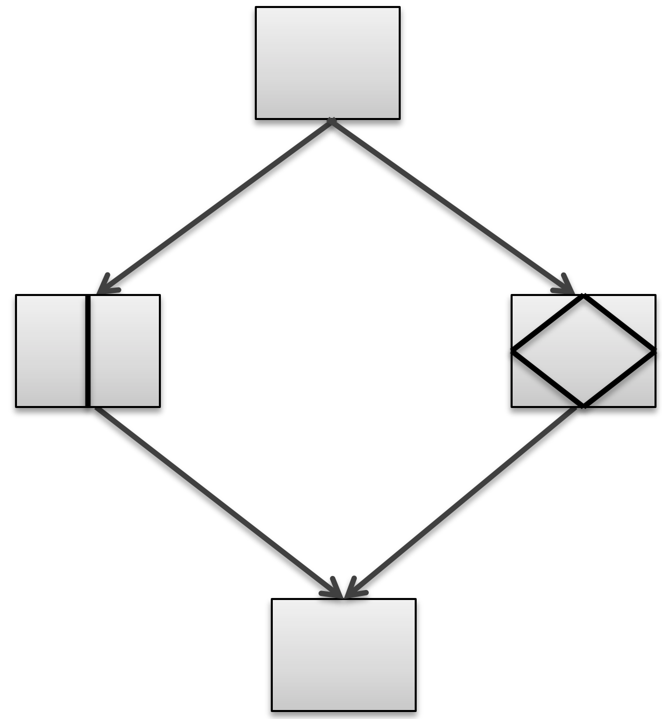
Series-parallel graphs arise from execution of functional programs with parallel pairs. Also known as well-structured, nested parallelism.

Series-Parallel Graphs Compose

one operation



two graphs
in sequence

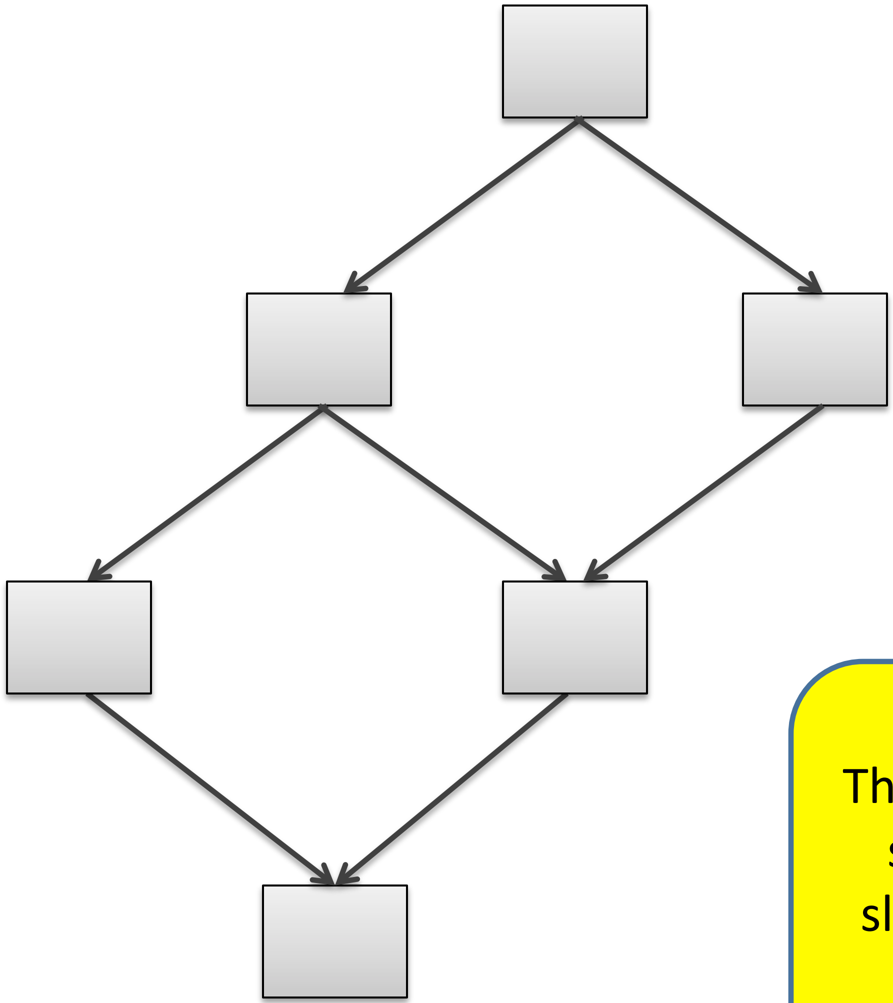


two graphs
in parallel

In general, a series-parallel graph has a source and a sink and is:

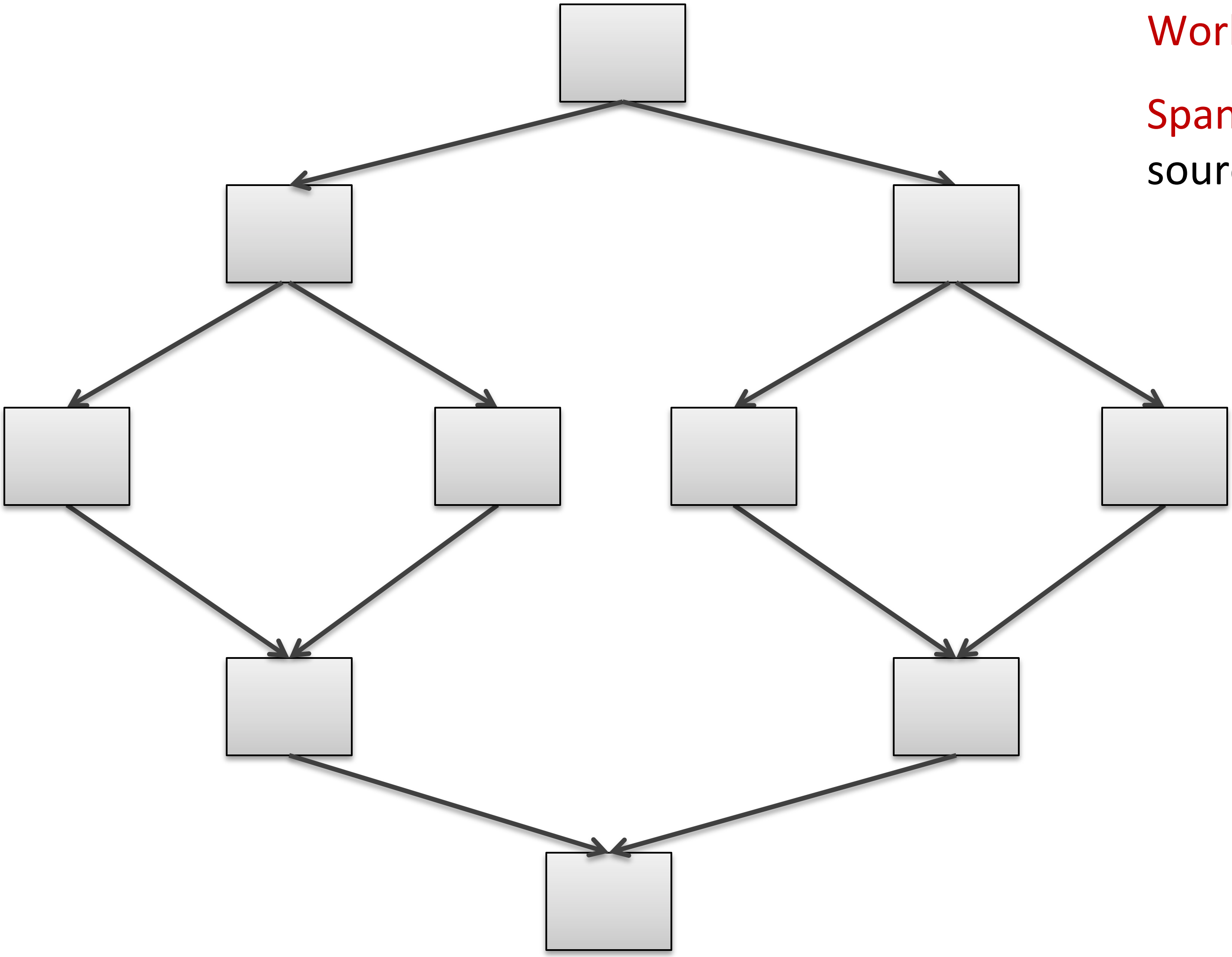
- a single node, or
- two series-parallel graphs in sequence, or
- two series-parallel graphs in parallel

Not a Series-Parallel Graph



However:
The results about greedy schedulers (next few slides) <u>do apply</u> to DAG schedules as well as series-parallel schedules!

Work and Span of Acyclic Graphs

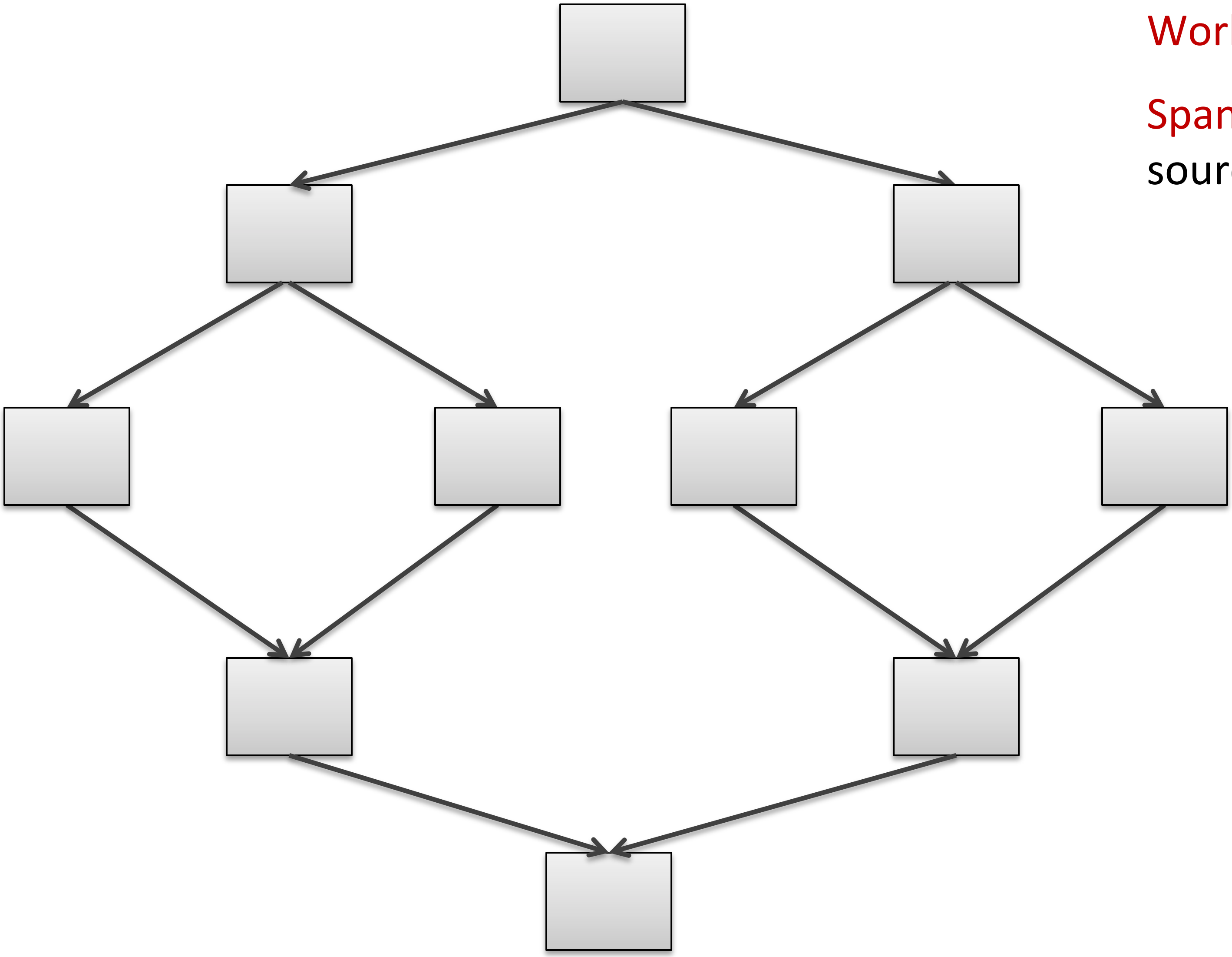


Let's assume each node costs 1.

Work: sum the nodes.

Span: longest path from source to sink.

Work and Span of Acyclic Graphs



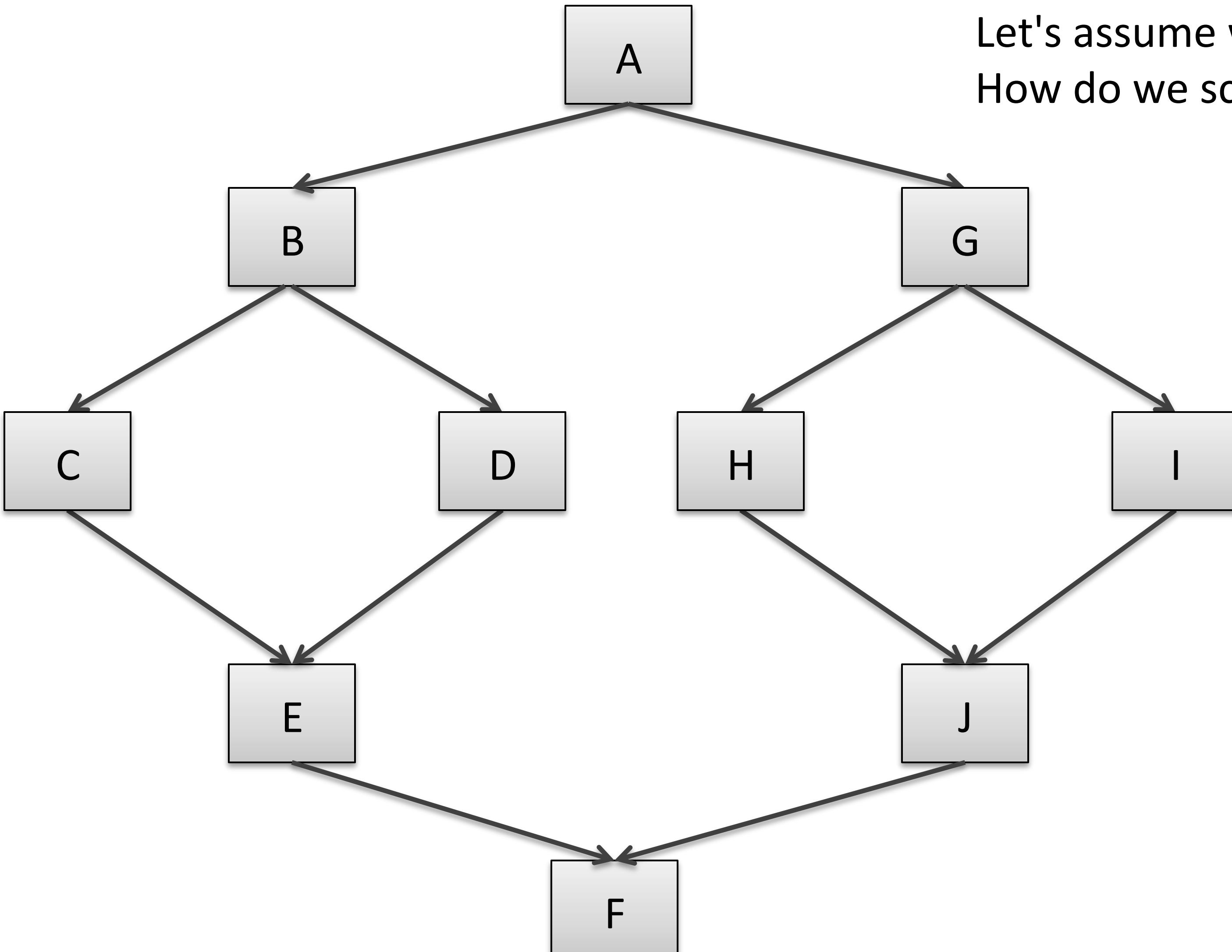
Let's assume each node costs 1.

Work: sum the nodes.

Span: longest path from source to sink.

work = 10
span = 5

Scheduling



Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Scheduling

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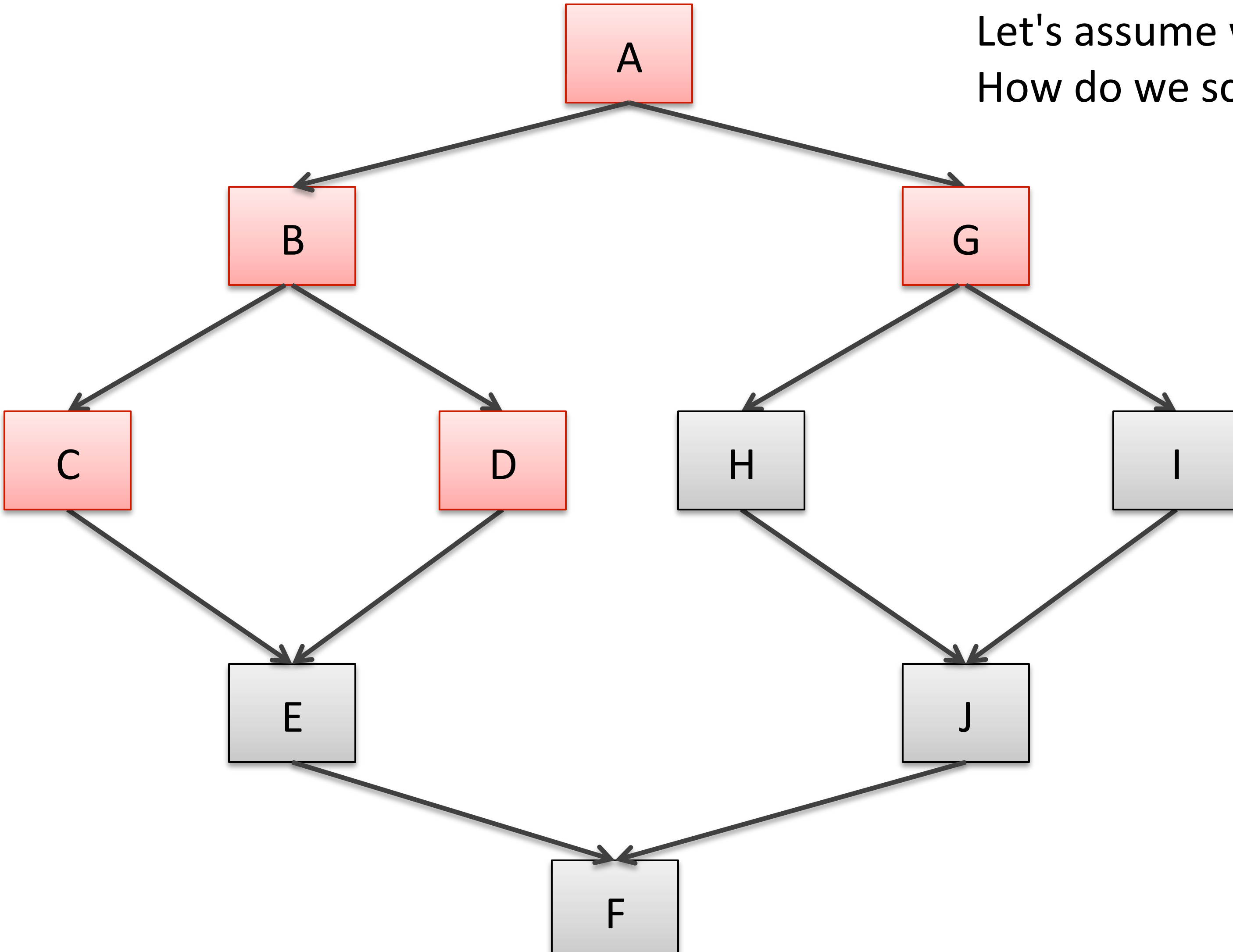
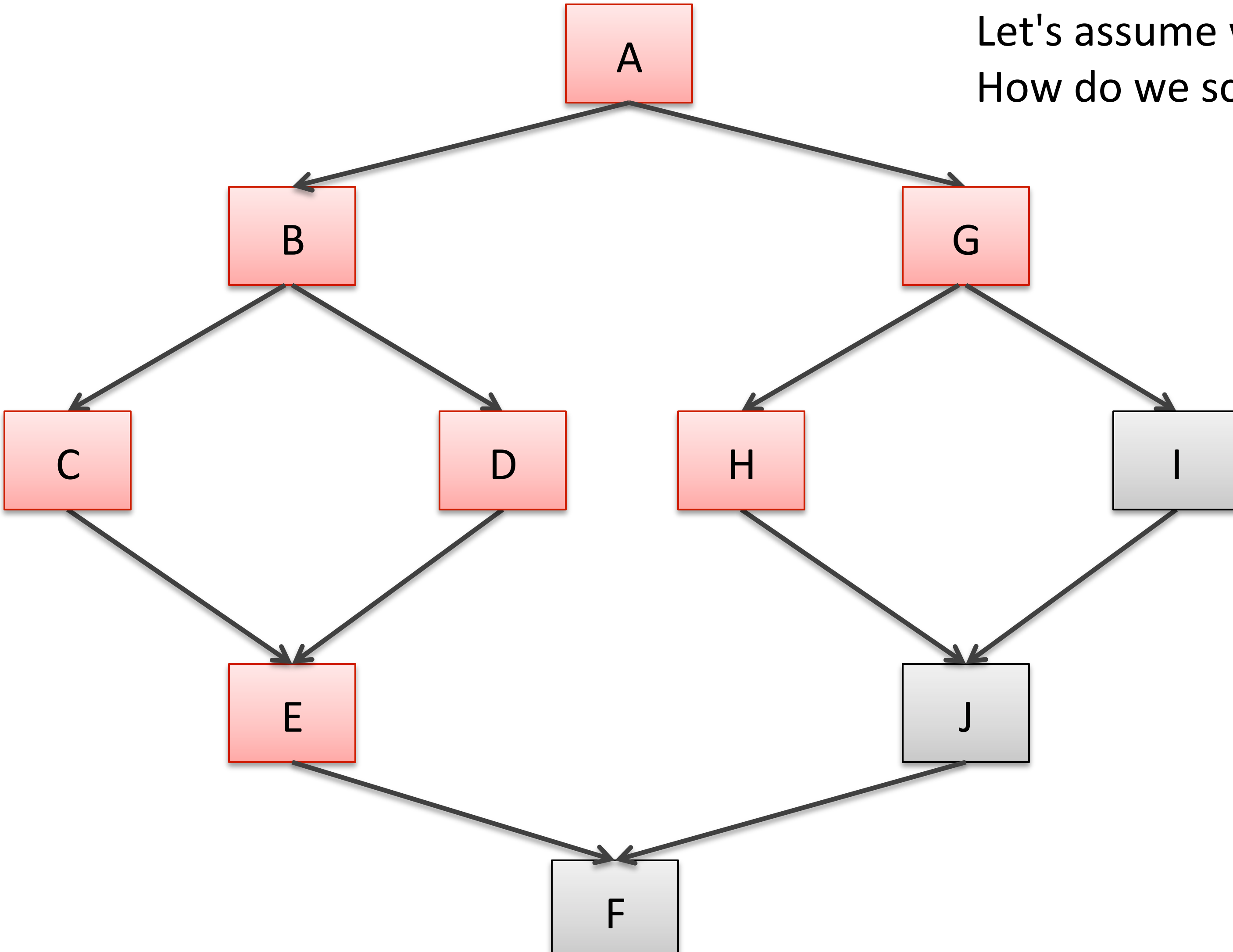
How do we schedule computation?



Option 1:
A
B G
C D

Scheduling

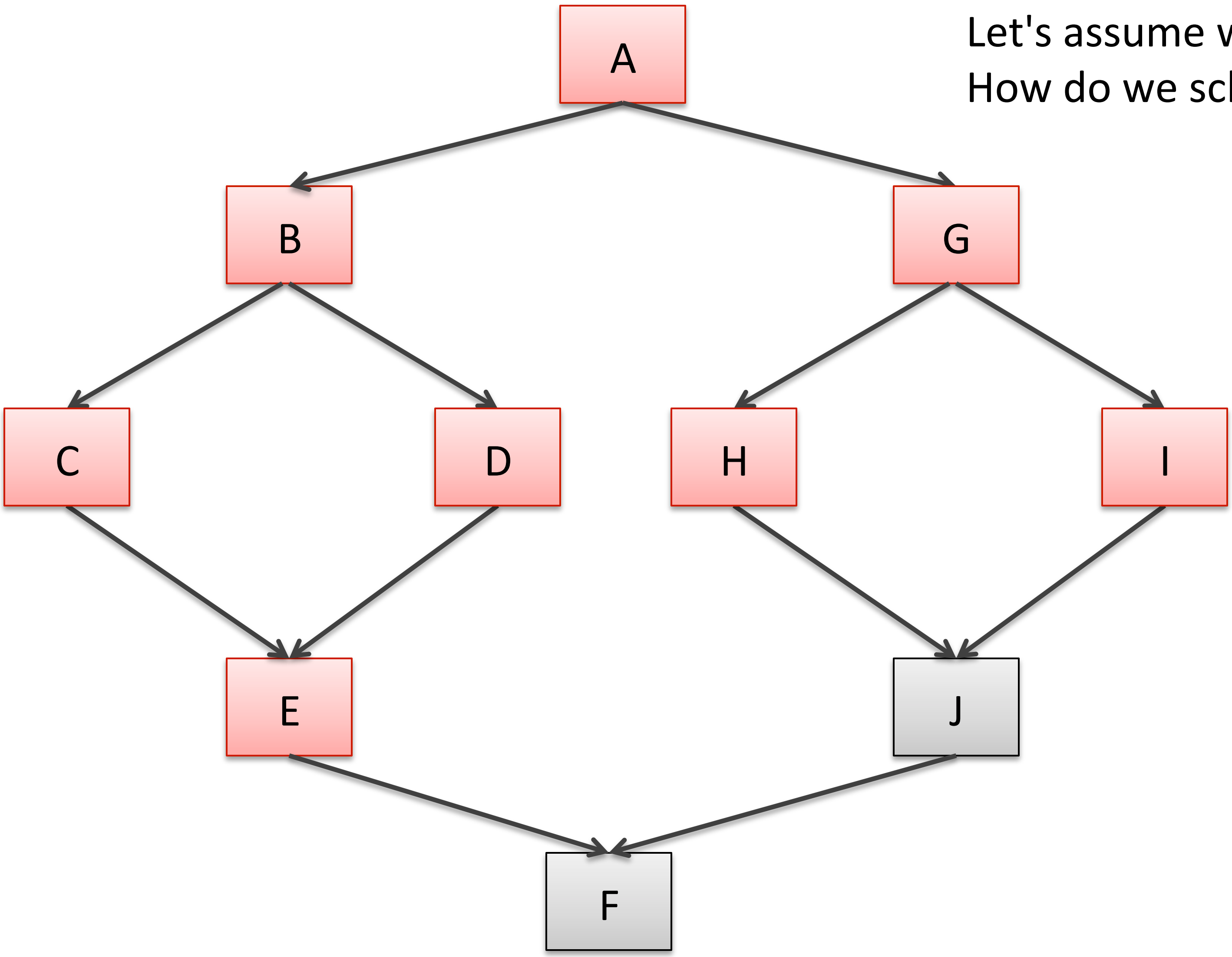


Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
E H

Scheduling



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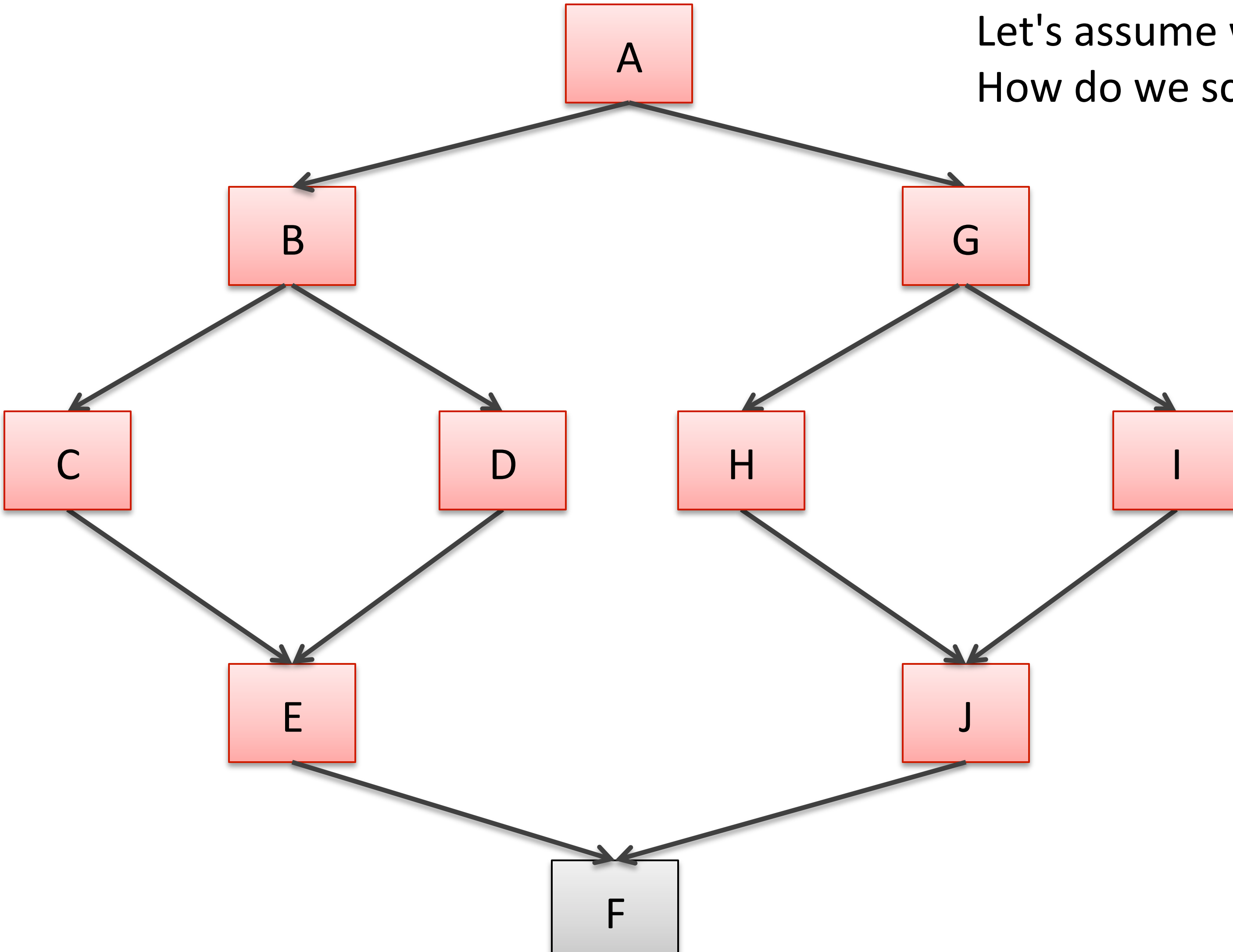
Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
E H
I

Scheduling



Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
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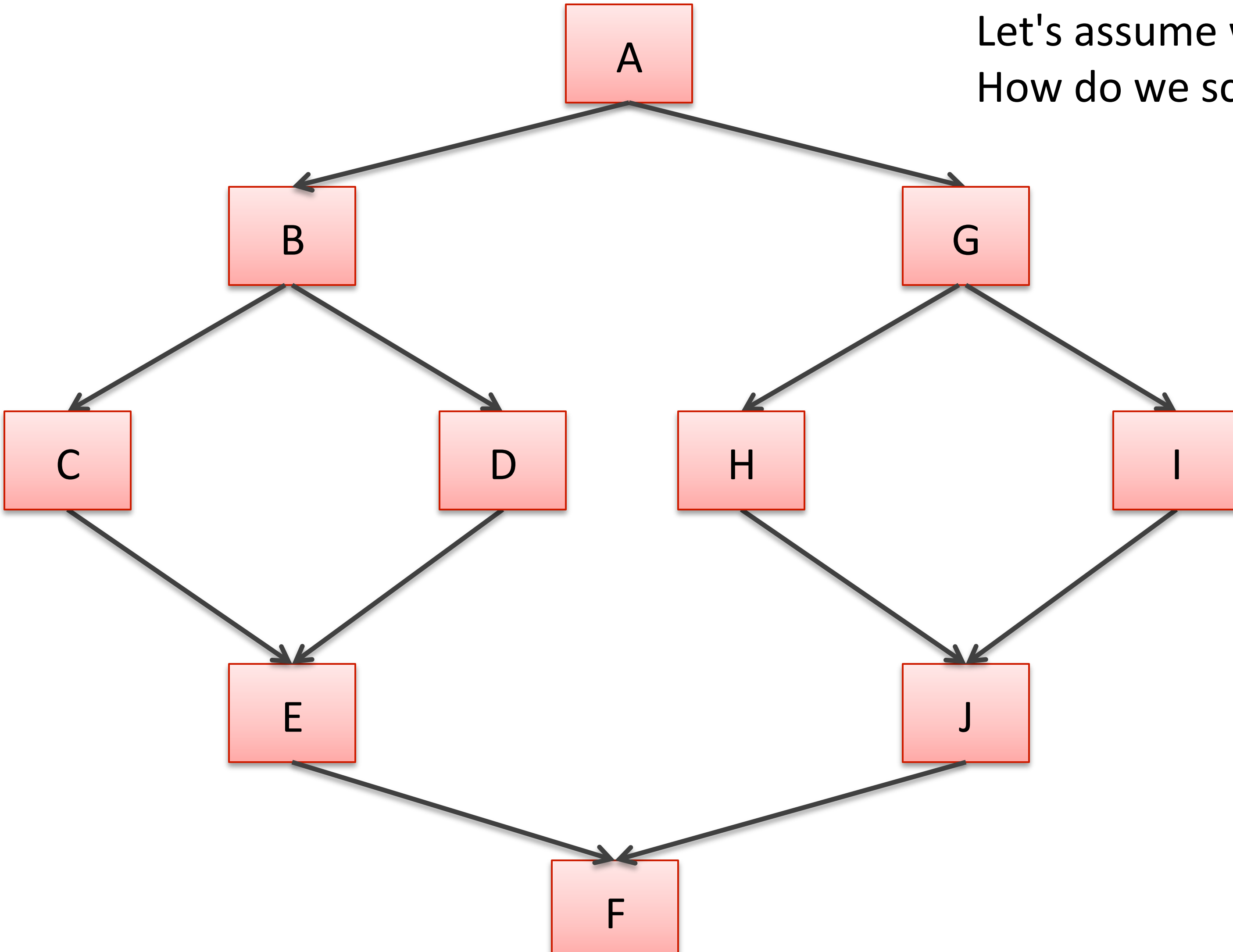
I
J

Scheduling

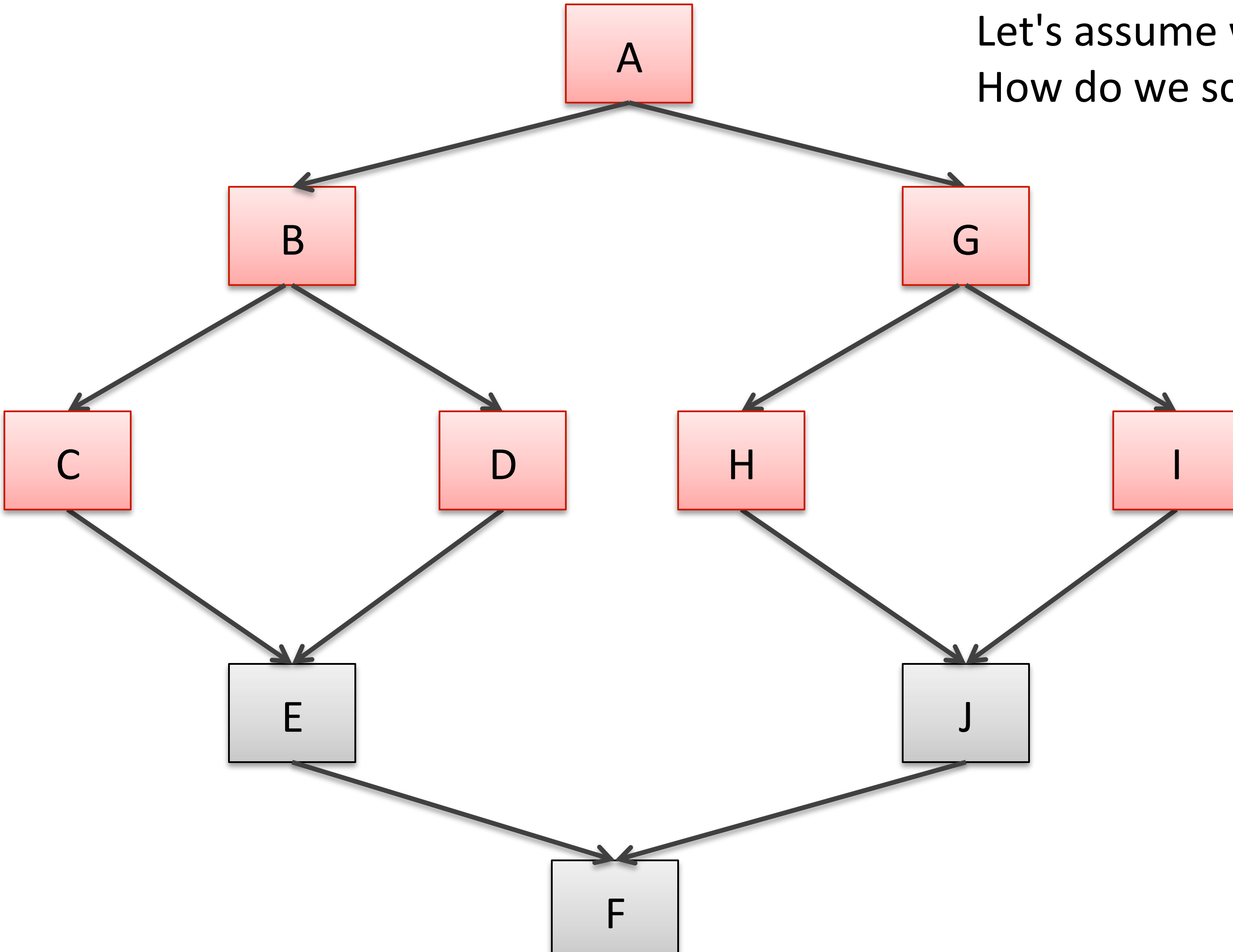
Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?



Option 1:
A
B G
C D
E H
I
J
F

Scheduling

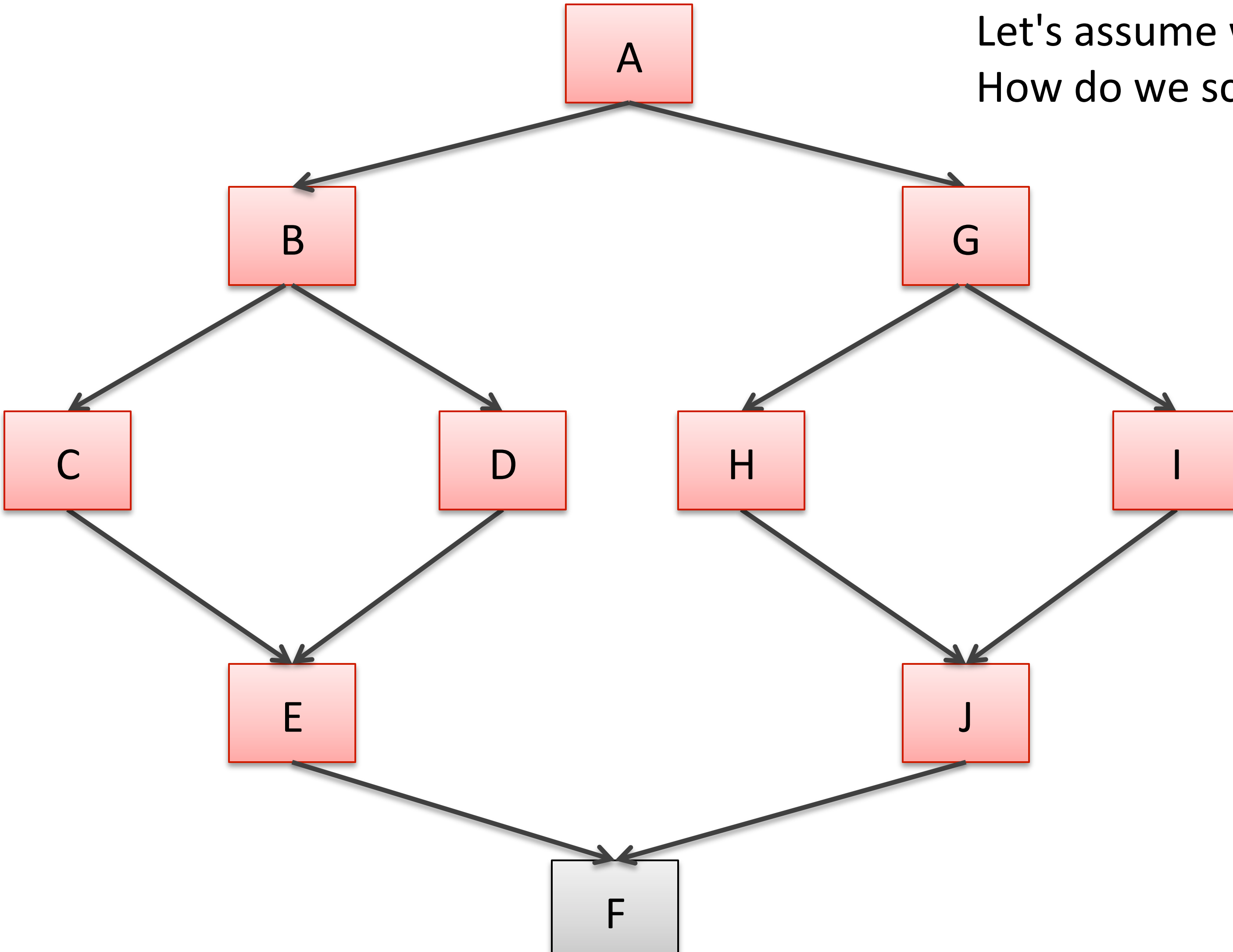


Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
~~E H~~ H I
~~I~~
~~J~~
~~F~~

Scheduling



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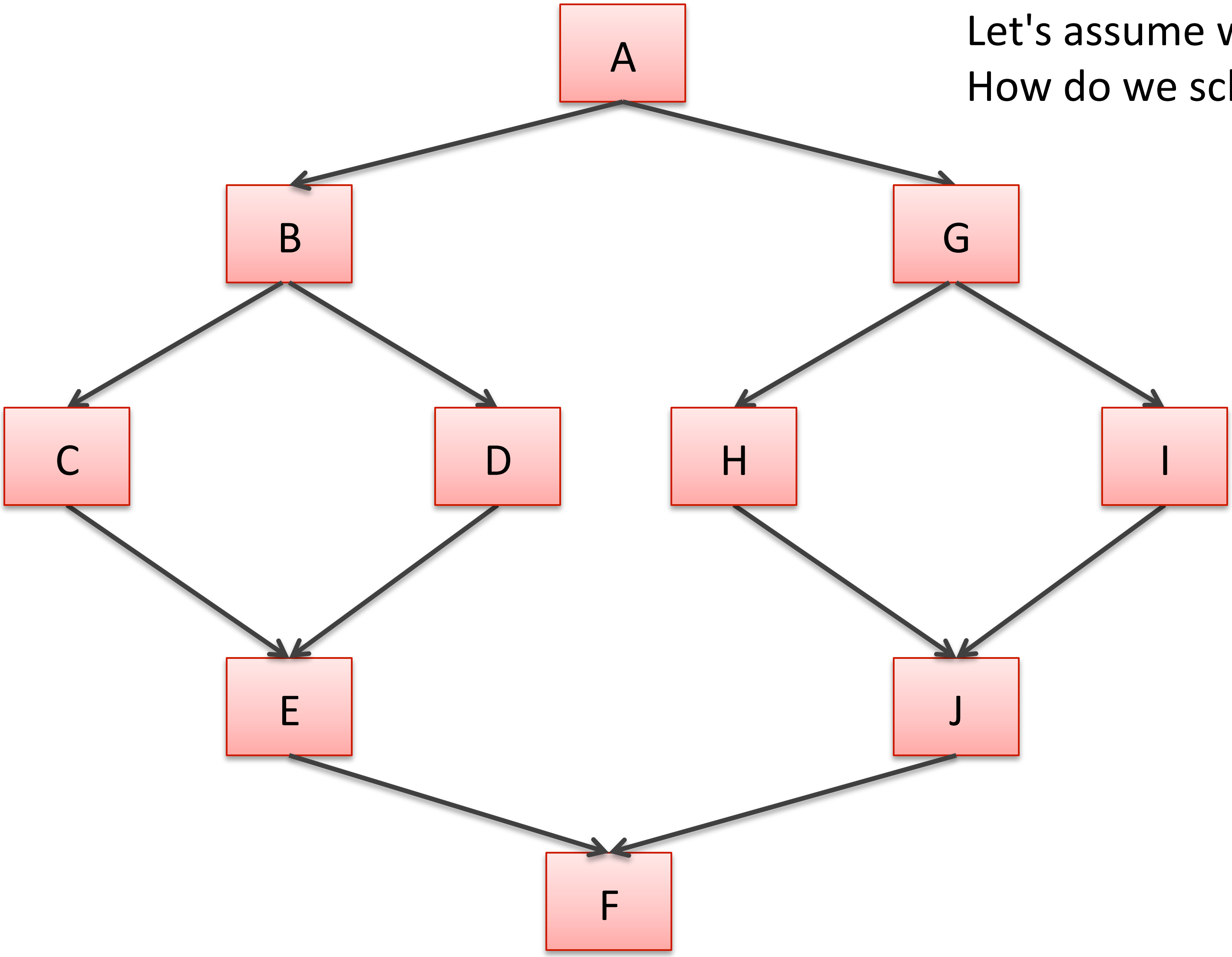
~~J~~
~~F~~

Scheduling

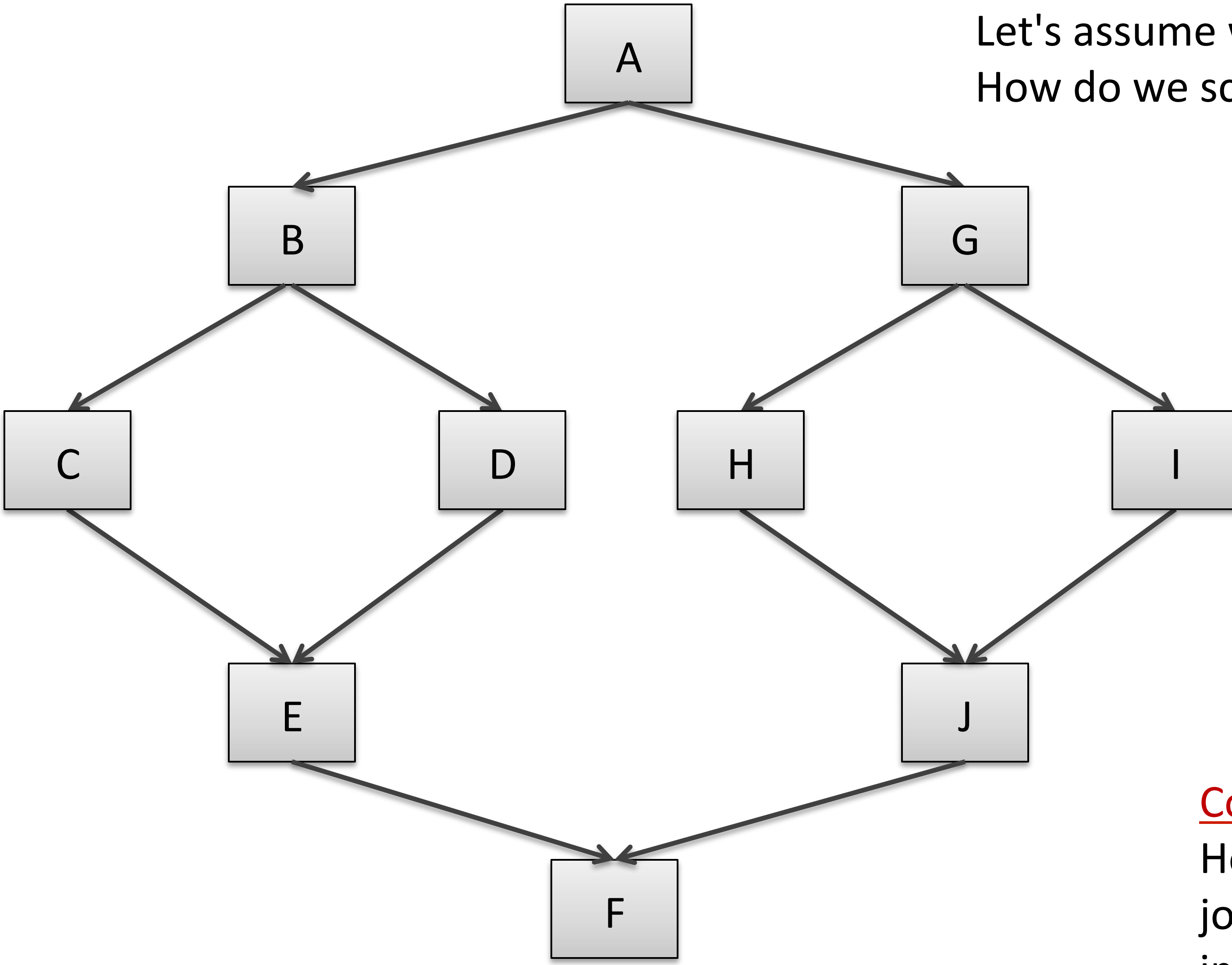


Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:
A
B G
C D
~~E H~~ H I
~~I~~ E J
~~J~~ F
~~F~~

Scheduling



Let's assume each node costs 1.

Let's assume we have 2 processors.
How do we schedule computation?

Option 1:

A	
B G	
C D	
~~E H~~	H I
~~I~~	E J
~~J~~	F
~~F~~	

<u>Conclusion:</u>
How you schedule jobs can have an impact on performance

Greedy Schedulers

- Greedy schedulers will schedule some task to a processor as soon as that processor is free.
  - Doesn't sound so smart!

- Properties (for p processors):
  - $T(p) < work/p + span$
    - won't be worse than dividing up the data perfectly between processors, except for the last little bit, which causes you to add the span on top of the perfect division
  - $T(p) >= \max(work/p, span)$
    - can't do better than perfect division between processors (work/p)
    - can't be faster than span

Greedy Schedulers

Properties (for p processors):

max(work/p, span) <= T(p) < work/p + span

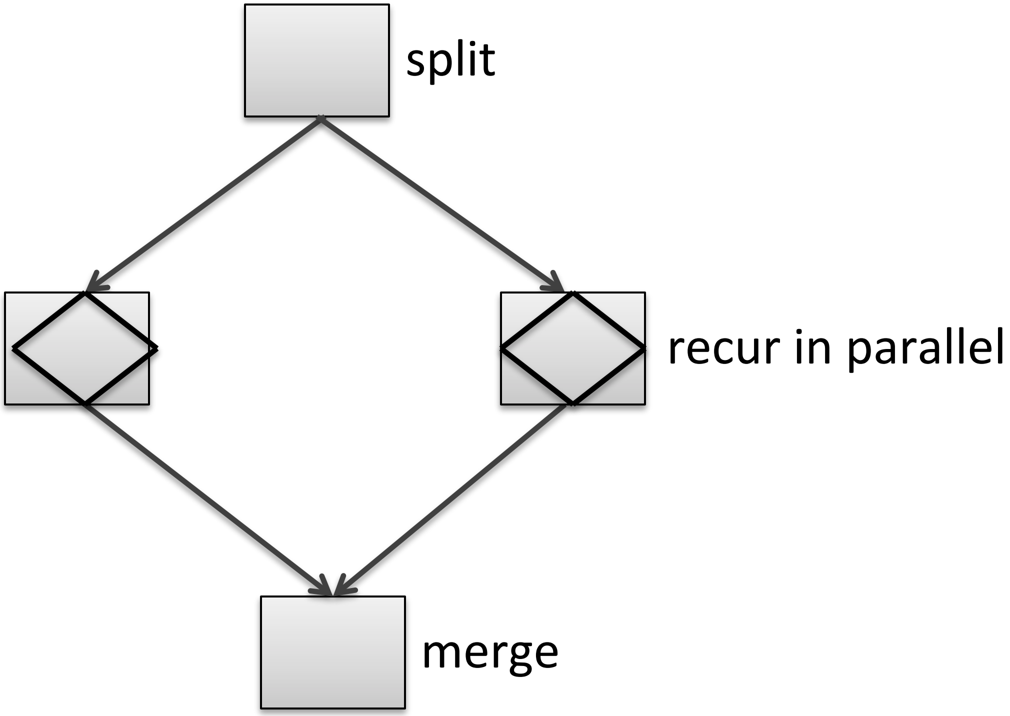
Consequences:

- as span gets small relative to work/p
  - work/p + span ==> work/p
  - max(work/p, span) ==> work/p
  - so T(p) ==> work/p -- greedy schedulers converge to the optimum!
- if span approaches the work
  - work/p + span ==> span
  - max(work/p, span) ==> span
  - so T(p) ==> span – greedy schedulers converge to the optimum!

COMPLEXITY OF PARALLEL PROGRAMS

Divide-and-Conquer Parallel Algorithms

- Split your input in 2 or more subproblems
- Solve the subproblems recursively in parallel
- Combine the results to solve the overall problem



Mergesort (on lists)

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
;;
```

for sequential mergesort, replace with: (mergesort sorted1,
mergesort sorted2)

Mergesort (on lists)

```
let rec split l =
  match l with
    []   -> ([]   , [])
  | [x] -> ([x] , [])
  | x :: y :: xs ->
      let (pile1, pile2) = split xs in
      (x :: pile1, y :: pile2)
```

```
let rec merge l1 l2 =
  match (l1, l2) with
    ([] , l2) -> l2
  | (l1 , []) -> l1
  | (x :: xs, y :: ys) ->
     if x < y then
       x :: merge xs l2
     else
       y :: merge l1 ys
```

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

work_mergesort(n) = work_split(n)
+ 2*work_mergesort(n/2)
+ work_merge(n)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []  -> []
  | [x] -> [x]
  | _   ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

work_mergesort(n) = work_split(n)
+ 2*work_mergesort(n/2)
+ work_merge(n)

read this as "approximately equal to"

= k1*n
+ 2*work_mergesort(n/2)
+ k2*n

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
     []   -> []
   | [x] -> [x]
   | _    ->
     let (pile1,pile2) = split l in
     let (sorted1,sorted2) =
       both mergesort pile1
            mergesort pile2
     in
     merge sorted1 sorted2
```

Assume input list of size n:

work_mergesort(n) = work_split(n)
+ 2*work_mergesort(n/2)
+ work_merge(n)

= k*n
+ 2*work_mergesort(n/2)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

work_mergesort(n) = work_split(n)
+ 2*work_mergesort(n/2)
+ work_merge(n)

= k*n
+ 2*work_mergesort(n/2)

= O(n log n)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

span_mergesort(n) = span_split(n)

+ max(span_mergesort(n/2), span_mergesort(n/2))

+ span_merge(n)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

span_mergesort(n) = k*n

+ span_mergesort(n/2)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
     []  -> []
   | [x] -> [x]
   | _   ->
     let (pile1,pile2) = split l in
     let (sorted1,sorted2) =
       both mergesort pile1
            mergesort pile2
     in
     merge sorted1 sorted2
```

Assume input list of size n:

span_mergesort(n) = k*n

+ k*(n/2 + n/4 + n/8 + ...)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []   -> []
  | [x] -> [x]
  | _    ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Assume input list of size n:

span_mergesort(n) = 2*k*n

= O(n)

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []  -> []
  | [x] -> [x]
  | _   ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Summary for input list of size n:

work_mergesort(n) = k*n*log n

span_mergesort(n) = k*n

parallelism?

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
    []  -> []
  | [x] -> [x]
  | _   ->
    let (pile1,pile2) = split l in
    let (sorted1,sorted2) =
      both mergesort pile1
           mergesort pile2
    in
    merge sorted1 sorted2
```

Summary for input list of size n:

work_mergesort(n) = k*n*log n

span_mergesort(n) = k*n

parallelism?

parallelism = work/span

= n*log n / n

= log n

when sorting 10 billion entries, can only make use of 30 machines

Complexity

```
let rec mergesort (l : int list) : int list =
  match l with
     []  -> []
   | [x] -> [x]
   | _   ->
     let (pile1,pile2) = split l in
     let (sorted1,sorted2) =
       both mergesort pile1
            mergesort pile2
     in
     merge sorted1 sorted2
```

Summary for input list of size n:

work_mergesort(n) = k*n*log n

span_mergesort(n) = k*n

splitting and merging take linear time – too long to get good speedups

parallelism?

parallelism = work/span

= n*log n / n

= log n

when sorting 10 billion entries, can only make use of 30 machines

Complexity

when sorting 10 billion entries,
can only make use of 30 machines/cores

data centers have 10s of 1000s of machines or more

Problem: splitting and merging take linear time – too long to get good speedups

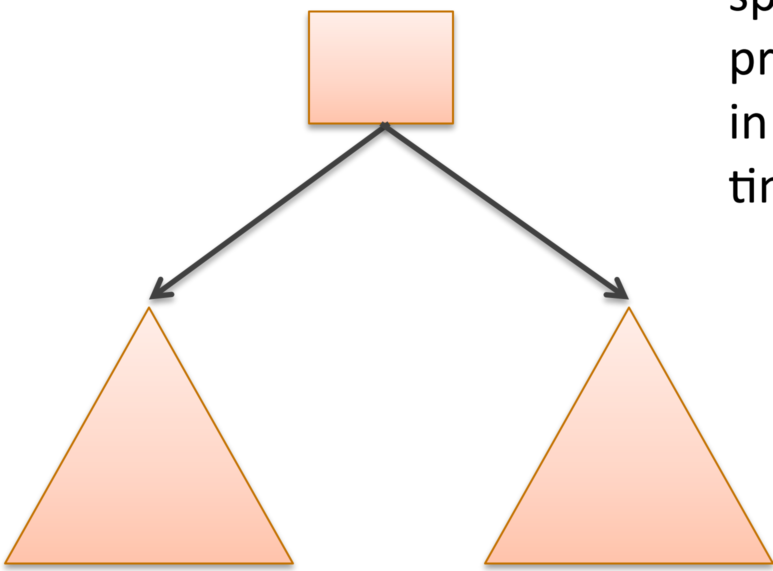
Problem: cutting a list in half takes at least time proportional to n/2

Problem: stitching 2 lists together of size n/2 takes n/2 time

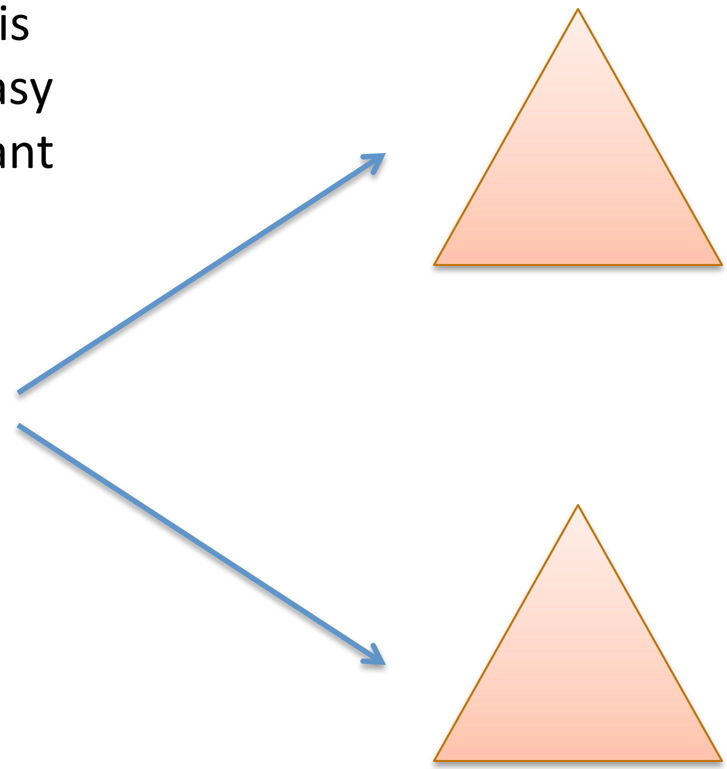
Conclusion: lists are a bad data structure to choose

Complexity

Consider balanced trees:



splitting is pretty easy in constant time



merging is harder, but can be done in poly-log time

Parallel TreeSort

```
type tree = Empty | Node of tree * int * tree

let node left i right = Node (left, i, right)

let one i = node Empty i Empty
```

- Problem: Given a balanced tree t, return a balanced tree with the same elements, in order:
  - elements in the left subtree are less than the root
  - elements in the right subtree are greater than the root

Parallel TreeSort

```
type tree = Empty | Node of tree * int * tree

let node left i right = Node (left, i, right)

let one i = node Empty i Empty
```

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        rebalance(merge (merge l' r') (one i))
```

We are going to ignore this.

Parallel TreeSort

```
type tree = Empty | Node of tree * int * tree

let node left i right = Node (left, i, right)

let one i = node Empty i Empty
```

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

Merging trees

- Subproblem: Given two sorted, balanced trees, l and r, create a new tree with the same elements that is also balanced and whose elements are in order.
- Uses split_at t i
  - divides t into items less than i and items greater than i

```
let rec merge (t1:tree) (t2:tree) : tree =
  match t1 with
      Empty -> t2
    | Node (l1, i, r1) ->
        let (l2, r2) = split_at t2 i in
        let (t1', t2') = both (merge l1) l2
                              (merge r1) r2
        in
        Node (t1', i, t2')
```

Splitting a tree

- Sub-problem: Divide t in to items less than i and items greater than i

```
let rec split_at t bound =
  match t with
      Empty -> (Empty, Empty)
    | Node (l, i, r) ->
        if bound < i then
          let (ll, lr) = split_at l bound in
          (ll, Node (lr, i,  r))
        else
          let (rl, rr) = split_at r bound in
          (Node (l, i, rl), rr)
```

Splitting a tree

- Sub-problem: Divide t in to items less than i and items greater than i

```
let rec split_at t bound =
  match t with
      Empty -> (Empty, Empty)
    | Node (l, i, r) ->
        if bound < i then
           let (ll, lr) = split_at l bound in
           (ll, Node (lr, i,  r))
        else
           let (rl, rr) = split_at r bound in
           (Node (l, i, rl), rr)
```

span (h) = k*h

where h is the height of the tree t
h = log(n) if t is balanced with n nodes

Span of Merge

```
let rec merge (t1:tree) (t2:tree) : tree =
  match t1 with
      Empty -> t2
    | Node (l1, i, r1) ->
        let (l2, r2) = split_at t2 i in
        let (t1', t2') = both (merge l1) l2
                              (merge r1) r2
        in
        Node (t1', i, t2')
```

let's assume t1 and t2 are balanced and have heights h1, h2 and h1 >= h2:

  span_merge(h1,h2)
= span_split(h2) + max(span_merge(h1-1), span_merge(h2-1))
= k*h2 + span_merge(h1-1)
= k*h2*h1

Span of Parallel TreeSort

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

let's assume:

- t is balanced with n nodes and height h = log n
- tsort returns balanced trees (l', r')
- merge returns balanced trees

Span of Parallel TreeSort

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

let's assume:

- t is balanced with n nodes and height h = log n
- tsort returns balanced trees (l', r')
- merge returns balanced trees

span_tsort(h)
= max(span_tsort(h-1),
span_tsort(h-1))
+ span_merge(h-1,h-1)
+ span_merge(h,1)

Span of Parallel TreeSort

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

let's assume:

- t is balanced with n nodes and height h = log n
- tsort returns balanced trees (l', r')
- merge returns balanced trees

span_tsort(h)
= max(span_tsort(h-1),
      span_tsort(h-1))
+ span_merge(h-1,h-1)
+ span_merge(h,1)
= span_tsort(h-1)
+ k*(h-1)*(h-1) + k*h

Span of Parallel TreeSort

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

let's assume:

- t is balanced with n nodes and height h = log n
- tsort returns balanced trees (l', r')
- merge returns balanced trees

span_tsort(h)
= max(span_tsort(h-1),
      span_tsort(h-1))
+ span_merge(h-1,h-1)
+ span_merge(h,1)
= span_tsort(h-1)
+ k*(h-1)*(h-1) + k*h
= k*h*h*h

Span of Parallel TreeSort

```
let rec tsort t =
  match t with
      Empty -> Empty
    | Node (l, i, r) ->
        let (l', r') = both tsort l
                            tsort r
        in
        merge (merge l' r') (one i)
```

let's assume:

- t is balanced with n nodes and height h = log n
- tsort returns balanced trees (l', r')
- merge returns balanced trees

span_tsort(h)
= max(span_tsort(h-1),
span_tsort(h-1))
+ span_merge(h-1,h-1)
+ span_merge(h,1)
= span_tsort(h-1)
+ k*(h-1)*(h-1) + k*h
= k*h*h*h

$= k*h^3$
$= O(\log^3 n)$

Summary of Parallel Sorting Exercise

Both parallel list sort and parallel tree sort follow a traditional parallel divide-and-conquer strategy.

By changing data structures from lists to trees, we were able to:

- split our data in half in constant span instead of linear span
- merge our data back together in $\log^3 n$ span instead of linear span

We get more parallelism:

- with lists: work/span = log n
  - make use of 30 machines when sorting 10 billion items
- with trees: work/span = $n \log n / \log^3 n = n / \log^2 n$
  - make use of millions* of machines when sorting 10 billion items
    - caveat: we didn't factor in data communication costs!

*Well, almost. What is $\log_2(10{,}000{,}000{,}000)$?

Summary: Work, Span, Parallelism

Series parallel-graphs describe the kinds of control structures that arise in pure functional programs with structured, parallel fork-join execution

- Work: total number/cost of operations
  - time program execution takes with 1 processor
  - Work(e1 || e2) = Work(e1) + Work(e2) + 1
- Span: length of the longest dependency chain
  - time program execution takes with infinite processors
  - Span (e1 || e2) = max (Span e1, Span e2) + 1
- Parallelism: Work / Span

Many parallel algorithms follow a divide-and-conquer strategy

- efficient algorithms divide quickly and merge quickly

Parallel Collections

Parallel Collections

One way to give programmers access to parallelism in a functional style (even in an imperative language) is to develop a library for programming parallel collections

Example collections: sets, tables, dictionaries, sequences

Example bulk operations: create, map, reduce, join, filter

Parallel Sequences

- Parallel sequences

$$< e1 , e2 , e3 , \ldots , en >$$

- Languages:
  - Nesl [Blelloch]
  - Data-parallel Haskell

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
```

work = $O(n \cdot work(f))$ span = $O(1 \cdot span(f))$

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
```

work = O(n·work(f))   span = O(1·span(f))

```
nth : 'a seq -> int -> 'a

nth <e0, e1, ..., e(n-1)> i == ei
```

work = O(1)   span = O(1)

Parallel Sequences: Selected Operations

```
tabulate : (int -> 'a) -> int -> 'a seq

tabulate f n  == <f 0, f 1, ..., f (n-1)>
```
work = $O(n \cdot \text{work}(f))$ span = $O(1 \cdot \text{span}(f))$

```
nth : 'a seq -> int -> 'a

nth <e0, e1, ..., e(n-1)> i == ei
```
work = $O(1)$ span = $O(1)$

```
length : 'a seq -> int

length <e0, e1, ..., e(n-1)> == n
```
work = $O(1)$ span = $O(1)$

Problems

Write a function that creates the sequence <0, ..., n-1>

Write a function such that given a sequence <v0, ..., vn-1>, maps f over each element of the sequence.

Work = O(n); Span = O(1) (if f is a constant-work function)

Write a function such that given a sequence <v1, ..., vn-1>, reverses the sequence.

Work = O(n); Span = O(1)

Try it!

Operations:

```
tabulate f n
nth i s
length s
```

Solutions

```
(* create n == <0, 1, ..., n-1> *)
let create n =
  tabulate (fun i -> i) n
```

```
(* map f <v0, ..., vn-1> == <f v0, ..., f vn-1> *)
let map f s =
  tabulate (fun i -> f (nth s i)) (length s)
```

```
(* reverse <v0, ..., vn-1> == <vn-1, ..., v0> *)
let reverse f s =
  let n = length s in
  tabulate (fun i -> nth s (n-i-1)) n
```

One more problem

- Consider the problem of determining whether a sequence of parentheses is balanced or not. For example:
  - balanced: ()()(())
  - not balanced: (or) or ()))
- Try formulating a divide-and-conquer parallel algorithm to solve this problem efficiently:

```
type paren = L | R       (* L(eft) or R(ight) paren *)

let balanced (ps : paren list) : bool = ...
```

- You will need another function on sequences:

```
(* split s n divides s into (s1, s2) such that s1 is
   the first n elements of s and s2 is the rest
   Work = O(n) Span = O(1) *)
split : 'a sequence -> int -> 'a sequence * 'a sequence
```