# A Functional Space Model

COS 326
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# Space

Understanding the space complexity of functional programs

- At least two interesting components:
  - the amount of *live space* at any instant in time
  - the *rate of allocation*
    - a function call may not change the amount of live space by much but may allocate at a substantial rate
    - because functional programs act by generating new data structures and discarding old ones, they often allocate a lot
      - » OCaml garbage collector is optimized with this in mind
      - » interesting fact: at the assembly level, the number of writes by a functional program is roughly the same as the number of writes by an imperative program

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      - » OCaml garbage collector is optimized with this in mind
      - » interesting fact: at the assembly level, the number of writes by a function program is roughly the same as the number of writes by an imperative program
- *What takes up space*?
  - conventional first-order data: tuples, lists, strings, datatypes
  - function representations (closures)
  - the call stack

# CONVENTIONAL DATA

# Blackboard!

Numbers

Tuples

Data types

Lists

# Space Model

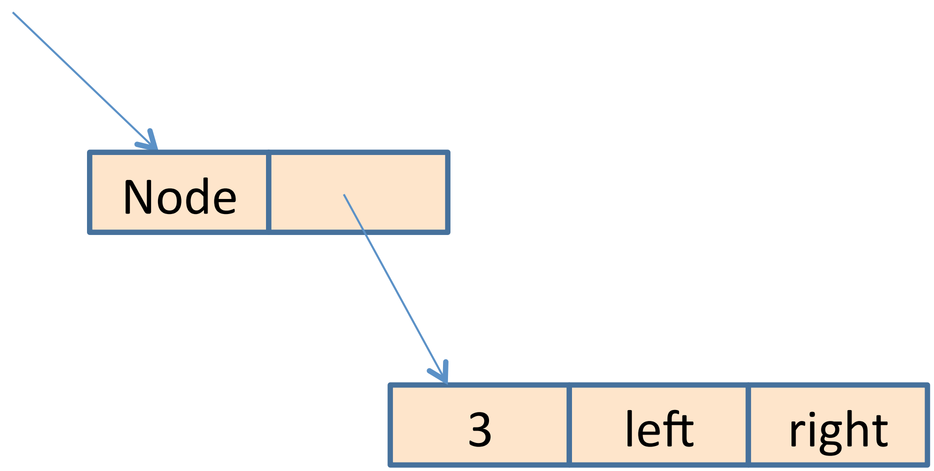
Data type representations:

```
type tree = Leaf | Node of int * tree * tree
```

Leaf:

0

Node(i, left, right):



# Allocating space

In C, you allocate when you call "malloc"

In Java, you allocate when you call "new"

What about ML?

# Allocating space

Whenever you *use a constructor*, space is allocated:

```
let rec insert (t:tree) (i:int) =
  match t with
    Leaf -> Node (i, Leaf, Leaf)
  | Node (j, left, right) ->
      if i <= j then
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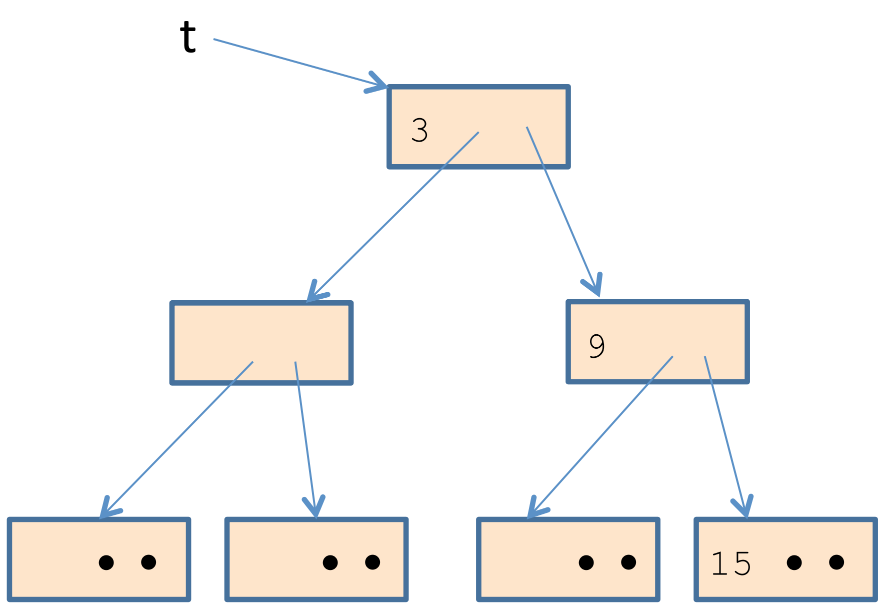

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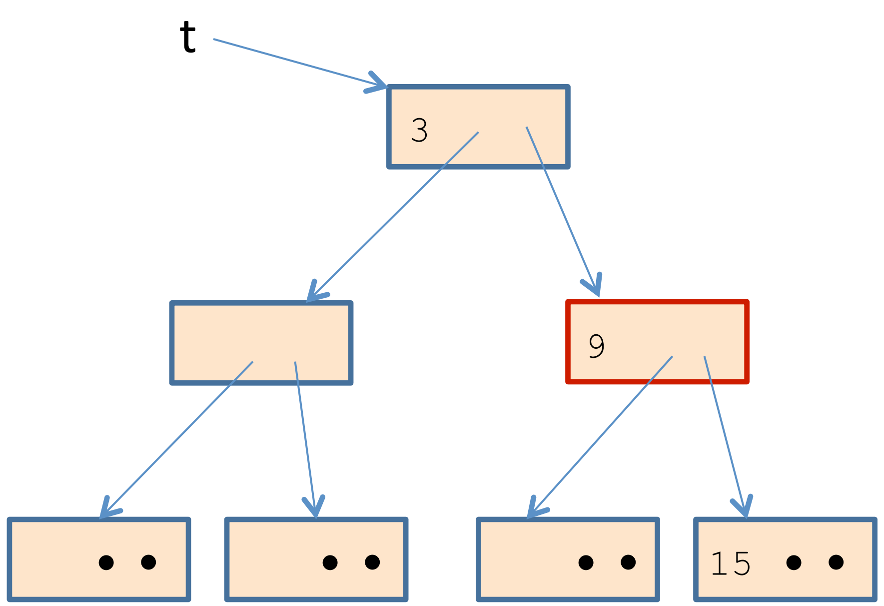

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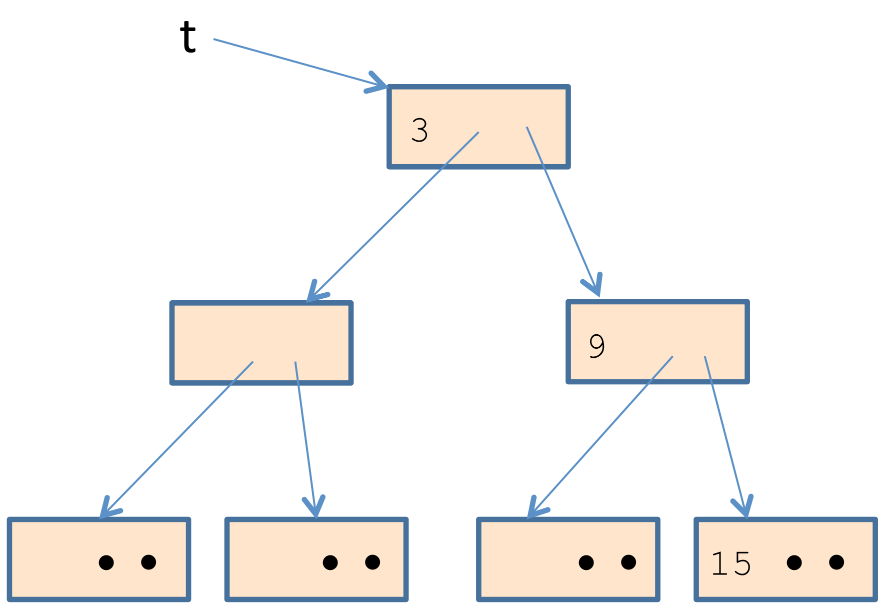


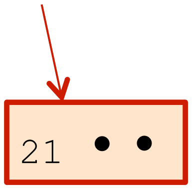

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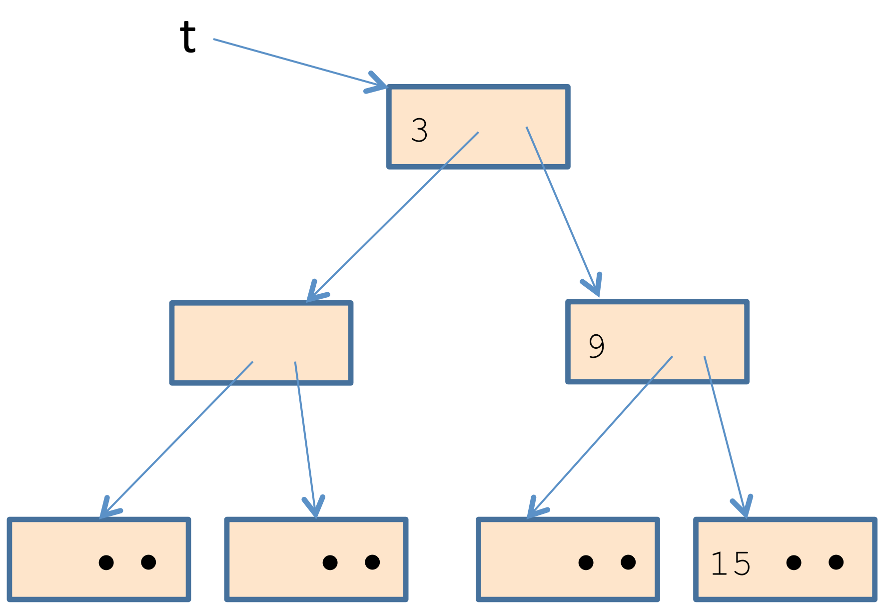


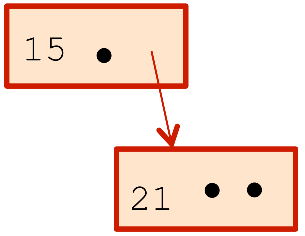

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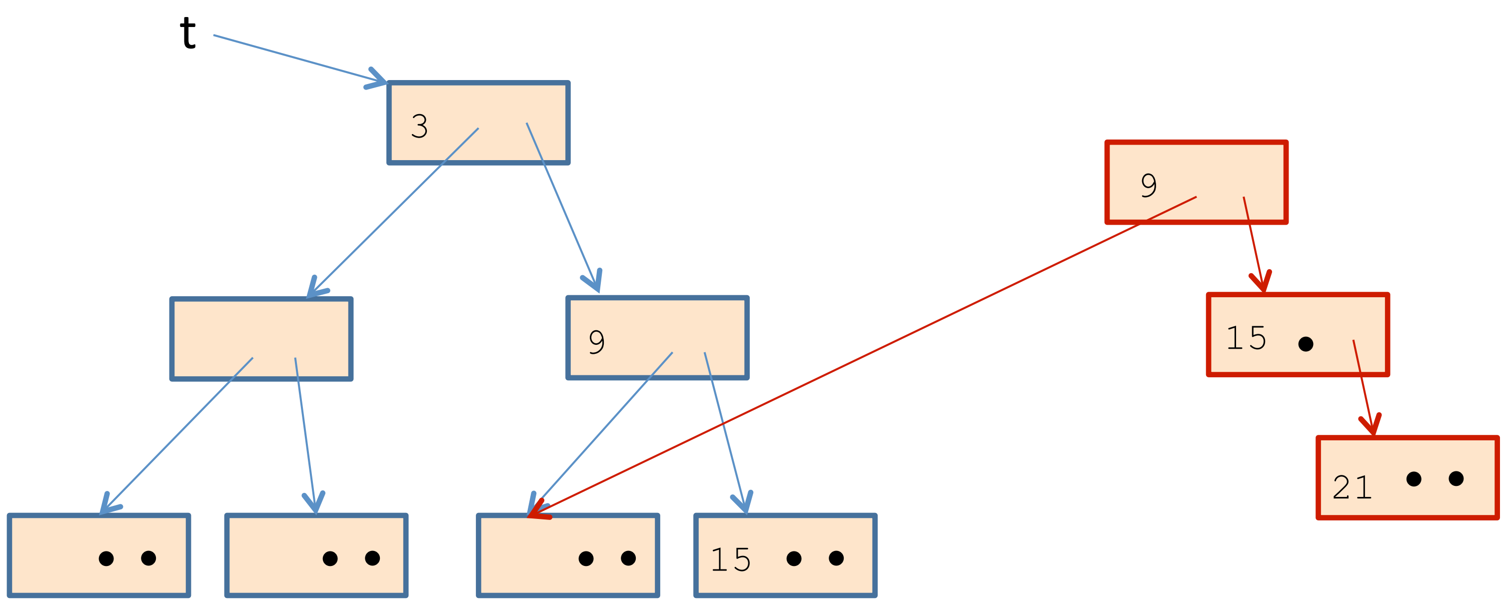

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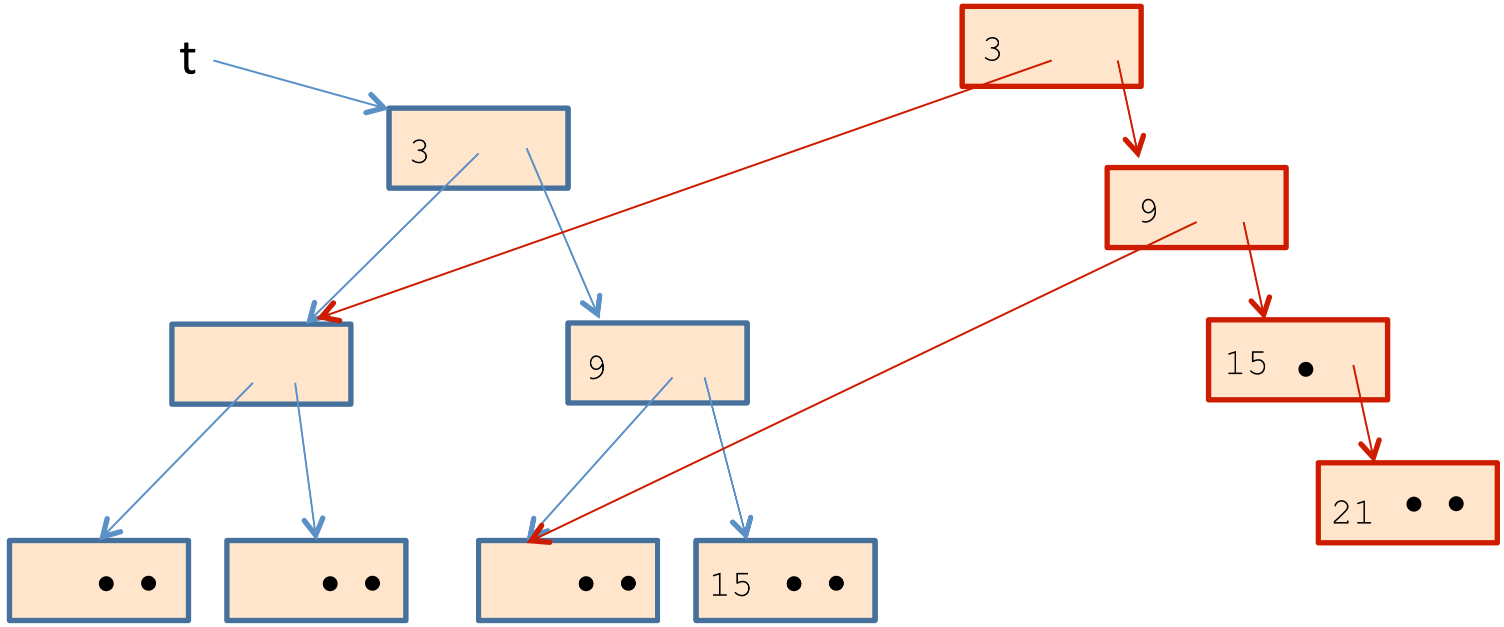

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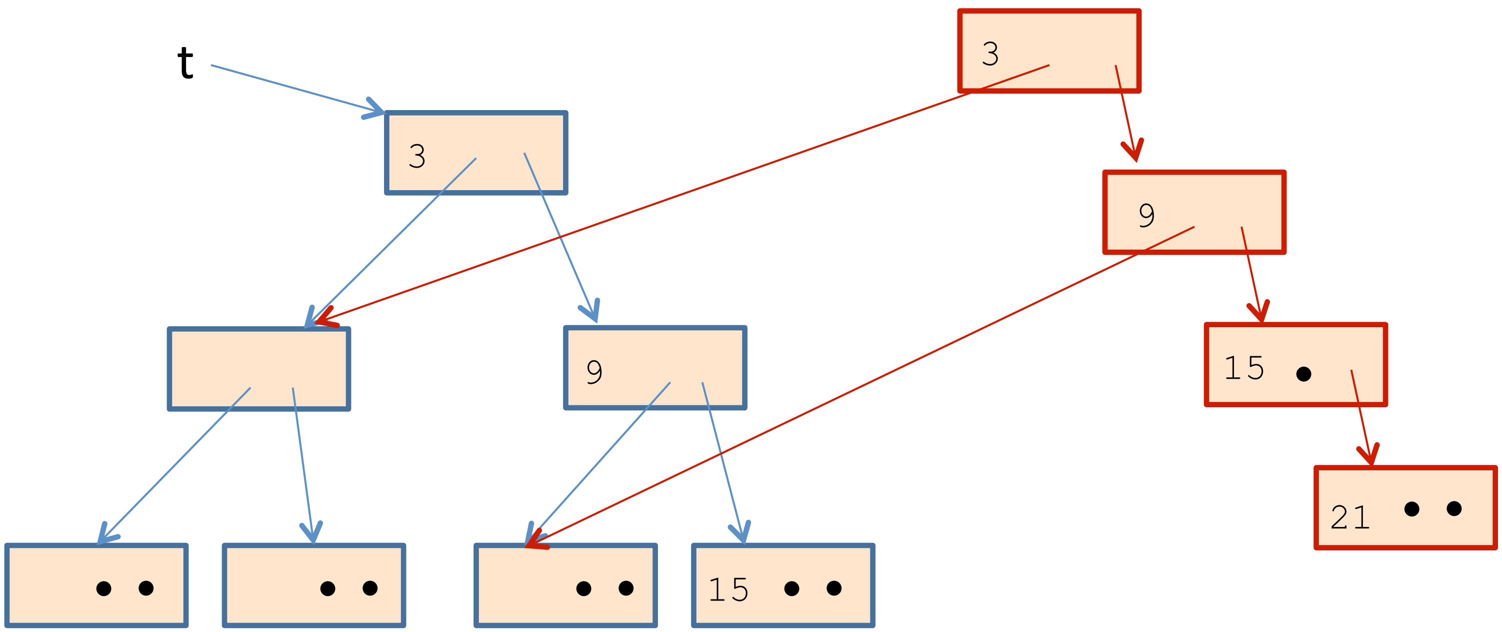
Total space allocated is proportional to the height of the tree.

~ log n, if tree with n nodes is balanced



# Net space allocated

The garbage collector reclaims unreachable data structures on the heap.

```
let fiddle (t: tree) =
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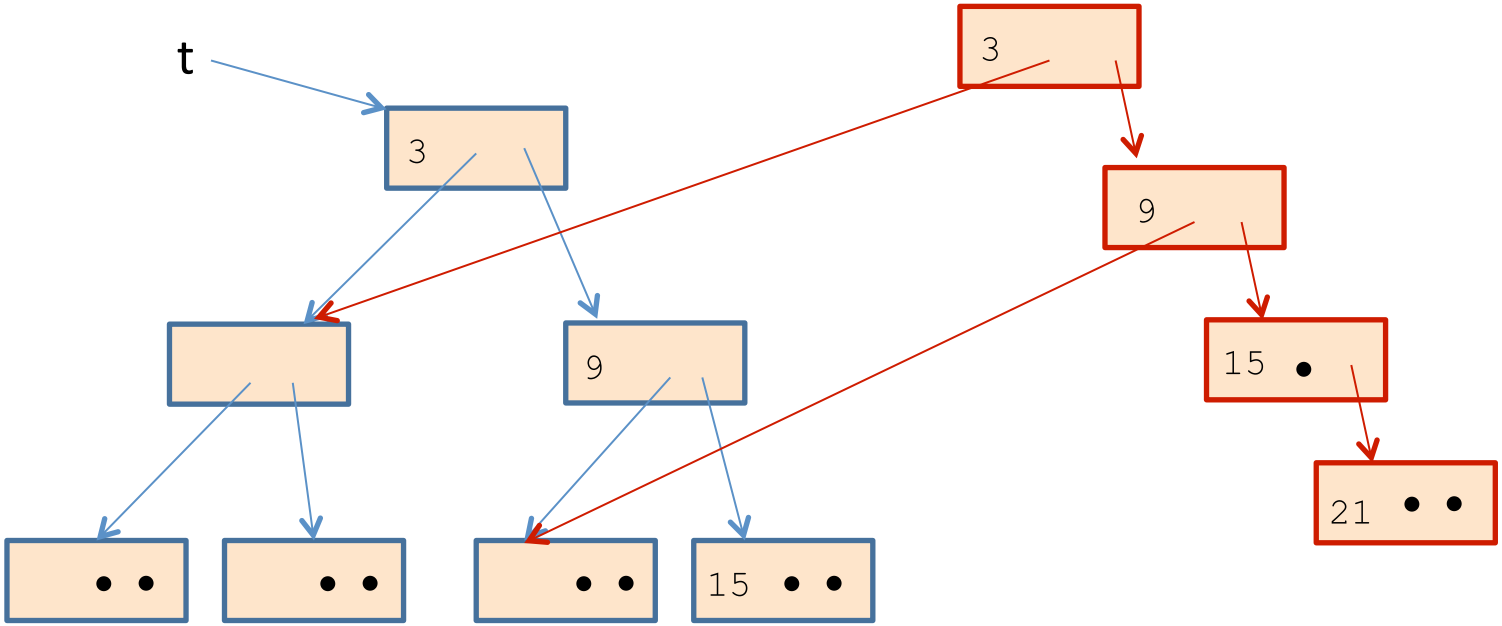
```

John McCarthy invented g.c. 1960



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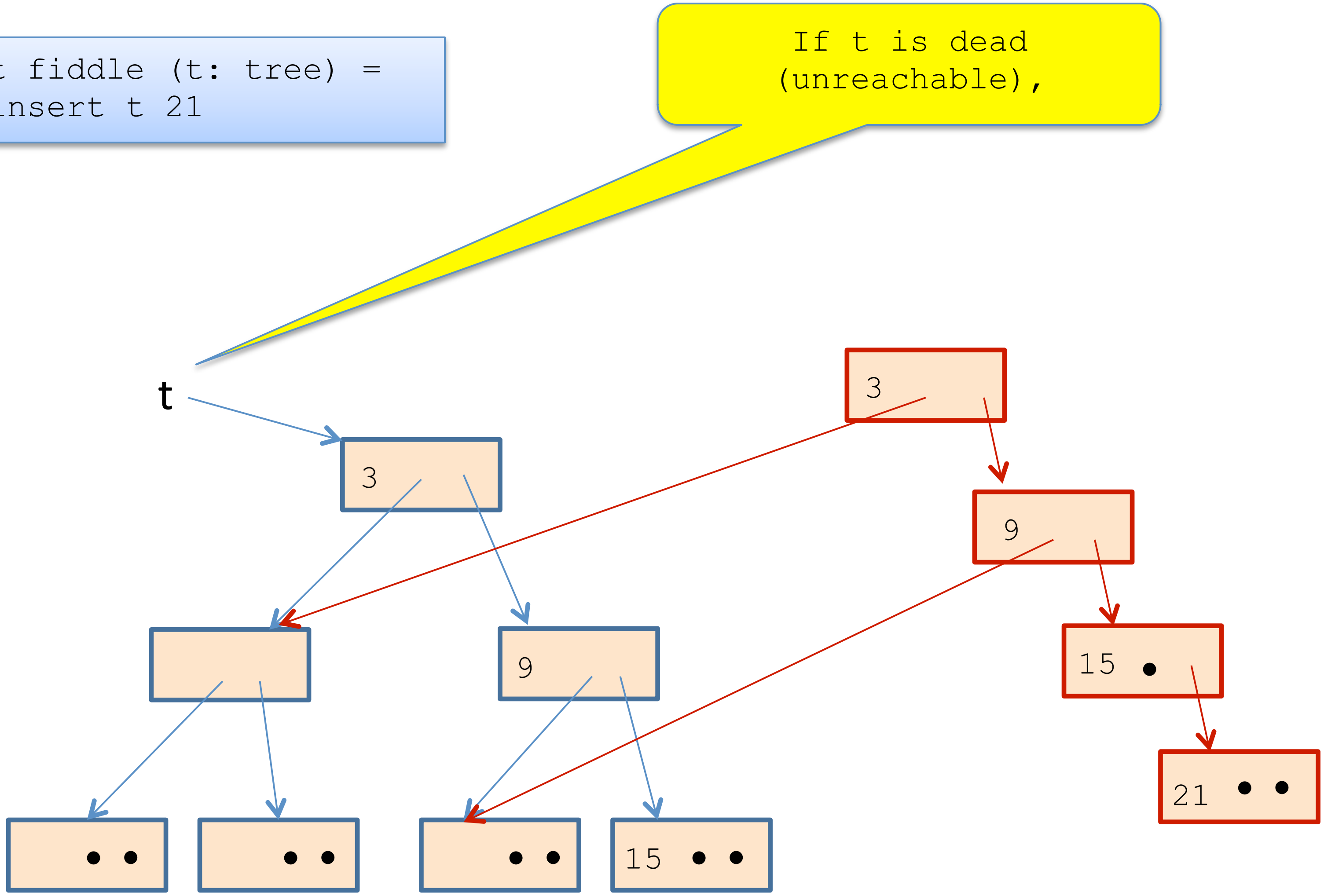

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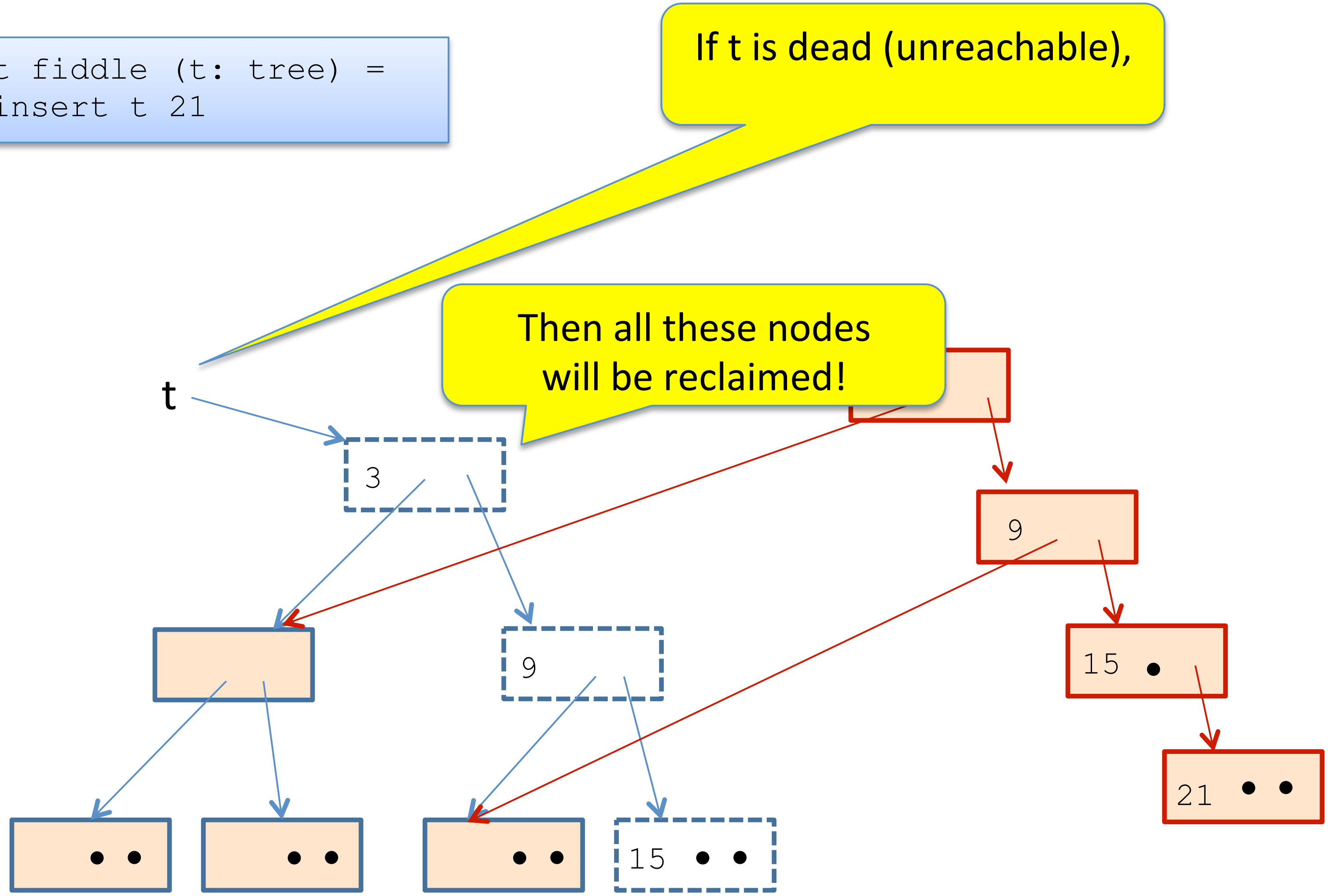

# Net space allocated

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```
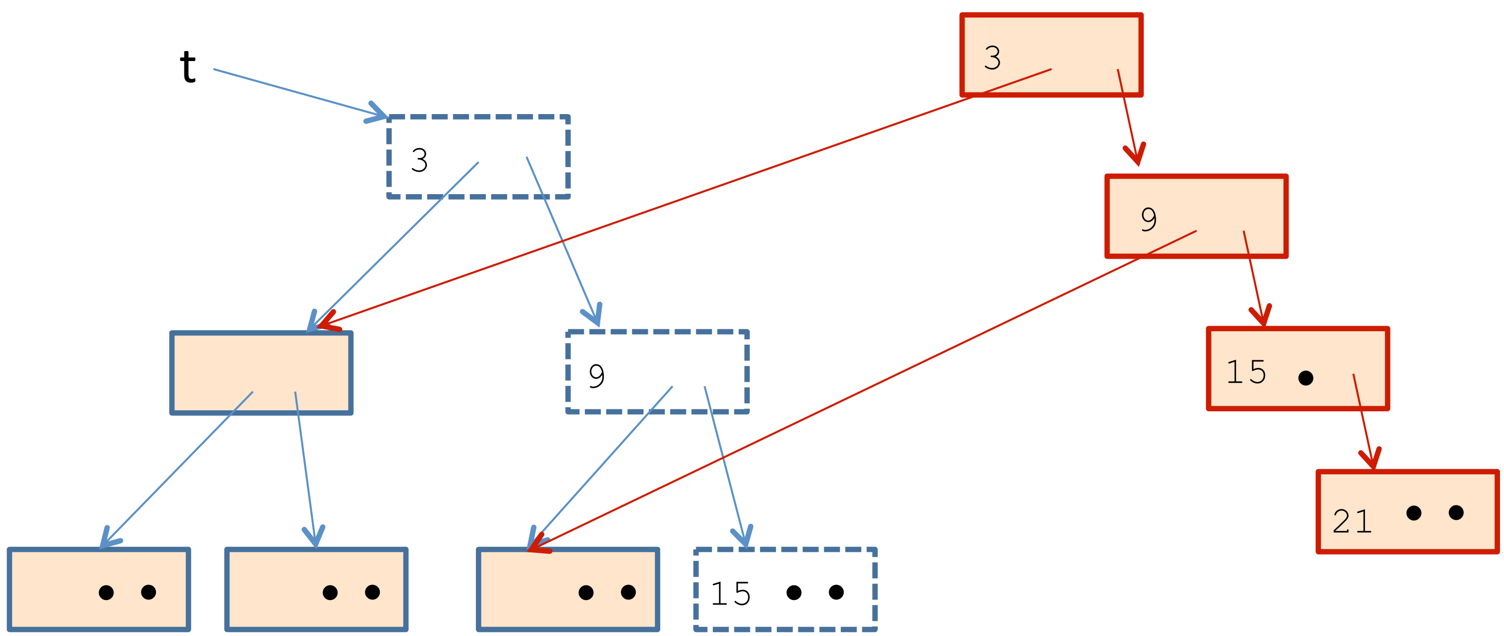

Net new space allocated:
1 node

(just like "imperative" version of binary search trees)



# Net space allocated

But what if you want to keep the old tree?

```
let faddle (t: tree) =
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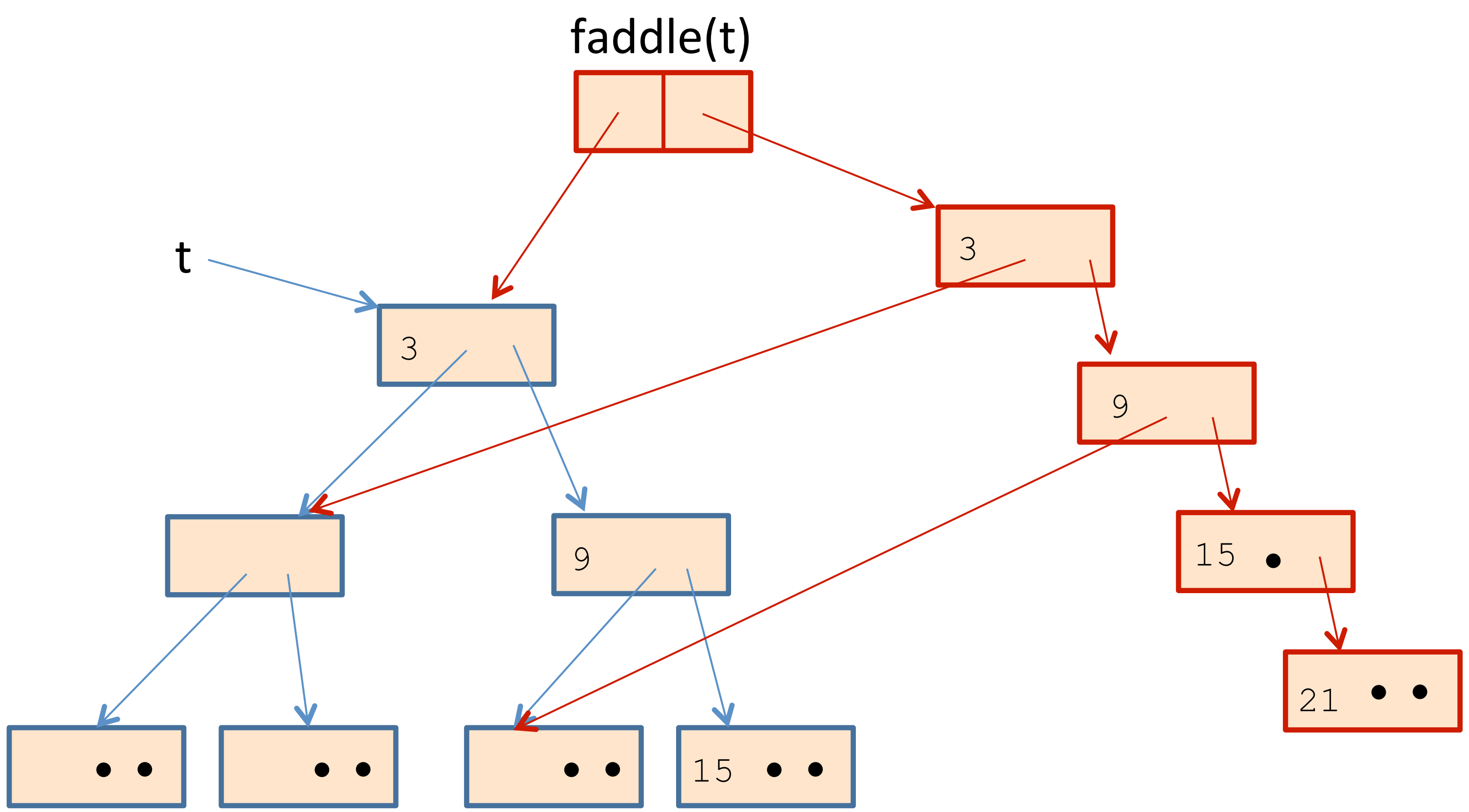

# Net space allocated

But what if you want to keep the old tree?

Net new space allocated:
log(N) nodes

but note: "imperative" version would have to copy the old tree, space cost N new nodes!

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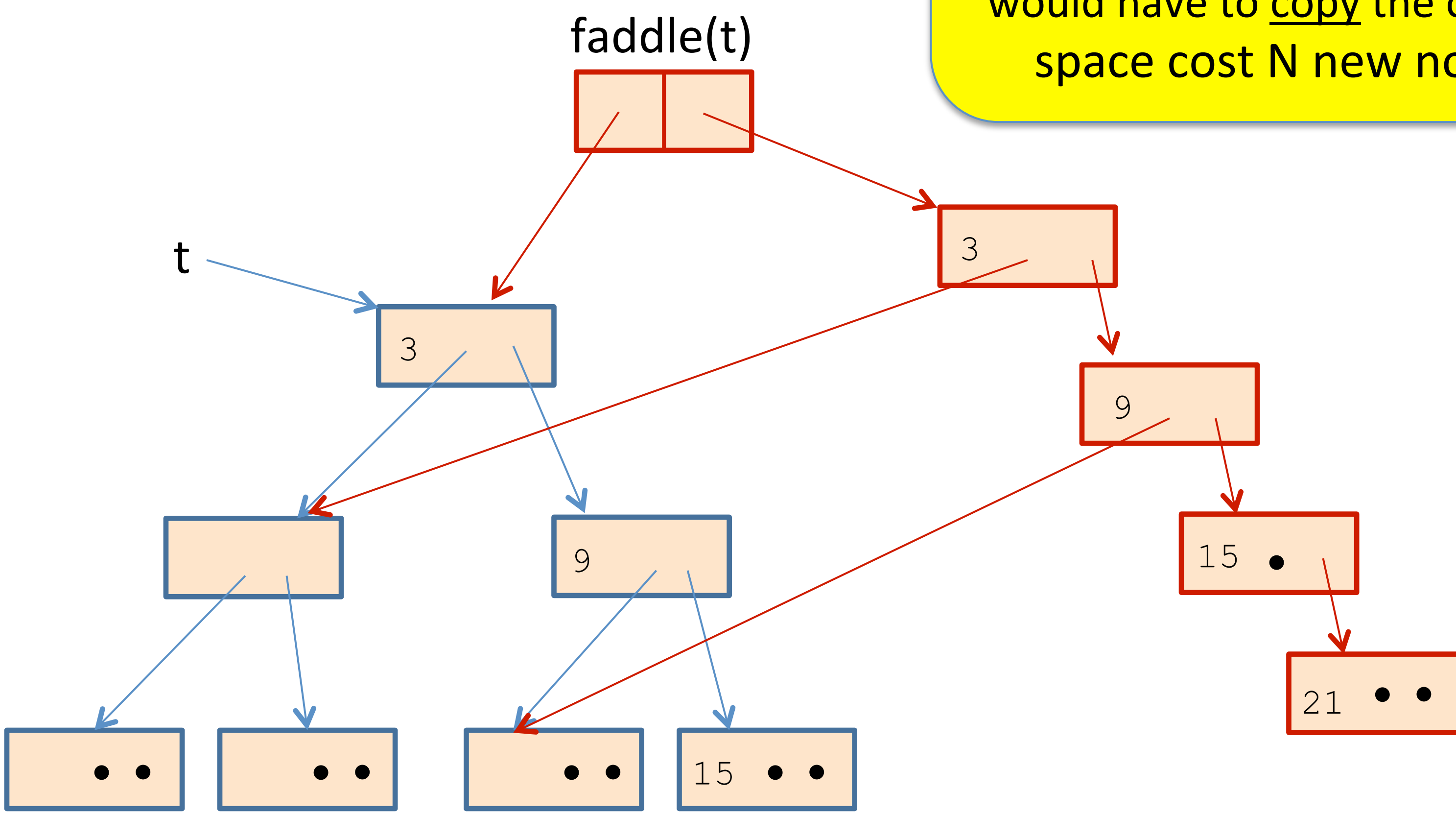

# Compare

```
let check_option (o:int option) : int option =
  match o with
    Some _ -> o
  | None -> failwith "found none"
;;
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let check_option (o:int option) : int option =
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allocates nothing when arg is Some i

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let cadd (c1:int*int) (c2:int*int) : int*int =
  let (x1,y1) = c1 in
  let (x2,y2) = c2 in
  (x1+x2, y1+y2)
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let double (c1:int*int) : int*int =
  let c2 = c1 in
  cadd c1 c2
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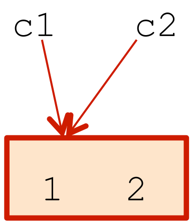


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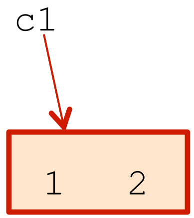

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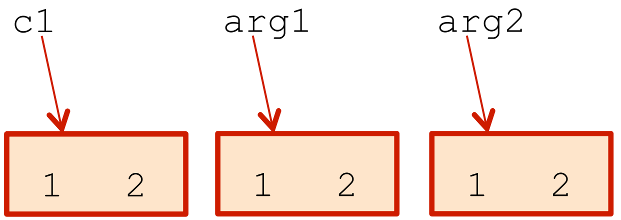

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allocates 2 pairs (unless the compiler happens to optimize…)

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double does not allocate

extracts components: it is a read

# FUNCTION CLOSURES

# Closures

Consider the following program:

```
let choose (arg:bool * int * int) : int -> int =
  let (b, x, y) = arg in
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choose (true, 1, 2);;
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# Substitution and Compiled Code

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compile →

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  ...
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```
choose:
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```
choose_subst:
  mov rb 0xF8[0]
  mov rx 0xF8[4]
  mov ry 0xF8[8]
  compare rb 0
  ...
  jmp ret
```

```
0xF8: 0
      1
      2
```

# Substitution and Compiled Code

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  mov rx
  mov ry
  ...
  jmp re

main:
  ...
  jmp choose
```

```
choose_subst:
  mov rb 0xF8[0]
  mo
  mo
  co
  ..
  jm
```

```
0xF8: 0
      1
```

```
choose_subst2:
  compare 1 0
  ...
  jmp ret
```

# What we aren't going to do

The substitution model of evaluation is *just a model*. It says that we generate new code at each step of a computation. We don't do that in reality. Too expensive!

The substitution model is a faithful model for reasoning about the relationship between inputs and outputs of a function but it doesn't tell us much about the resources that are used along the way.

I'm going to tell you a little bit about how ML programs are compiled so you can understand how much space your programs will use. Understanding the space consumption of your programs is an important component in making these programs more efficient.

# Compiling functions

General tactic: Reduce the problem of compiling ML-like functions to the problem of compiling C-like functions.

Some functions are already C-like:

```
let add (x:int*int) : int =
  let (y,z) = x in
  y + z
;;
```

→

```
# argument in r1
# return address in r0

add:
  ld r2, r1[0]      # y in r2
  ld r3, r1[4]      # z in r3
  add r4, r2, r3    # sum in r4
  jmp r0
```

# But what about nested, higher-order functions?

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x)
  else
    (fun n -> n + y)
;;
```

?

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    f1
  else
    f2
;;
```

?

```
let f1 n = n + x;;
```

?

```
let f2 n = n + y;;
```

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```

?

```
let f1 n = n + x;;
```

?

```
let f2 n = n + y;;
```

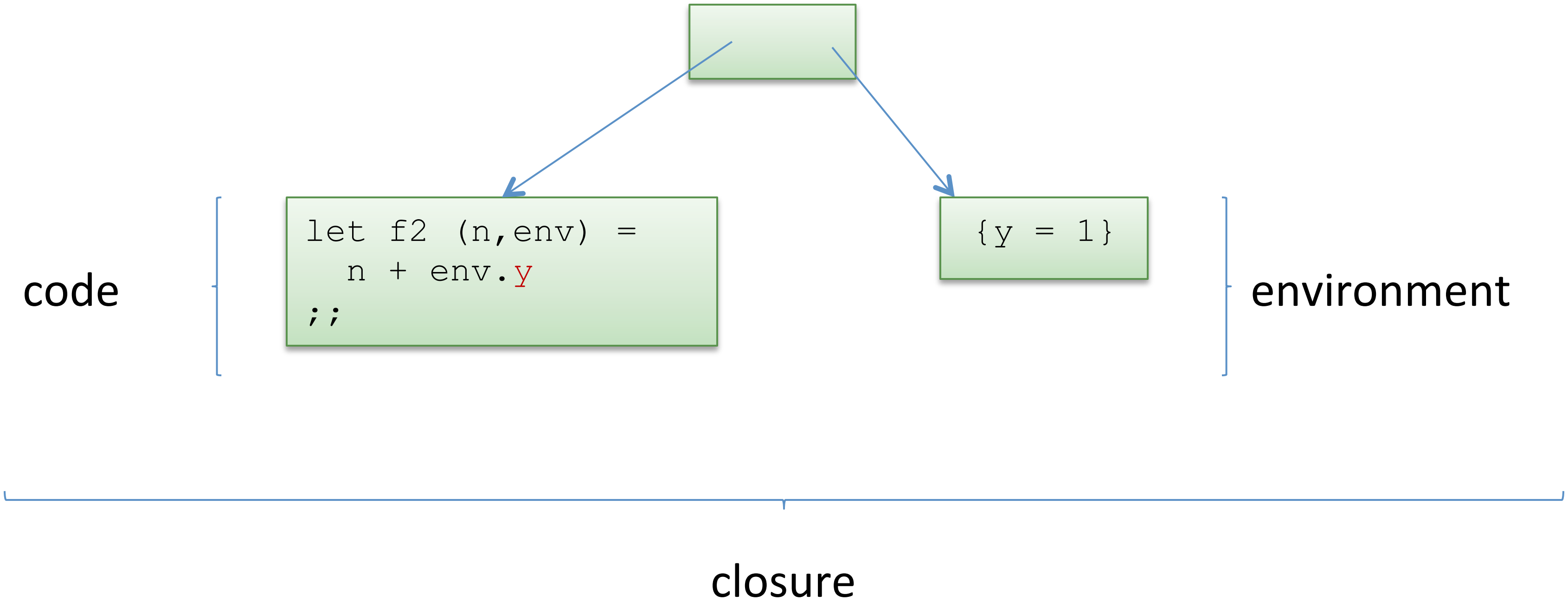
Darn!  *Doesn't work naively*. Nested functions contain *free variables*.
Simple unnesting leaves them undefined.

# But what about nested, higher-order functions?

We can't execute a function like the following:

```
let f2 n = n + y;;
```

But we can execute a *closure* which is a pair of some code and an environment:



# Closure Conversion

*Closure conversion* (also called lambda lifting) converts open, nested functions into closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

# Closure Conversion

Closure conversion (also called lambda lifting) converts open, nested functions in to closed, top-level functions.

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let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

add environment parameter

create closures

use environment variables instead of free variables

# Closure Conversion

Closure conversion converts open, nested functions in to closed, top-level functions.

```
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  let (b, x, y) = arg in
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;;
```

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

add environment parameter

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure        = (choose, ())           in (* create closure *)
let (c_code, c_cenv) = c_closure              in (* extract code, env *)
let f_closure        = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
let (f_code, f_env)  = f_closure              in (* extract code, env *)
f_code (3, f_env)                                (* call f code *)
;;
```

# Closure Conversion

Closure conversion converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

add environment parameter

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

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(choose (true,1,2)) 3
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```
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let (f_code, f_env)    = f_closure                in (* extract code, env *)
f_code (3, f_env)                                    (* call f code *)
```

# Closure Conversion

Closure conversion converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

add environment parameter

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure          = (choose, ())              in (* create closure *)
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let f_closure          = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
let (f_code, f_env)    = f_closure                 in (* extract code, env *)
f_code (3, f_env)                                     (* call f code *)
```

# Closure Conversion

Closure conversion converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

add environment parameter

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure          = (choose, ())               in (* create closure *)
let (c_code, c_cenv) = c_closure                  in (* extract code, env *)
let f_closure          = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
let (f_code, f_env)  = f_closure                  in (* extract code, env *)
f_code (3, f_env)                                    (* call f code *)
```

# One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't—because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, F1 {xe=x; ye=y})
  else
    (f2, F2 {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

```
type f1_env = {x1:int; y1:int}      type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {y2:int}              type f2_clos = (int * f2_env -> int) * f2_env
```

# One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, F1 {xe=x; ye=y})
  else
    (f2, F2 {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

Solution 0: Don't bother to typecheck after closure conversion.

After all, the source program was well typed (checked by the source-language ML typechecker), and the compiler (with its *closure conversion* algorithm) cannot possibly have produced a program with the wrong behavior.

That is, consider the post-closure-converted language to be an *untyped* language.

This is the traditional solution, and it's not stupid. But can we do better?

# One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, F1 {x1=x; y2=y})
  else
    (f2, F2 {y2=y})
;;
```

```
let f1 (n,env) =
  match env with
    F1 e -> n + e.x1 + e.y2
  | F2 _ -> failwith "bad env!"
;;
```

```
let f2 (n,env) =
  match env with
    F1 _ -> failwith "bad env!"
  | F2 e -> n + e.y2
;;
```

```
type f1_env = {x1:int; y1:int}     type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {y2:int}             type f2_clos = (int * f2_env -> int) * f2_env
```

fix I:

```
type env = F1 of f1_env | F2 of f2_env
type f1_clos = (int * env -> int) * env
type f2_clos = (int * env -> int) * env
```

# One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

```
type f1_env = {xe:int; ye:int}      type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {xe:int}              type f2_clos = (int * f2_env -> int) * f2_env
```

fix II:

```
type f1_env = {xe:int; ye:int}
type f2_env = {xe:int}
type f1_clos = ∃env.(int * env -> int) * env
type f2_clos = ∃env.(int * env -> int) * env
```

# One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

"*From System F to Typed Assembly Language*,"
-- Morrisett, Walker et al.

```
type f1_env = {xe:int; ye:int}     type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {xe:int}             type f2_clos = (int * f2_env -> int) * f2_env
```

fix II:

```
type f1_env = {xe:int; ye:int}
type f2_env = {xe:int}
type f1_clos = ∃env.(int * env -> int) * env
type f2_clos = ∃env.(int * env -> int) * env
```

# Aside: Existential Types

map has a *universal* polymorphic type:

map : ('a -> 'b) -> 'a list -> 'b list        "for *all* types 'a and for *all* types 'b, ..."

when we closure-convert a function that has type int -> int, we get a function with *existential* polymorphic type:

∃ 'a. ((int * 'a) -> int) * 'a        "there *exists some* type 'a such that, ..."

In OCaml, we can approximate existential types using datatypes (a data type allows you to say "there exists a type 'a drawn from one of the following finite number of options."  In Haskell, you've got the real thing.

# Closure Conversion: Summary

| (before) | (after) |
| --- | --- |

All function definitions equipped with extra env parameter:

| `let f arg = ...` | `let f_code (arg, env) = ...` |
| --- | --- |

All free variables obtained from parameters or environment:

| `x` | `env.cx` |
| --- | --- |

All functions values paired with environment:

| `f` | `(f_code, {cx1=v1; ...; cxn=vn})` |
| --- | --- |

All function calls extract code and environment and call code:

| `f e` | `let (f_code, f_env) = f in`<br>`f_code (e, f_env)` |
| --- | --- |

# The Space Cost of Closures

The space cost of a closure

= the cost of the pair of code and environment pointers (2 words)

+ the cost of the data referred to by function free variables

(1 word for each free variable)

# Assignment #4

An environment-based interpreter:

- Instead of substitution, build up environment.
  - just a list of variable-value pairs
- When you reach a free variable, look in environment for its value.
- To evaluate a recursive function, create a closure data structure
  - pair current environment with recursive code
- To evaluate a function call, extract environment and code from closure, pass environment and argument to code

# TAIL CALLS AND CONTINUATIONS

# Some Innocuous Code

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

Let’s try it.

(Go to tail.ml)

# Some Other Code

Four functions:  Green works on big inputs; Red doesn't.

```
let sum_to2 (n: int) : int =
  let rec aux (n:int) (a:int) : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

```
let rec sum2 (l:int list) : int =
  match l with
      [] -> 0
    | hd::tail -> hd + sum2 tail
;;
```

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;
```

```
let sum (l:int list) : int =
  let rec aux (l:int list) (a:int) : int =
    match l with
        [] -> a
      | hd::tail -> aux tail (a+hd)
  in
  aux l 0
;;
```

# Some Other Code

Four functions: Green works on big inputs; Red doesn't.

```
let sum_to2 (n: int) : int =
  let rec aux (n:int) (a:int) : int =
    if n > 0 then
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    else a
  in
  aux n 0
;;
```

```
let rec sum2 (l:int list) : int =
  match l with
      [] -> 0
    | hd::tail -> hd + sum2 tail
;;
```

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;
```

```
let sum (l:int list) : int =
  let rec aux (l:int list) (a:int) : int =
    match l with
        [] -> a
      | hd::tail -> aux tail (a+hd)
  in
  aux l 0
;;
```

code that works:
*no computation after recursive function call*

# Tail Recursion

A *tail-recursive function* does no work after it calls itself recursively.

Not tail-recursive, the substitution model:

```
    sum_to 1000000
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

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A *tail-recursive function* does no work after it calls itself recursively.

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```
    sum_to 1000000
-->
    1000000 + sum_to 99999
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

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A *tail-recursive function* does no work after it calls itself recursively.

Not tail-recursive, the substitution model:

```
    sum_to 1000000
-->
    1000000 + sum_to 99999
-->
    1000000 + 99999 + sum_to 99998
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

expression size grows
at every recursive call ...

lots of adding to do after
the call returns"

# Tail Recursion

A *tail-recursive function* does no work after it calls itself recursively.

Not tail-recursive, the substitution model:

```
    sum_to 1000000
-->
    1000000 + sum_to 99999
-->
    1000000 + 99999 + sum_to 99998
-->
    ...
-->
    1000000 + 99999 + 99998 + ... + sum_to 0
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

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A *tail-recursive function* does no work after it calls itself recursively.

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```
    sum_to 1000000
-->
    1000000 + sum_to 99999
-->
    1000000 + 99999 + sum_to 99998
-->
    ...
-->
    1000000 + 99999 + 99998 + ... + sum_to 0
-->
    1000000 + 99999 + 99998 + ... + 0
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

recursion
finally bottoms out

# Tail Recursion

A *tail-recursive function* does no work after it calls itself recursively.

Not tail-recursive, the substitution model:

```
    sum_to 1000000
-->
    1000000 + sum_to 99999
-->
    1000000 + 99999 + sum_to 99998
-->
    ...
-->
    1000000 + 99999 + 99998 + ... + sum_to 0
-->
    1000000 + 99999 + 99998 + ... + 0
-->
    ... add it all back up ...
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

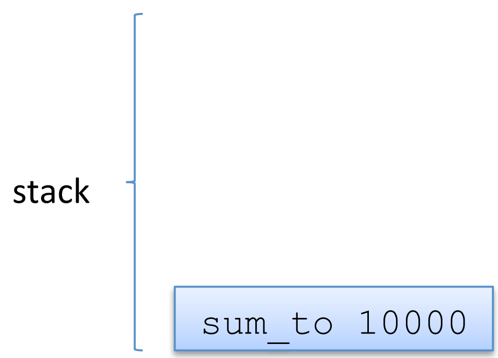
sum big_int;;
```

do a long series
of additions to get
back an int

# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
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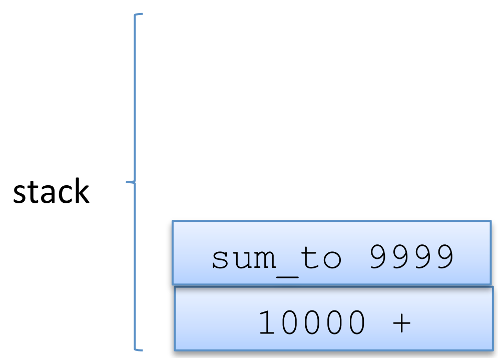
;;

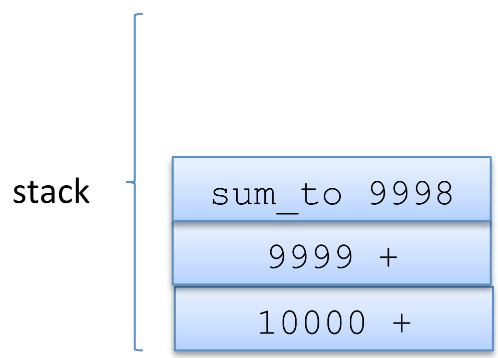
sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

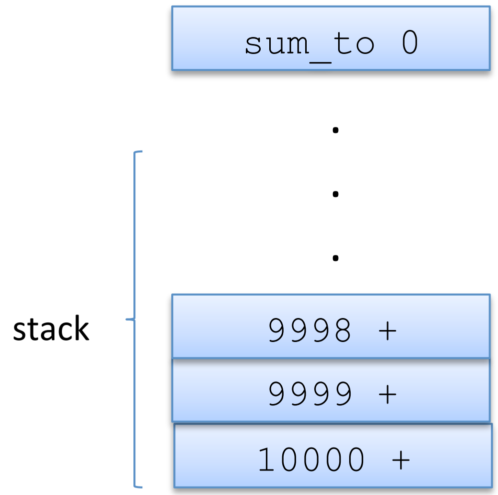
sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

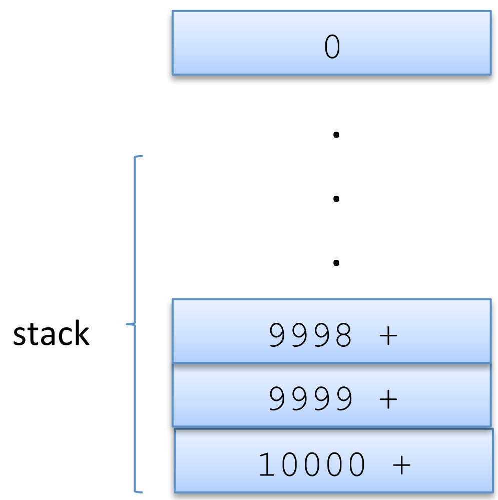
sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

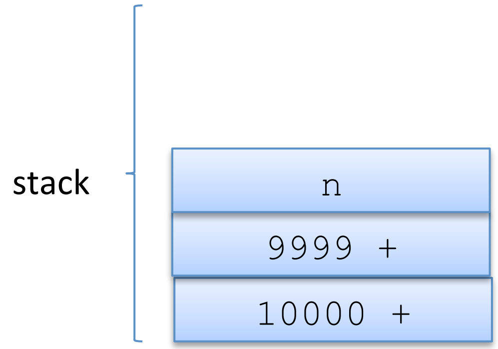
sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

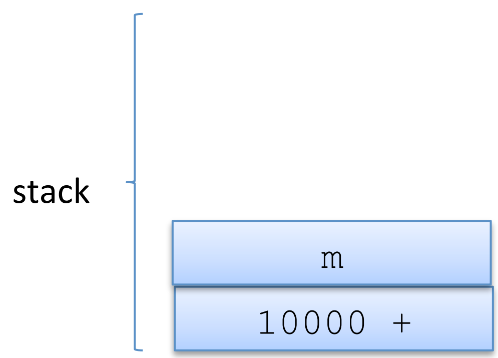
sum_to 10000
```



# Non-tail recursive

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

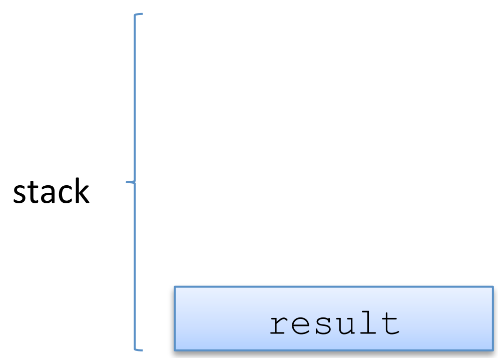
sum_to 10000
```



# Non-tail recursive

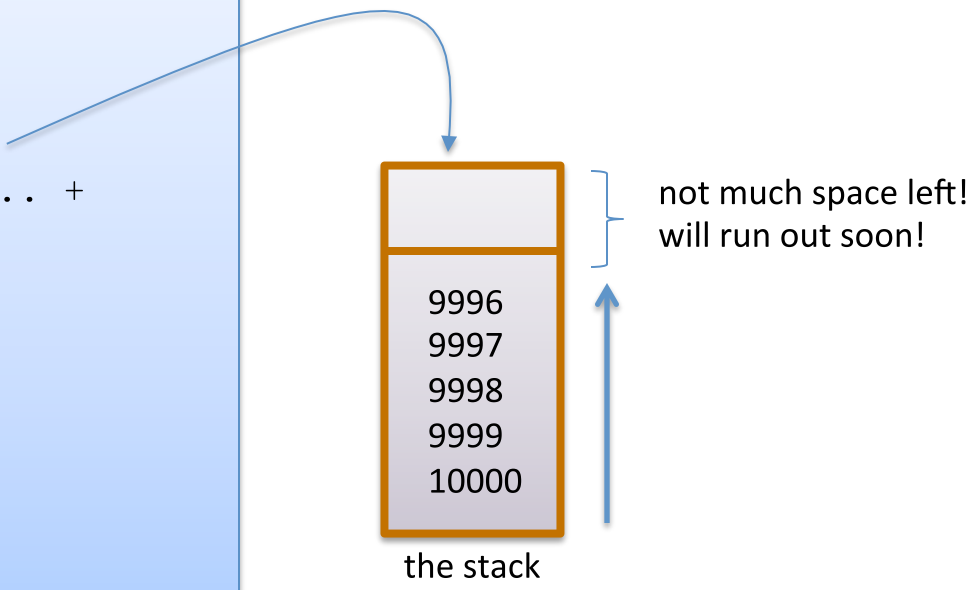
```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

sum_to 100
```



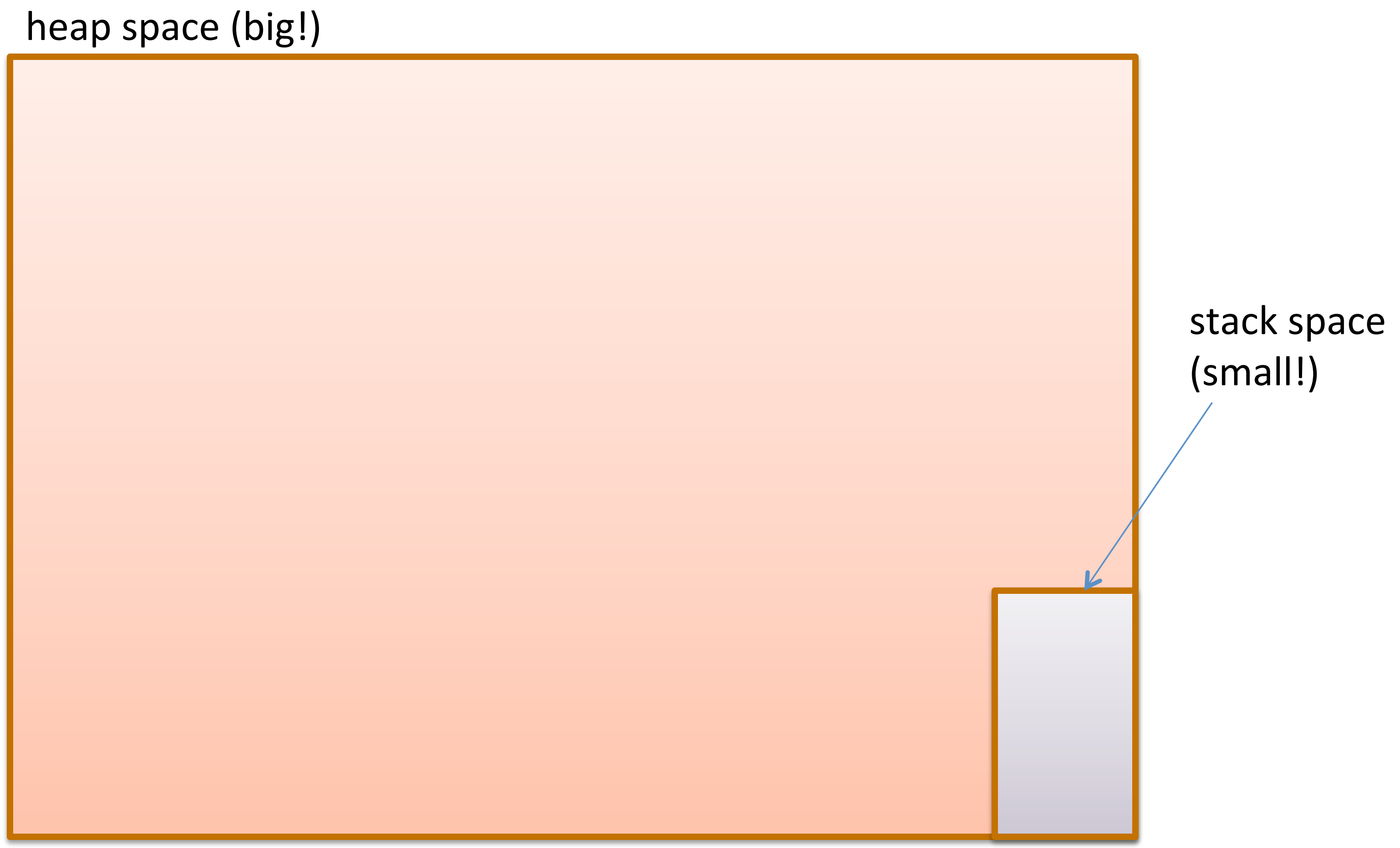
# Data Needed on Return Saved on Stack

```
    sum_to 10000
-->
    ...
-->
    10000 + 9999 + 9998 + 9997 + ... +
-->
    ...
-->
    ...
```




every non-tail call puts the data from the calling context on the stack

# Memory is partitioned: Stack and Heap



# Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

Tail-recursive:

```
sum_to2 1000000
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                  : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

# Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

Tail-recursive:

```
    sum_to2 1000000
-->
    aux 1000000 0
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                  : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

# Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

Tail-recursive:

```
   sum_to2 1000000
-->
   aux 1000000 0
-->
   aux 99999 1000000
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
               : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

# Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

Tail-recursive:

```
    sum_to2 1000000
-->
    aux 1000000 0
-->
    aux 99999 1000000
-->
    aux 99998 1999999
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                  : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

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Tail-recursive:

```
    sum_to2 1000000
-->
    aux 1000000 0
-->
    aux 99999 1000000
-->
    aux 99998 1999999
-->
    ...
-->
    aux 0 (-363189984)
-->
    -363189984
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                 : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

constant size expression in the substitution model

(addition overflow occurred at some point)

# Tail Recursion

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```
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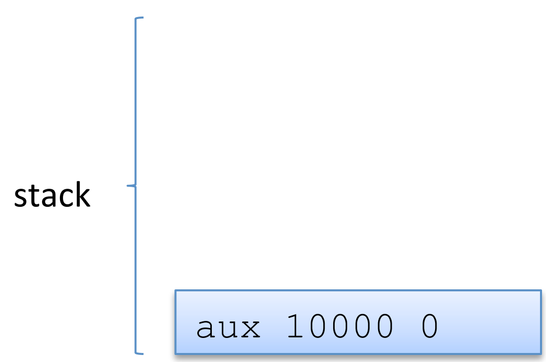

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    else a
  in
  aux n 0
;;
```

stack

```
aux 9999 10000
```

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```
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;;
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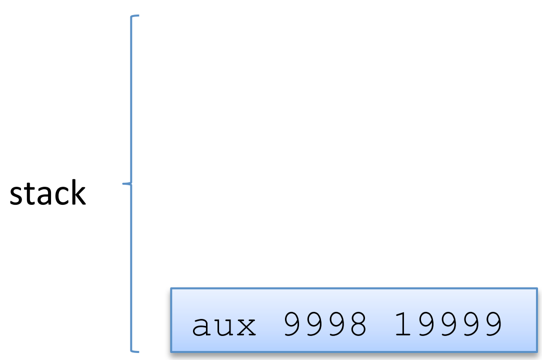

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(* sum of 0..n *)

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  in
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;;
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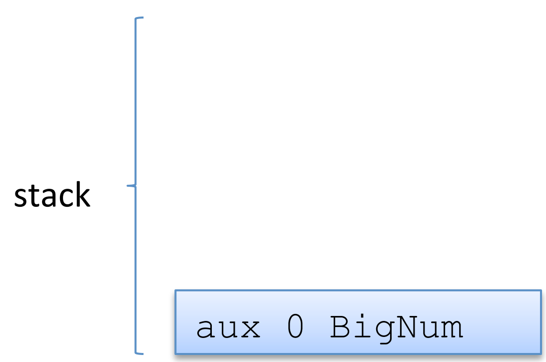

# Question

We used human ingenuity to do the tail-call transform.

Is there a mechanical procedure to transform *any* recursive function in to a tail-recursive one?

not only is sum2 tail-recursive but it reimplements an algorithm that took *linear space* (on the stack) using an algorithm that executes in *constant space*!

```
let rec sum_to (n: int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;
```

human ingenuity

```
let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int) : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

# CONTINUATION-PASSING STYLE
# CPS!

# CPS

CPS:

- Short for *Continuation-Passing Style*
- Every function takes a *continuation* (a function) as an argument that expresses "what to do next"
- CPS functions only call other functions as the last thing they do
- All CPS functions are tail-recursive

Goal:

- Find a mechanical way to translate any function in to CPS

# Serial Killer or PL Researcher?

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Gordon Plotkin
Programming languages researcher
Invented CPS conversion.

Call-by-Name, Call-by Value
and the Lambda Calculus. TCS, 1975.

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Serial Killer

Killed a teenager at a campsite
in the Adirondacks in 1974.
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# Question

Can any non-tail-recursive function be transformed in to a tail-recursive one? Yes, if we can capture the *differential* between a tail-recursive function and a non-tail-recursive one.

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

Idea: Focus on what happens after the recursive call.

# Question

Can any non-tail-recursive function be transformed in to a tail-recursive one? Yes, if we can capture the *differential* between a tail-recursive function and a non-tail-recursive one.

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

what happens next

Idea: Focus on what happens after the recursive call.

Extracting that piece:

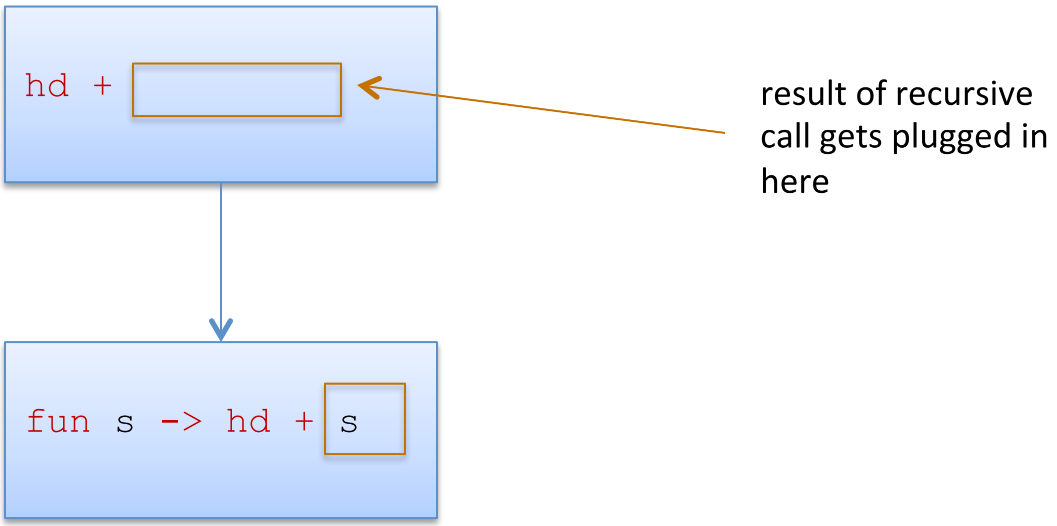
```
hd + [          ]
```

result of recursive call gets plugged in here

How do we capture it?

# Question

How do we capture that computation?



# Question

How do we capture that computation?

```
hd + [        ]
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> ???) ;;
```

# Question

How do we capture that computation?

```
hd + [      ]
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
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;;
```

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type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
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  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;
```

# Question

How do we capture that computation?

```
hd + [      ]
```

```
fun s -> hd + [s]
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + [sum tail]
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = ??
```

# Question

How do we capture that computation?

```
hd + [    ]
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
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;;
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let sum (l:int list) : int = sum_cont l (fun s -> s)
```

# Execution

```
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```

```
    sum [1;2]
```

# Execution

```
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let rec sum_cont (l:int list) (k:cont): int =
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  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

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```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
```

# Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
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let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
```

# Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
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```

```
    sum [1;2]
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    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
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-->
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-->
    (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s)) 0
```

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let rec sum_cont (l:int list) (k:cont): int =
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let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
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-->
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-->
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-->
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```

# Execution

```
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```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
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-->
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```

# Execution

```
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-->
    (fun s -> (fun s -> s) (1 + s)) (2 + 0))
-->
    (fun s -> s) (1 + (2 + 0))
-->
    1 + (2 + 0)
-->
    3
```

# Question

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
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let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    ...
-->
    3
```

Where did the stack space go?

```
sum_cont []
  (fun s3 ->
    (fun s2 ->
      (fun s1 -> s1) (hd1 + s2)
    ) (hd2 + s3)
  )
```

function inside function inside function inside expression

→

each function is a closure; points to the closure inside it

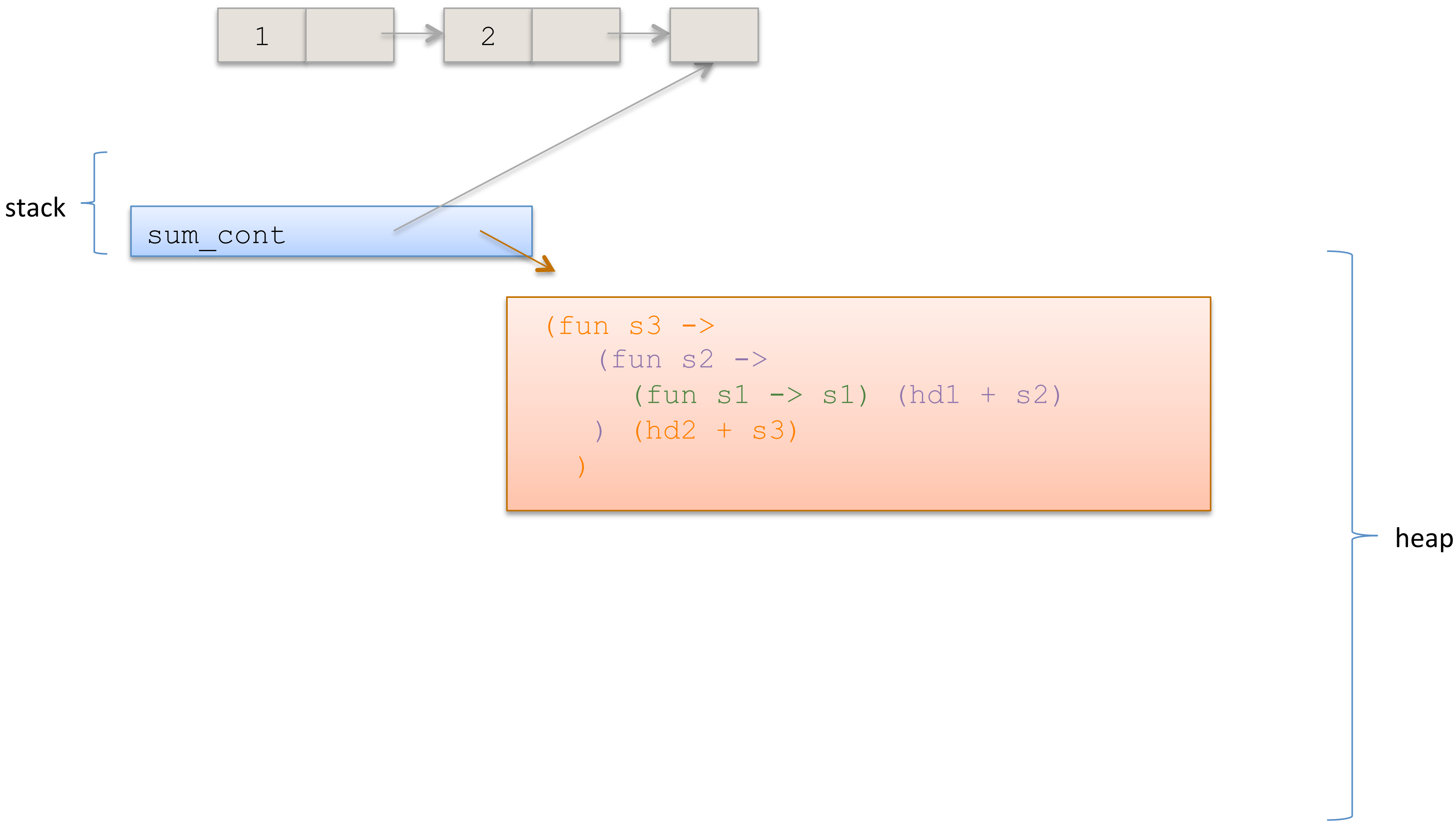
→

a stack of closures on the heap

function inside
function inside
function inside
expression

⇒ a stack of closures on the heap

```
sum_cont []
  (fun s3 ->
    (fun s2 ->
      (fun s1 -> s1) (hd1 + s2)
    ) (hd2 + s3)
  )
```



function inside
function inside
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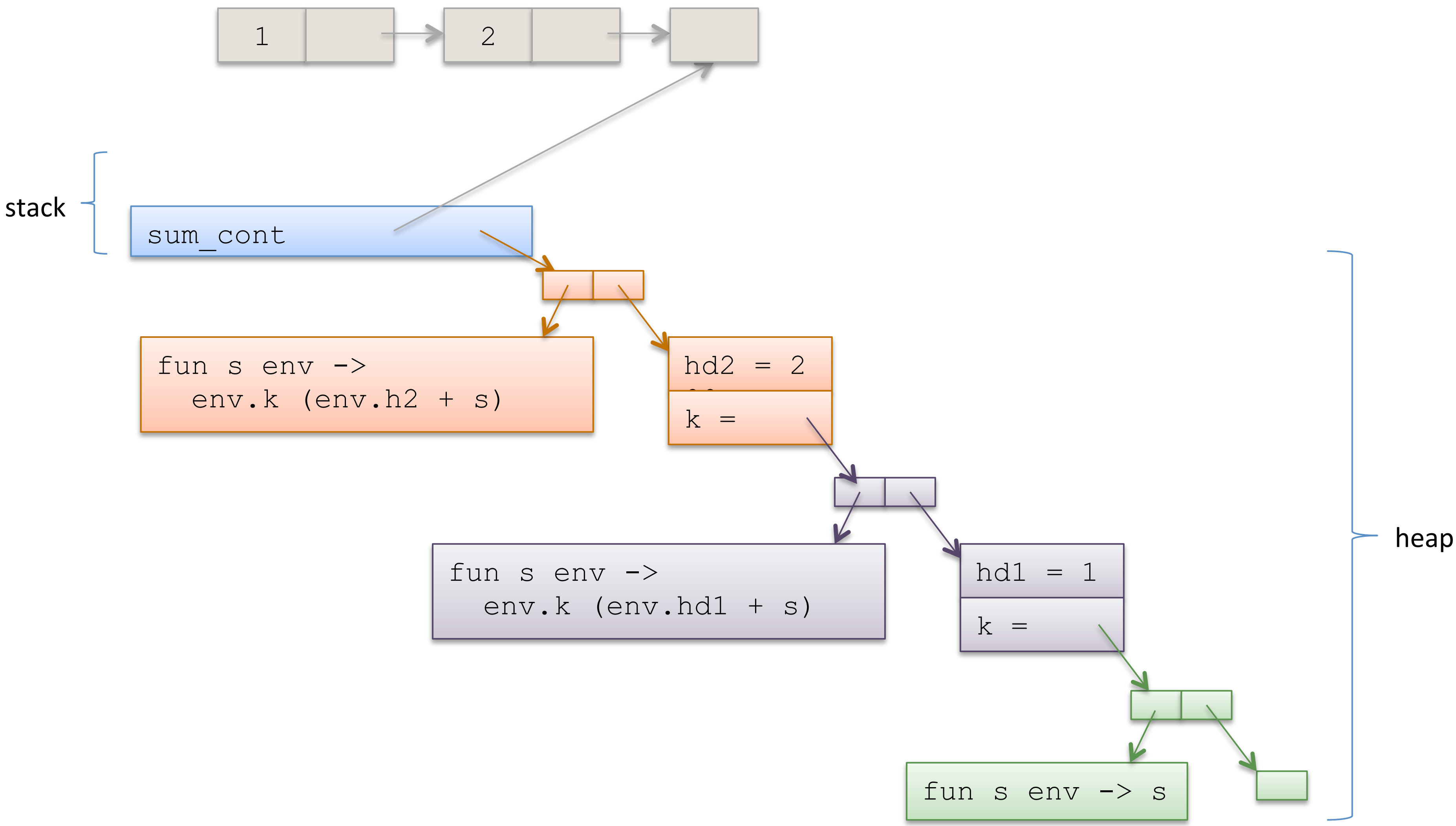

# Continuation-passing style

```
let rec sum_cont (l:int list) (k:cont): int =
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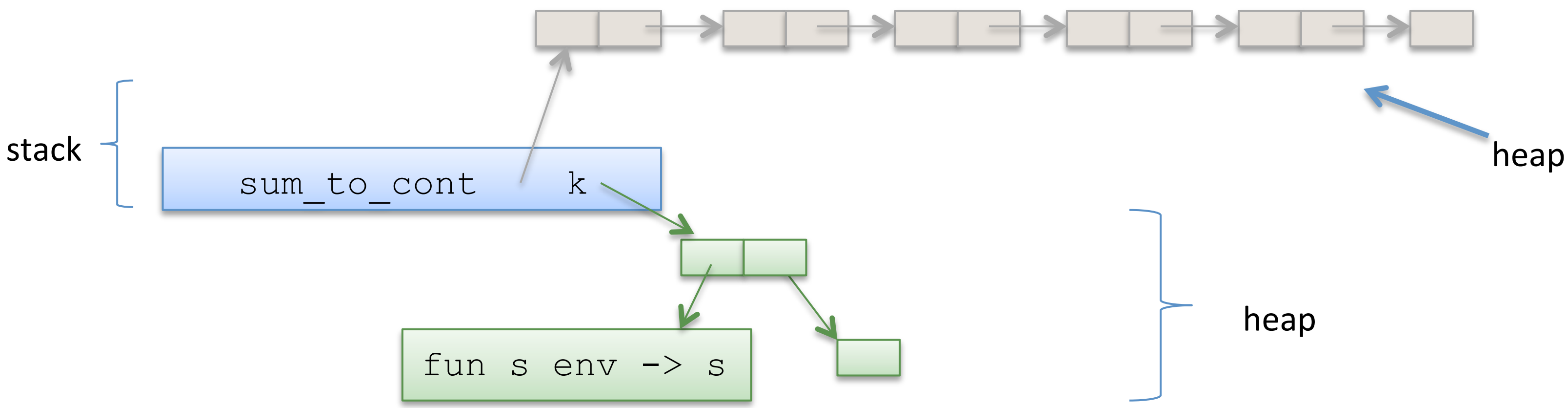

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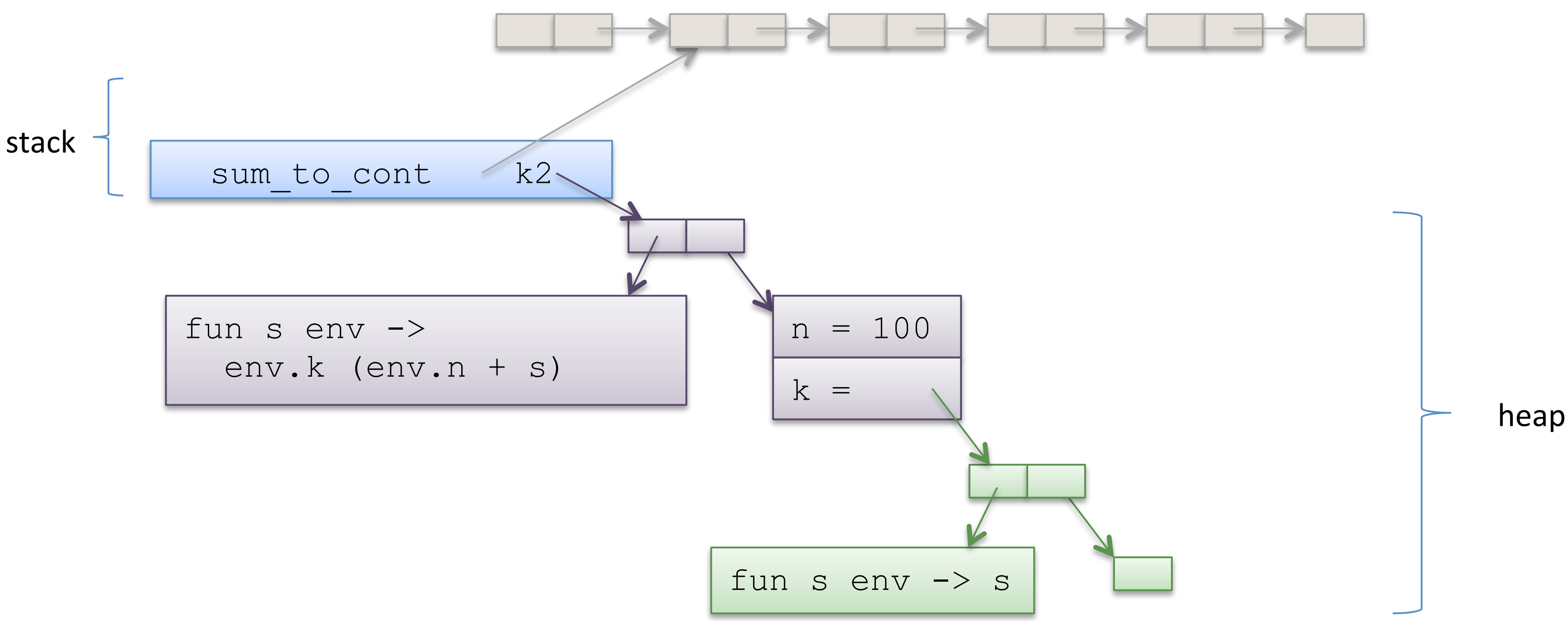
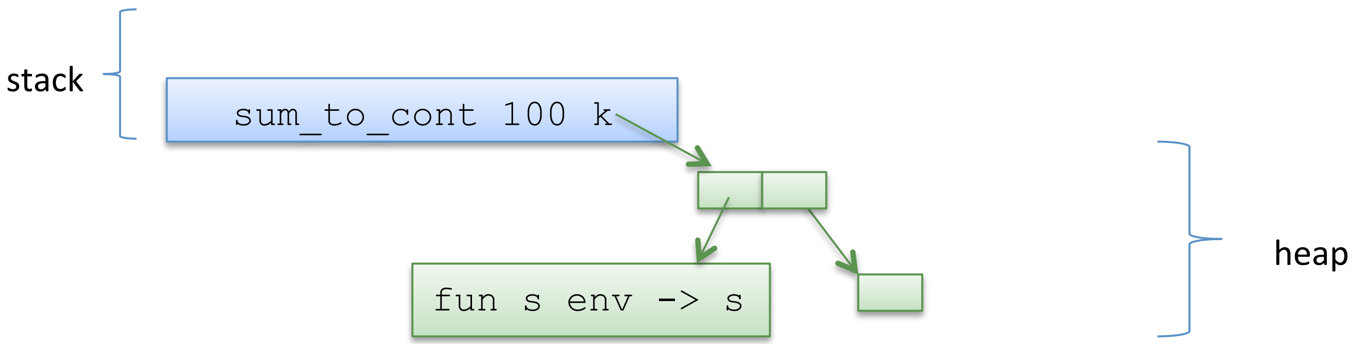


# Continuation-passing style

```
let rec sum_to_cont (n:int) (k:int->int) : int =
  if n > 0 then
    sum_to_cont (n-1) (fun s -> k (n+s))
  else
    k 0  ;;

sum_to_cont 100 (fun s -> s)
```



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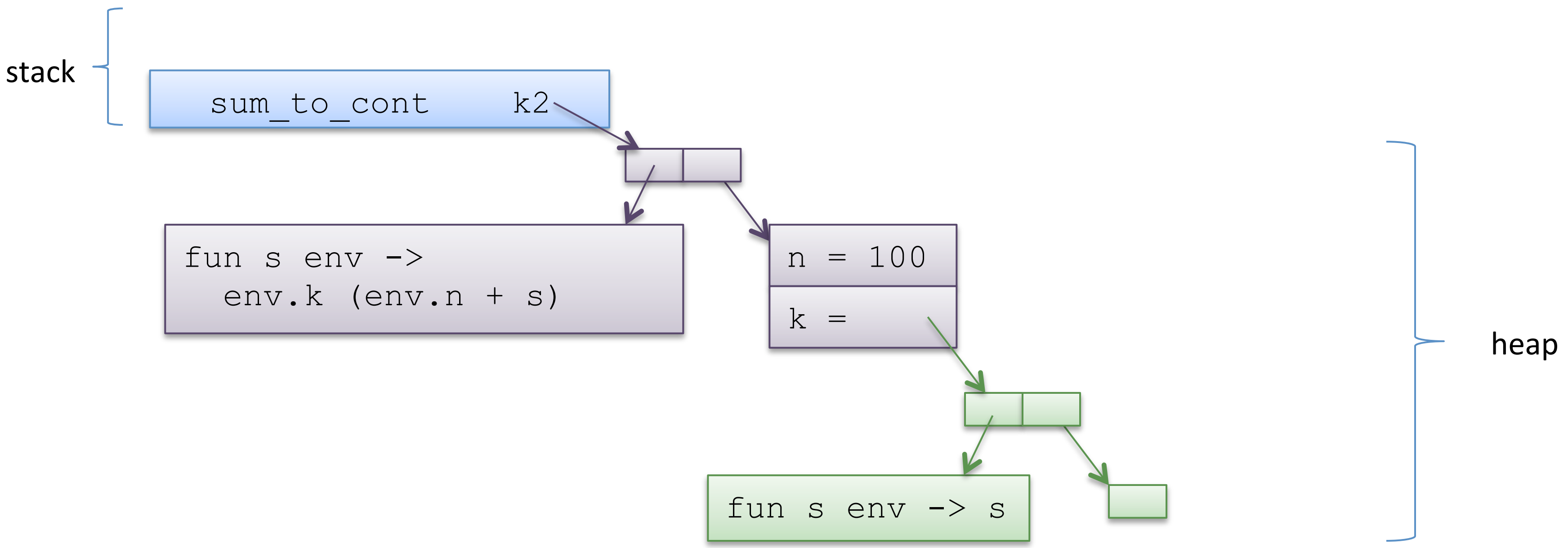

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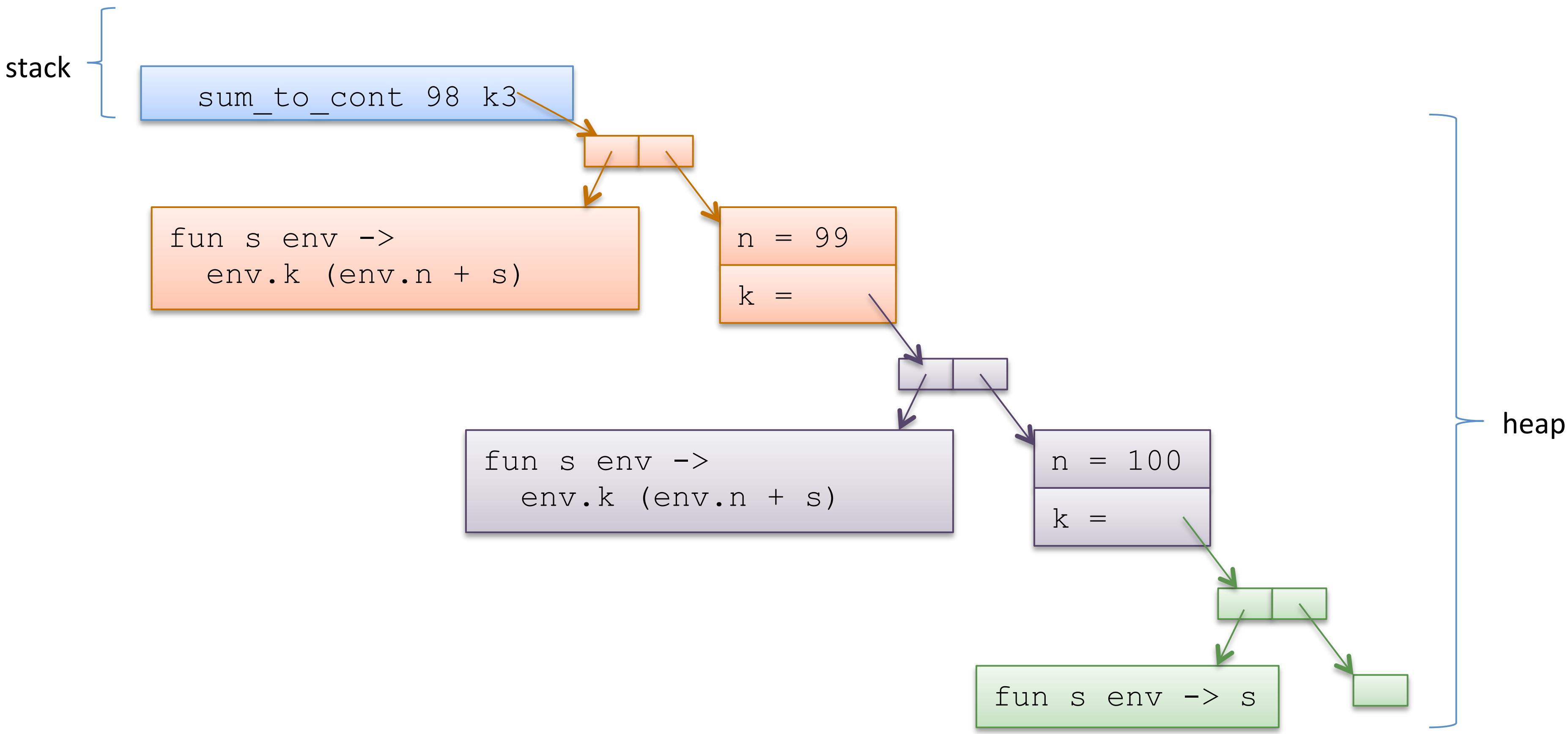
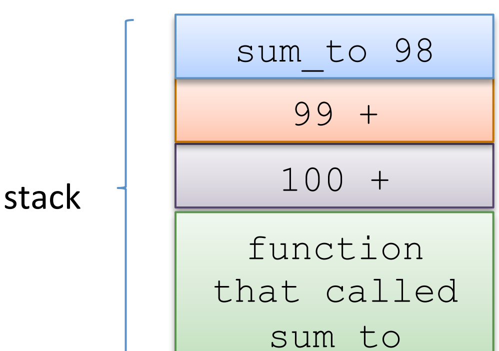


# Back to stacks

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;

sum_to 100
```



# Back to stacks

```
let rec sum_to (n:int) : int =
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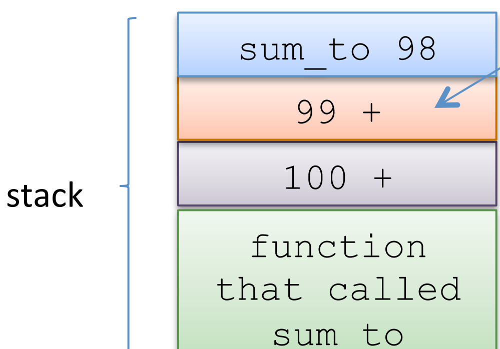
  else
    0
;;

sum_to 100
```

but how do you really implement that?



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```
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```

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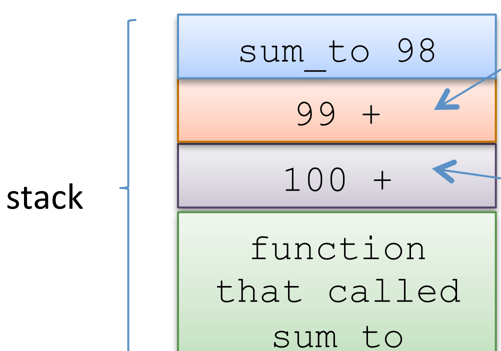


there is two bits of information here:

(1) some state (n=100) we had to remember
(2) some code we have to run later

# Back to stacks

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
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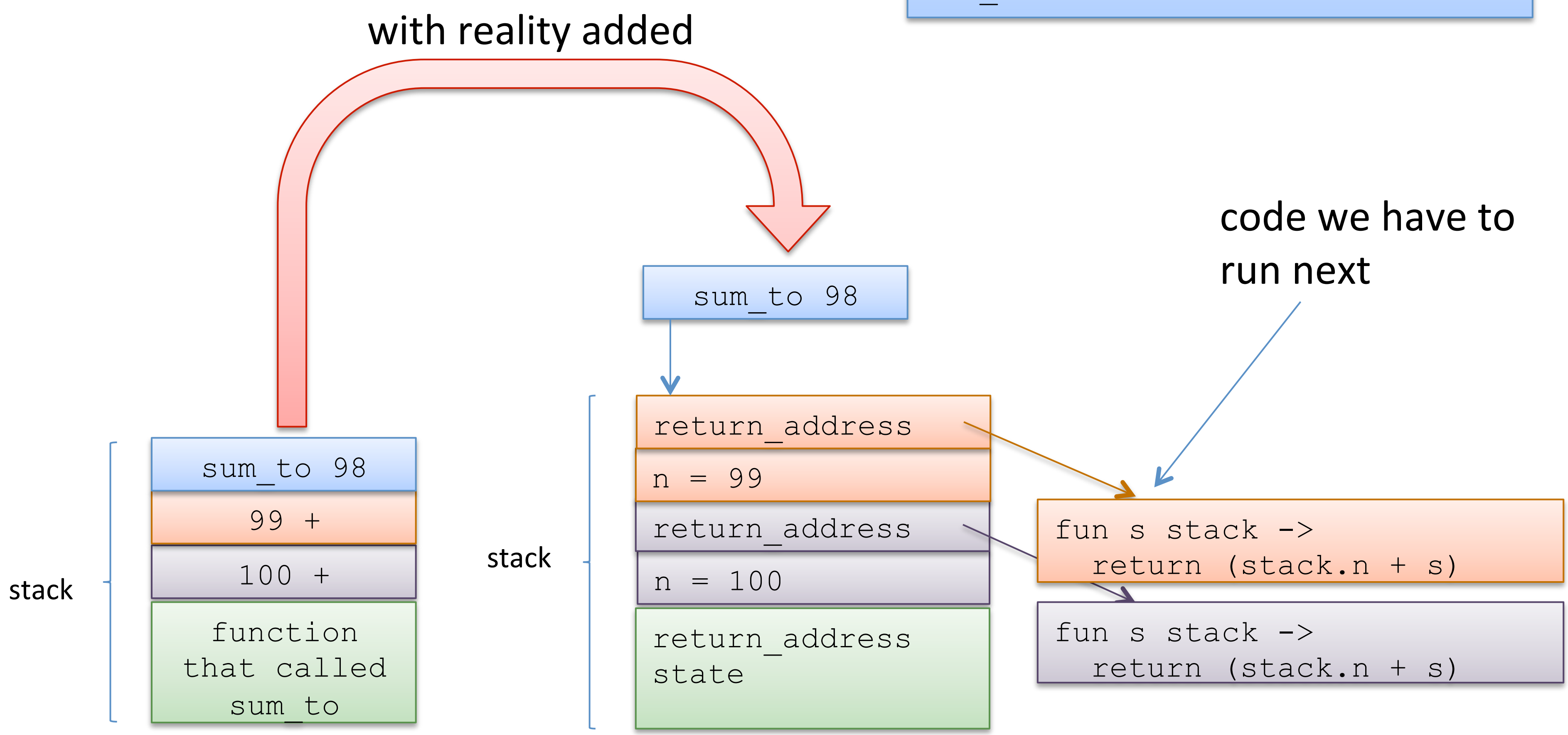
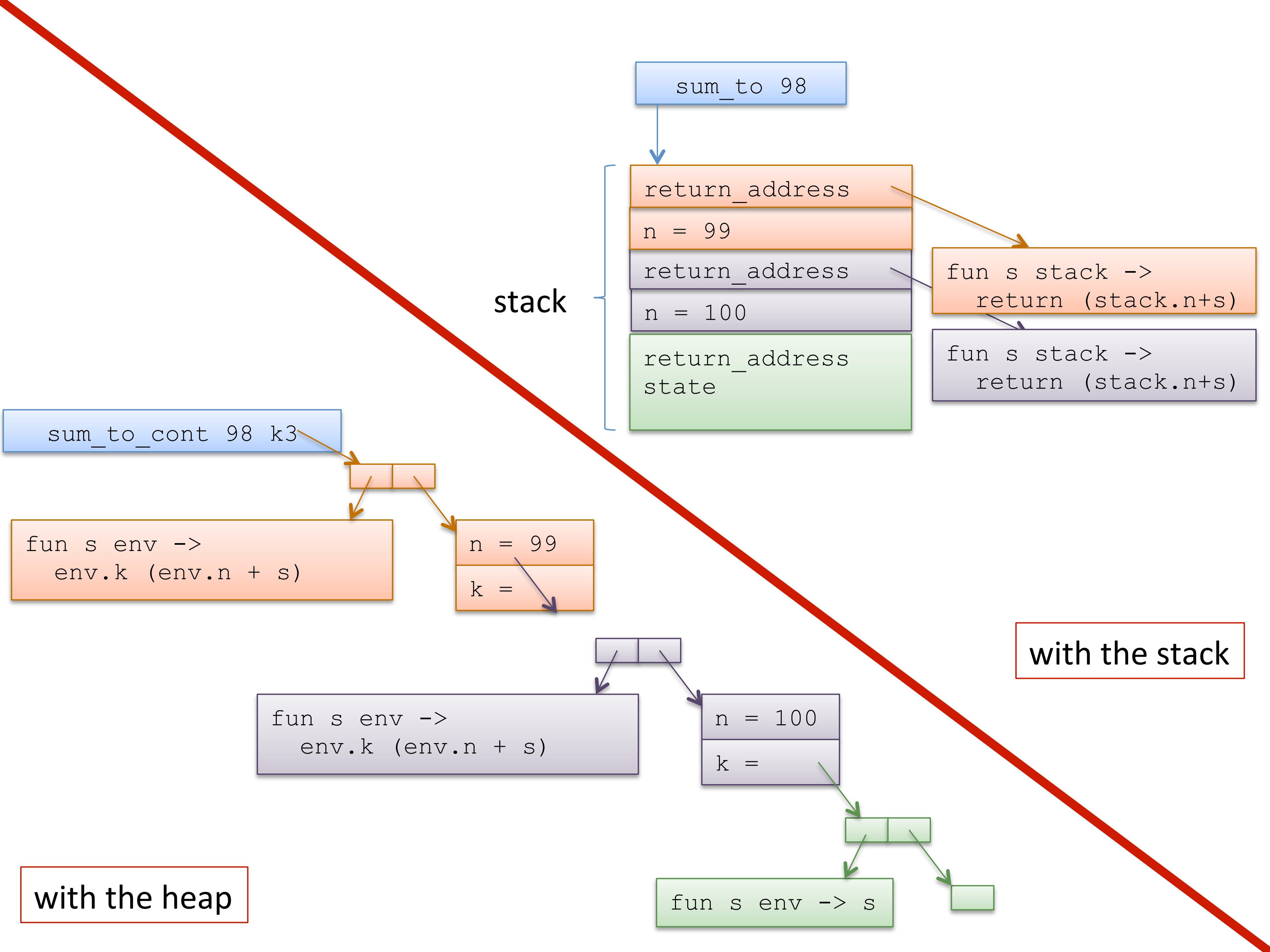
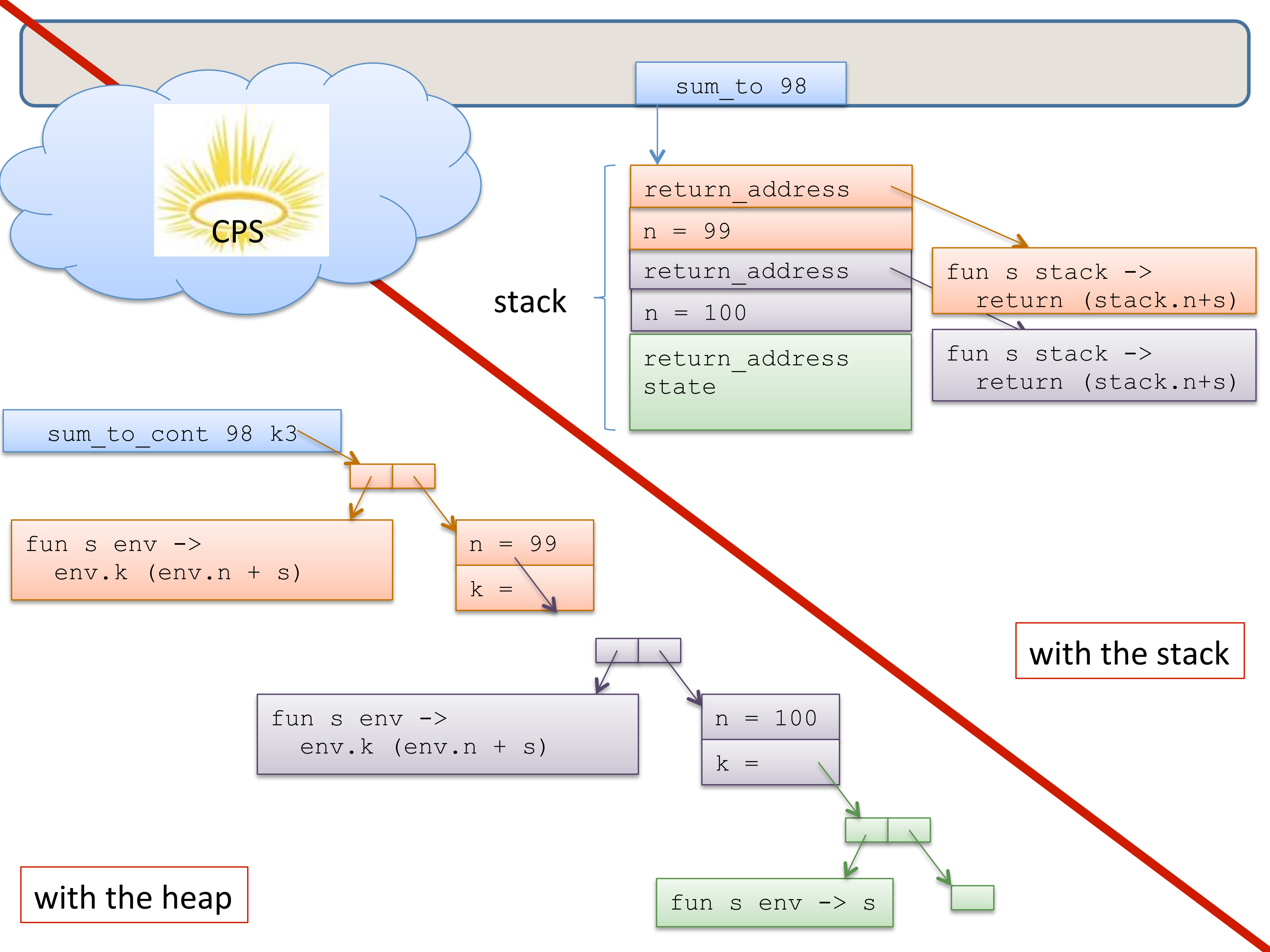
;;

sum_to 100
```




sum_to 98
stack
return_address
n = 99
return_address
n = 100
return_address
state
fun s stack ->
return (stack.n+s)
fun s stack ->
return (stack.n+s)
with the stack
sum_to_cont 98 k3
fun s env ->
env.k (env.n + s)
n = 99
k =
fun s env ->
env.k (env.n + s)
n = 100
k =
fun s env -> s
with the heap



sum_to 98
CPS
return_address
n = 99
return_address
n = 100
return_address
state
stack
fun s stack ->
return (stack.n+s)
fun s stack ->
return (stack.n+s)
sum_to_cont 98 k3
fun s env ->
env.k (env.n + s)
n = 99
k =
fun s env ->
env.k (env.n + s)
n = 100
k =
fun s env -> s
with the stack
with the heap

# Why CPS?

Continuation-passing style is *inevitable*.

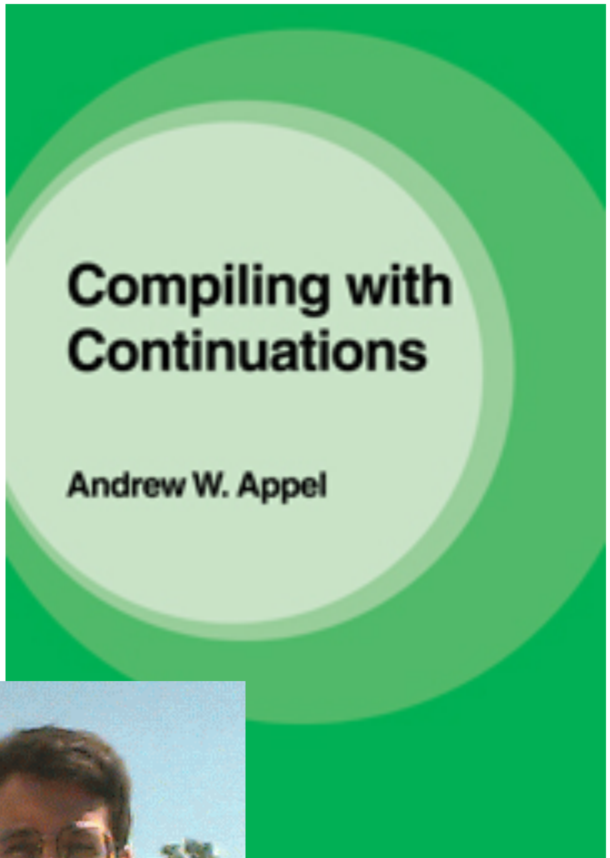
It does not matter whether you program in Java or C or OCaml -- there's code around that tells you "*what to do next*"

- If you explicitly CPS-convert your code, "*what to do next*" is stored on the heap
- If you don't, it's stored on the stack

If you take a conventional compilers class, the continuation will be called a *return address* (but you'll know what it really is!)

The idea of a *continuation* is much more general!

# Standard ML of New Jersey




Your compiler can put all the continuations in the heap so you don't have to (and you don't run out of stack space)!

Other pros:

- light-weight concurrent threads

Some cons:

- hardware architectures optimized to use a stack
- need tight integration with a good garbage collector

see

Empirical and Analytic Study of Stack versus Heap Cost for Languages with Closures. Shao & Appel

# Call-backs: Another use of continuations

Call-backs:

```
request_url : url -> (html -> 'a) -> 'a

request_url http://www.stuff.com/i.html
      (fun html -> process html)
```

continuation

# Summary

CPS is interesting and important:

- *unavoidable*
  - assembly language is continuation-passing
- *theoretical ramifications*
  - fixes evaluation order
  - call-by-value evaluation == call-by-name evaluation
- *efficiency*
  - generic way to create tail-recursive functions
  - Appel's SML/NJ compiler based on this style
- *continuation-based programming*
  - call-backs
  - programming with "*what to do next*"
- *implementation-technique for concurrency*

# Overall Summary

We developed techniques for reasoning about the space costs of functional programs

- the cost of *manipulating data types* like tuples and trees
- the cost of allocating and *using function closures*
- the cost of *tail-recursive* and non-tail-recursive *functions*

We also talked about some important program transformations:

- *closure conversion* makes nested functions with free variables in to pairs of closed code and environment
- the *continuation-passing style* (CPS) transformation turns non-tail-recursive functions in to tail-recursive ones that use no stack space
  - the stack gets moved in to the function closure
- since stack space is often small compared with heap space, it is often necessary to use *continuations and tail recursion*
  - but full CPS-converted programs are unreadable: use judgement

# Challenge: CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

Hint 1: introduce one let expression for each function call:

let x = incr left i in ...

Hint 2: you will need two continuations

# CORRECTNESS OF A CPS TRANSFORM

# Are the two functions the same?

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum2 (l:int list) : int = sum_cont l (fun s -> s)
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

Here, it is really pretty tricky to be sure you've done it right if you don't prove it. Let's try to prove this theorem and see what happens:

```
for all l:int list,
  sum_cont l (fun x -> x) == sum l
```

# Attempting a Proof

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail
```

# Attempting a Proof

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Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail

   sum_cont (hd::tail) (fun s -> s)
==
```

# Attempting a Proof

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail

   sum_cont (hd::tail) (fun s -> s)
== sum_cont tail (fn s' -> (fn s -> s) (hd + s'))   (eval)
```

# Attempting a Proof

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail

   sum_cont (hd::tail) (fun s -> s)
== sum_cont tail (fn s' -> (fn s -> s) (hd + s'))   (eval)
== sum_cont tail (fn s' -> hd + s')                 (eval -- hd + s' valuable)
```

# Need to Generalize the Theorem and IH

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail

   sum_cont (hd::tail) (fun s -> s)
== sum_cont tail (fn s' -> (fn s -> s) (hd + s'))   (eval)
== sum_cont tail (fn s' -> hd + s')                 (eval -- hd + s' valuable)

== darn!
```

we'd like to use the IH, but we can't!
we might like:

sum_cont tail (fn s' -> hd + s') == sum tail

... but that's not even true

not the identity continuation (fun s -> s) like the IH requires

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...      (eval)
  == k 0                                            (eval)
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...      (eval)
  == k 0                                            (eval)



  == k (sum [])
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...      (eval)
  == k 0                                            (eval)

  == k (0)                                          (eval, reverse)
  == k (match [] with [] -> 0 | hd::tail -> ...)    (eval, reverse)
  == k (sum [])

case done!
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:    for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))     (eval)
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun s -> k (hd+s))
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun s -> k (hd+s))
  == k (hd + (sum tail))                      (eval, since sum total and
                                                     and sum tail valuable)
```

# Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun s -> k (hd+s))
  == k (hd + (sum tail))                      (eval, since sum total and
                                                     and sum tail valuable)
  == k (sum (hd:tail))                        (eval sum, reverse)

case done!
QED!
```

# Finishing Up

Ok, now what we have is a proof of this theorem:

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

We can use that general theorem to get what we really want:

```
for all l:int list,
   sum2 l
== sum_cont l (fun s -> s)       (by eval sum2)
== (fun s -> s) (sum l)         (by theorem, instantiating k with (fun s -> s)
== sum l                        (by eval, since sum l valuable)
```

So, we've show that the function sum2, which is tail-recursive, is functionally equivalent to  the non-tail-recursive function sum.

# SUMMARY

# Summary of the CPS Proof

We tried to prove the *specific* theorem we wanted:

```
for all l:int list, sum_cont l (fun s -> s) == sum l
```

But it didn't work because in the middle of the proof, *the IH didn't apply* -- inside our function we had the wrong kind of continuation -- not (fun s -> s) like our IH required. So we had to *prove a more general theorem* about *all* continuations.

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

This is a common occurrence -- *generalizing the induction hypothesis* -- and it requires human ingenuity. It's why proving theorems is hard. It's also why writing programs is hard -- you have to make the proofs and programs work more generally, around every iteration of a loop.

# Overall Summary

We developed techniques for reasoning about the space costs of functional programs

- the cost of *manipulating data types* like tuples and trees
- the cost of allocating and *using function closures*
- the cost of *tail-recursive* and non-tail-recursive *functions*

We also talked about some important program transformations:

- *closure conversion* makes nested functions with free variables into pairs of closed code and environment
- the *continuation-passing style* (CPS) transformation turns non-tail-recursive functions in to tail-recursive ones that use no stack space
  - the stack gets moved in to the function closure
- since stack space is often small compared with heap space, it is often necessary to use *continuations and tail recursion*
  - but full CPS-converted programs are unreadable: use judgement

# Challenge: CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

(see solution after the next slide)

Solution:

# CPS CONVERT THE INCR FUNCTION

# CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

```
type cont = tree -> tree ;;

let rec incr_cps (t:tree) (i:int) (k:cont) : tree =
  match t with
    Leaf -> k Leaf
  | Node (j,left,right) -> ...
;;
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

first continuation:

```
Node (i+j, ___________ , incr right i)
```

second continuation:

```
Node (i+j, left_done, ______________ )
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr i left, incr i right)
;;
```

first continuation:

```
fun left_done -> Node (i+j, left_done , incr right i)
```

second continuation:

```
fun right_done -> k (Node (i+j, left_done, right_done))
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

second continuation *inside* first continuation:

```
fun left_done ->
  let k2 =
    (fun right_done ->
       k (Node (i+j, left_done, right_done))
     )
  in
  incr right i k2
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

```
type cont = tree -> tree ;;

let rec incr_cps (t:tree) (i:int) (k:cont) : tree =
  match t with
    Leaf -> k Leaf
  | Node (j,left,right) ->
      let k1 = (fun left_done ->
                  let k2 = (fun right_done ->
                              k (Node (i+j, left_done, right_done)))
                  in
                  incr_cps right i k2
                )
      in
      incr_cps left i k1
;;


let incr_tail (t:tree) (i:int) : tree = incr_cps t i (fun t -> t);;
```