An OCaml definition of OCaml evaluation, or,

Implementing OCaml in OCaml

COS 326
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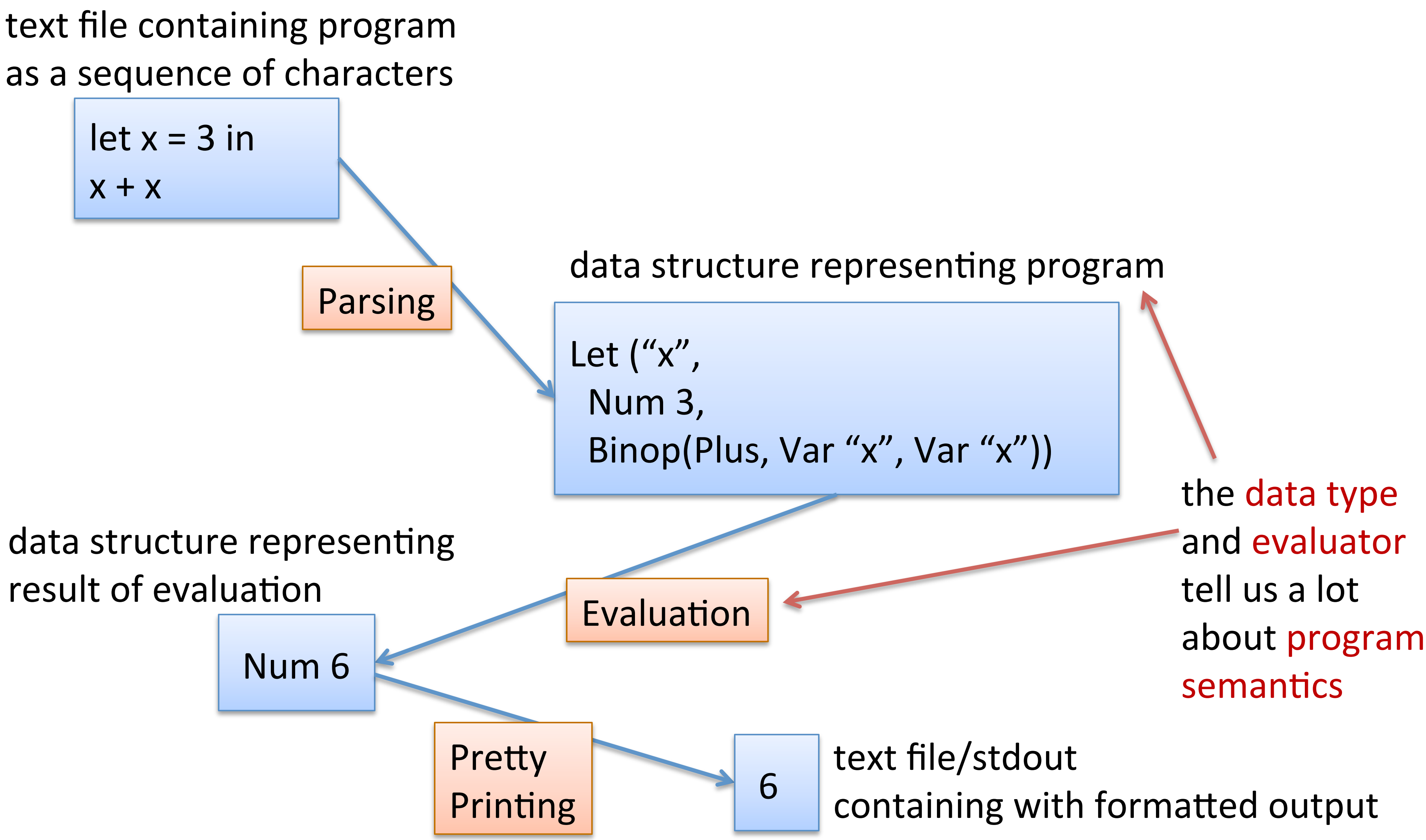
Implementing an Interpreter



Making These Ideas Precise

We can define a datatype for simple OCaml expressions:

```
type variable = string

type op = Plus | Minus | Times | …

type exp =
  | Int_e of int
  | Op_e  of exp * op * exp
  | Var_e of variable
  | Let_e of variable * exp * exp

type value = exp
```

Making These Ideas Precise

We can define a datatype for simple OCaml expressions:

```
type variable = string
type op = Plus | Minus | Times | …
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of variable
   | Let_e of variable * exp * exp
type value = exp

let e1 = Int_e 3
let e2 = Int_e 17
let e3 = Op_e (e1, Plus, e2)
```

represents “3 + 17”

Making These Ideas Precise

We can represent the OCaml program:

```
let x = 30 in
  let y =
    (let z = 3 in
     z*4)
  in
  y+y;;
```

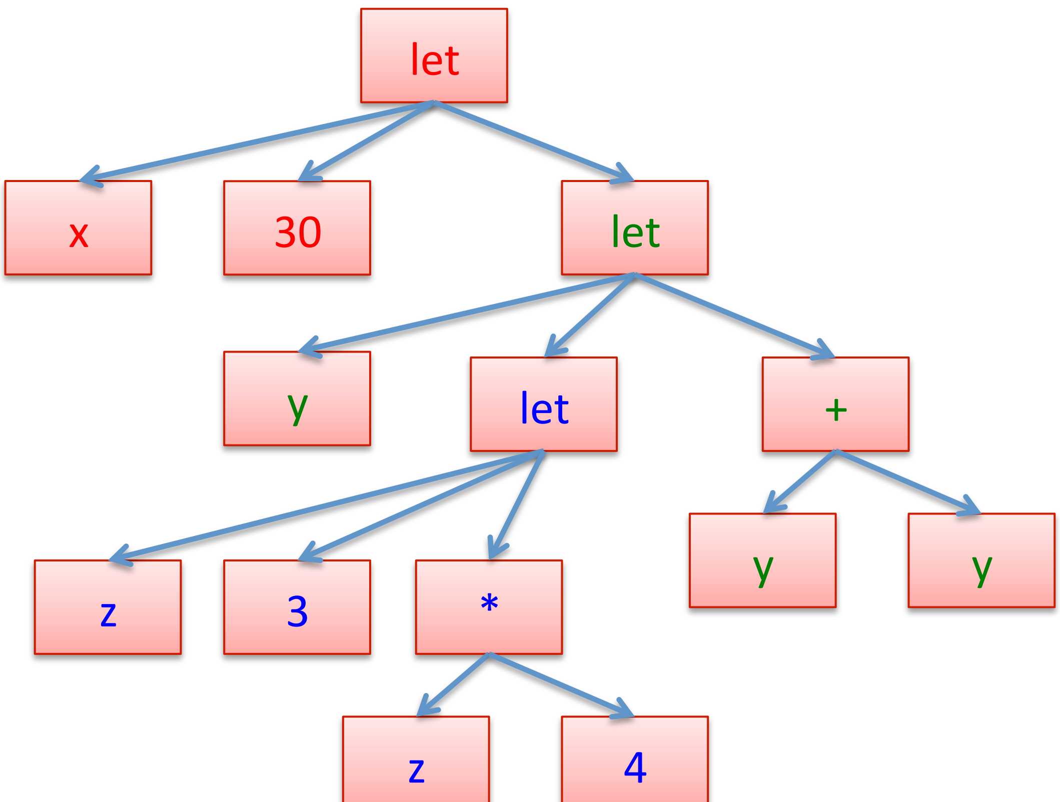
as an exp value:

```
   Let_e(“x”, Int_e 30,
         Let_e(“y”,
               Let_e(“z”, Int_e 3,
                     Op_e(Var_e “z”, Times, Int_e 4)),
         Op_e(Var_e “y”, Plus, Var_e “y”)
```

Making These Ideas Precise

Notice how this reflects the “tree”:

```
Let_e("x",Int_e 30,
      Let_e("y",Let_e("z",Int_e 3,
                      Op_e(Var_e "z", Times, Int_e 4)),
          Op_e(Var_e "y", Plus, Var_e "y")
```



Binding occurrences versus applied occurrences


```
type exp =
   | Int_e of int
   | Op_e of exp * op * exp
   | Var_e of variable
   | Let_e of variable * exp * exp
```

This is a use of a variable

This is a binding occurrence of a variable

Some Useful Auxiliary Functions

nested “|” pattern
(can’t use variables)

```
let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | ( Op_e  _
    | Let_e _
    | Var_e _ ) -> false


eval_op     : value -> op -> value -> exp

(* substitute v x e:
   replace free occurrences of x with v in e *)
substitute : value -> variable -> exp -> exp
```

A Simple Evaluator


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i ->
  | Op_e(e1,op,e2) ->



  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->




  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        let v1 = eval e1 in
        let v2 = eval e2 in
        eval_op v1 op v2
  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) ->
         let v1 = eval e1 in
         let v2 = eval e2 in
         eval_op v1 op v2
   | Let_e(x,e1,e2) ->
         let v1  = eval e1 in
         let e2' = substitute v1 x e2 in
         eval e2'
```

Shorter but Dangerous


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
        eval (substitute (eval e1) x e2)
```

Why?

Simpler but Dangerous

```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) ->
          eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) ->
          eval (substitute (eval e1) x e2)
```

Which gets evaluated first?
Does OCaml use left-to-right eval order or right-to-left?
Always use OCaml let if you want to specify evaluation order.

Simpler but Dangerous


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
        eval (substitute (eval e1) x e2)
```

Since the language we are interpreting is *pure* (no effects), it won't matter which expression gets evaluated first. We'll produce the same answer in either case.

Limitations of metacircular interpreters


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variab    -> exp   > e

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) ->
         let v1 = eval e1 in
         let v2 = eval e2 in
         eval_op v1 op v2
   | Let_e(x,e1,e2) ->
         let v1  = eval e1
         let e2' = substit
         eval e2'
```

Which gets evaluated first, (eval e1) or (eval e2) ? Seems obvious, right? But that's because we assume OCaml has call-by-value evaluation! If it were call-by-name, then this ordering of **let**s would not guarantee order of evaluation.

*Moral: using a language to define **its own semantics** can have limitations.*

Back to the eval function…


```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
```

(same as the one a couple of slides ago)

Oops! We Missed a Case:

```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> ???
```

We should never encounter a variable – they should have been substituted with a value! (This is a type-error.)

We could leave out the case for variables if we ***type check before evaluating.*** *(which we should definitely do!)*

But that will create a mess of Ocaml warnings – bad style. (Bad for debugging.)

We Could Use Options:


```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp option =
  match e with
  | Int_e i -> Some(Int_e i)
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> None
```

But this isn't quite right – we need to match on the recursive calls to eval to make sure we get Some value!

Exceptions


```
exception UnboundVariable of variable

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
```

Instead, we can throw an exception.

Exceptions

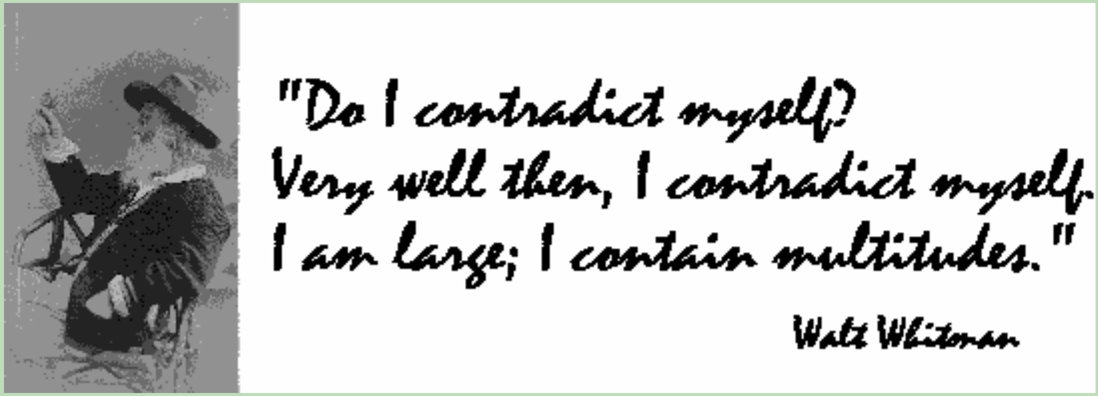
```
exception UnboundVariable of variable

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
```

Note that an exception declaration is a lot like a datatype declaration. Really, we are extending one big datatype (exn) with a new constructor (UnboundVariable).

Later on, we'll see how to catch an exception.

Exception or option?

In a previous lecture, I explained why to use an option type (or a Success/Failure datatype) instead of raising an exception.



No! I do not contradict myself!

- The other example was for errors that *will occur* (because the input might be ill formatted).
- This example is for errors that *cannot occur* (unless the program itself has a bug).

AUXILIARY FUNCTIONS

Evaluating the Primitive Operations


```
let eval_op (v1:exp) (op:operand) (v2:exp) : exp =
  match v1, op, v2 with
   | Int_e i, Plus,  Int_e j -> Int_e (i+j)
   | Int_e i, Minus, Int_e j -> Int_e (i-j)
   | Int_e i, Times, Int_e j -> Int_e (i*j)
   | _ ,(Plus | Minus | Times), _ ->
     if is_value v1 && is_value v2 then raise TypeError
     else raise NotValue
```

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x) ;;
```

Substitution

Want to replace x (and only x) with v.

```
let substitute (v:exp) (x:variable) (e:exp) : exp =

    ...

;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ ->
         | Op_e(e1,op,e2) ->
         | Var_e y ->               ... use x ...
         | Let_e (y,e1,e2) ->       ... use x ...



  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) ->
         | Var_e y ->
         | Let_e (y,e1,e2) ->




  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y ->
         | Let_e (y,e1,e2) ->




  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->




  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     subst e2)
  in
  subst e
;;
```

WRONG!

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
             Let_e (y,
                    if x = y then e1 else subst e1,
                    if x = y then e2 else subst e2)
  in
  subst e
;;
```

wrong

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```


evaluation/subsitution must implement our variable *scoping* rules correctly

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```

If x and y are the same variable, then y *shadows* x.

SCALING UP THE LANGUAGE

(MORE FEATURES, MORE FUN)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

Scaling up the Language

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

OCaml's
fun x -> e
is represented as
Fun_e(x,e)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
    | Var_e of variable | Let_e of variable * exp * exp
    | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

A function call
fact 3
is implemented as
FunCall_e (Var_e “fact”, Int_e 3)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp;;

let is_value (e:exp) : bool =
   match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | (   Op_e (_,_,_)
      | Let_e (_,_,_)
      | Var_e _
      | FunCall_e (_,_) ) -> false ;;
```

Functions are values!

Easy exam question:
What value does the OCaml interpreter produce when you enter (fun x -> 3) in to the prompt?
Answer: the value produced is (fun x -> 3)

Scaling up the Language:

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp;;

let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | (  Op_e (_,_,_)
      | Let_e (_,_,_)
      | Var_e _
      | FunCall_e (_,_) ) -> false ;;
```

Function calls are not values.

Scaling up the Language:


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
        | Fun_e (x,e), v2 -> eval (substitute v2 x e)
        | _ -> raise TypeError)
```

Scaling up the Language:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

values (including functions) always evaluate to themselves.

Scaling up the Language:


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

To evaluate a function call, we first evaluate both e1 and e2 to values.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

e1 had better evaluate to a function value, else we have a type error.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

Then we substitute e2's value (v2) for x in e and evaluate the resulting expression.

Simplifying a little


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
         | Fun_e (x,e) -> eval (substitute (eval e2) x e)
         | _ -> raise TypeError)
```

We don't really need
to pattern-match on e2.
Just evaluate here

Simplifying a little


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (ef,e1) ->
        (match eval ef with
          | Fun_e (x,e2) -> eval (substitute (eval e1) x e2)
          | _ -> raise TypeError)
```

This looks like
the case for let!

Let and Lambda


```
let x = 1 in x+41
-->
1+41
-->
42
```

```
(fun x -> x+41) 1
-->
1+41
-->
42
```

In general:

```
(fun x -> e2) e1    ==   let x = e1 in e2
```

So we could write:


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (FunCall (Fun_e (x,e2), e1))
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (ef,e2) ->
       (match eval ef with
        | Fun_e (x,e1) -> eval (substitute (eval e1) x e2)
        | _ -> raise TypeError)
```

In programming-languages speak: "Let is *syntactic sugar* for a function call"

Syntactic sugar: A new feature defined by a simple, local transformation.

Recursive definitions


```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp ;;
```

```
let rec f x = f (x+1) in f 3
```

(rewrite) →

```
let f = (rec f x -> f (x+1)) in
f 3
```

(alpha-convert) →

```
let g = (rec f x -> f (x+1)) in
g 3
```

(implement) →

```
Let_e ("g,
  Rec_e ("f", "x",
    FunCall_e (Var_e "f", Op_e (Var_e "x", Plus, Int_e 1))
 ),
 FunCall (Var_e "g", Int_e 3)
)
```

Recursive definitions

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp ;;
```

```
let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | Rec_e of (_,_,_) -> true
   | (Op_e (_,_,_) | Let_e (_,_,_) |
      Var_e _ | FunCall_e (_,_) ) -> false ;;
```

Recursive definitions

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp ;;
```

```
let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | Rec_e o    _,_,_) -> true
   | (Op_e (_,       | Let_e (_,_,_) |
     Var_e                                   ;;
```

Fun_e (x, body) == Rec_e("unused", x, body)

A better IR would just delete Fun_e – avoid unnecessary redundancy

Interlude: Notation for Substitution

"Substitute value v for variable x in expression e:" e [v / x]

examples of substitution:

(x + y) [7/y]	is	(x + 7)
(let x =30 in let y=40 in x + y) [7/y]	is	(let x =30 in let y=40 in x + y)
(let y = y in let y = y in y + y) [7/y]	is	(let y = 7 in let y = y in y + y)

Evaluating Recursive Functions

Basic evaluation rule for recursive functions:

```
(rec f x = body) arg    -->    body [arg/x] [rec f x = body/f]
```

argument value substituted for parameter

entire function substituted for function name

Evaluating Recursive Functions

Start out with
a let bound to
a recursive function:

```
let g =
  rec f x ->
    if x <= 0 then x
    else x + f (x-1)
in g 3
```

The Substitution:

```
g 3 [rec f x ->
       if x <= 0 then x
       else x + f (x-1)   / g]
```

The Result:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

Evaluating Recursive Functions

Recursive Function Call:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

The Substitution:

```
(if x <= 0 then x else x + f (x-1))
  [ rec f x ->
       if x <= 0 then x
       else x + f (x-1)   / f ]
  [ 3 / x ]
```

Substitute argument for parameter

Substitute entire function for function name

The Result:

```
(if 3 <= 0 then 3 else 3 +
   (rec f x ->
       if x <= 0 then x
       else x + f (x-1)) (3-1))
```

Evaluating Recursive Functions


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1 with
         | Fun_e (x,e) ->
             let v = eval e2 in
             substitute e x v

         | (Rec_e (f,x,e)) as f_val ->
             let v = eval e2 in
             substitute f_val f (substitute v x e)

         | _ -> raise TypeError)
```

pattern as *x*

match the pattern
and binds x to value

More Evaluation


```
(rec fact n = if n <= 1 then 1 else n * fact(n-1)) 3
-->
if 3 < 1 then 1 else
  3 * (rec fact n = if ... then ... else ...) (3-1)
-->
3 * (rec fact n = if … ) (3-1)
-->
3 * (rec fact n = if … ) 2
-->
3 * (if 2 <= 1 then 1 else 2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(1))
-->
3 * 2 * (if 1 <= 1 then 1 else 1 * (rec fact ...)(1-1))
-->
3 * 2 * 1
```

A MATHEMATICAL DEFINITION* OF OCAML EVALUATION

* it's a partial definition and this is a big topic; for more, see COS 510

From Code to Abstract Specification

OCaml code can give a language semantics

- advantage: it can be executed, so we can try it out
- advantage: it is amazingly concise
  - especially compared to what you would have written in Java
- disadvantage: it is a little ugly to operate over concrete ML datatypes like "Op_e(e1,Plus,e2)" as opposed to "e1 + e2"
- disadvantage: defining a language in itself is a logical fallacy

From Code to Abstract Specification

PL researchers have developed their own standard notation for writing down how programs execute

- it has a mathematical "feel" that makes PL researchers feel special and gives us *goosebumps* inside
- it operates over abstract expression syntax like "e1 + e2"
- it is useful to know this notation if you want to read specifications of programming language semantics
  - e.g.: Standard ML (of which OCaml is a descendent) has a formal definition given in this notation (and C, and Java; but not OCaml…)
  - e.g.: most papers in the conference POPL (ACM Principles of Prog. Lang.)

Goal

Our goal is to explain how an expression e evaluates to a value v.

In other words, we want to define a mathematical *relation* between pairs of expressions and values.

Formal Inference Rules

We define the "evaluates to" relation using a set of (inductive) rules that allow us to *prove* that a particular (expression, value) pair is part of the relation.

A rule looks like this:

$$\frac{\text{premise 1} \qquad \text{premise 2} \qquad \ldots \qquad \text{premise 3}}{\text{conclusion}}$$

You read a rule like this:

- "if premise 1 can be proven and premise 2 can be proven and ... and premise n can be proven then conclusion can be proven"

Some rules have no premises

- this means their conclusions are always true
- we call such rules "axioms" or "base cases"

An example rule

As a rule:

$$\frac{\text{e1 --> v1} \qquad \text{e2 --> v2} \qquad \text{eval\_op (v1, op, v2) == v'}}{\text{e1 op e2 --> v'}}$$

In English:

"If e1 evaluates to v1
and e2 evaluates to v2
and eval_op (v1, op, v2) is equal to v'
then
e1 op e2 evaluates to v'

In code:

```
let rec eval (e:exp) : exp =
  match e with
  | Op_e(e1,op,e2) -> let v1 = eval e1 in
                      let v2 = eval e2 in
                      let v' = eval_op v1 op v2 in
                      v'
```

An example rule

As a rule:

$$\frac{i \in Z}{i \dashrightarrow i}$$

asserts i is an integer

In English:

"If the expression is an integer value, it evaluates to itself."

In code:

```
let rec eval (e:exp) : exp =
  match e with
 | Int_e i -> Int_e i
 ...
```

An example rule concerning evaluation

As a rule:

$$\frac{\text{e1 --> v1} \qquad \text{e2 [v1/x] --> v2}}{\text{let x = e1 in e2 --> v2}}$$

In English:

> "If e1 evaluates to v1 (which is a *value*)
> and e2 with v1 substituted for x evaluates to v2
> then let x=e1 in e2 evaluates to v2."

In code:

```
let rec eval (e:exp) : exp =
  match e with
 | Let_e(x,e1,e2) -> let v1 = eval e1 in
                     eval (substitute v1 x e2)
 ...
```

An example rule concerning evaluation

As a rule:

$$\frac{}{\lambda x.e \;\texttt{-->}\; \lambda x.e}$$

typical "lambda" notation for a function with argument x, body e

In English:

"A function value evaluates to itself."

In code:

```
let rec eval (e:exp) : exp =
  match e with
 ...
 | Fun_e (x,e) -> Fun_e (x,e)
 ...
```

An example rule concerning evaluation

As a rule:

$$\frac{e1 \dashrightarrow \lambda x.e \qquad e2 \dashrightarrow v2 \qquad e[v2/x] \dashrightarrow v}{e1\ e2 \dashrightarrow v}$$

In English:

> "if e1 evaluates to a function with argument x and body e
> and e2 evaluates to a value v2
> and e with v2 substituted for x evaluates to v
> then e1 applied to e2 evaluates to v"

In code:

```
let rec eval (e:exp) : exp =
  match e with
 ..
| FunCall_e (e1,e2) ->
        (match eval e1 with
          | Fun_e (x,e) -> eval (substitute (eval e2) x e)
          | ...)
...
```

An example rule concerning evaluation

As a rule:

$$\frac{e1 \text{ --> } rec\ f\ x = e \qquad e2 \text{ --> } v \qquad e[rec\ f\ x = e/f][v/x] \text{ --> } v2}{e1\ e2 \text{ --> } v2}$$

In English:

“uggh”

In code:

```
let rec eval (e:exp) : exp =
  match e with
     ...
   | (Rec_e (f,x,e)) as f_val ->
        let v = eval e2 in
        substitute f_val (substitute v x e) g
```

Comparison: Code vs. Rules

complete eval code:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
        | Fun_e (x,e) -> eval (Let_e (x,e2,e))
        | _ -> raise TypeError)
   | LetRec_e (x,e1,e2) ->
      (Rec_e (f,x,e)) as f_val ->
        let v = eval e2 in
        substitute f_val f (substitute v x e)
```

complete set of rules:

$$\frac{i \in Z}{i \;\texttt{-->}\; i}$$

$$\frac{e1 \;\texttt{-->}\; v1 \qquad e2 \;\texttt{-->}\; v2 \qquad eval\_op\,(v1, op, v2) == v}{e1\ op\ e2 \;\texttt{-->}\; v}$$

$$\frac{e1 \;\texttt{-->}\; v1 \qquad e2\,[v1/x] \;\texttt{-->}\; v2}{let\ x = e1\ in\ e2 \;\texttt{-->}\; v2}$$

$$\frac{}{\lambda x.e \;\texttt{-->}\; \lambda x.e}$$

$$\frac{e1 \;\texttt{-->}\; \lambda x.e \qquad e2 \;\texttt{-->}\; v2 \qquad e[v2/x] \;\texttt{-->}\; v}{e1\ e2 \;\texttt{-->}\; v}$$

$$\frac{e1 \;\texttt{-->}\; rec\ f\ x = e \qquad e2 \;\texttt{-->}\; v2 \qquad e[rec\ f\ x = e/f][v2/x] \;\texttt{-->}\; v3}{e1\ e2 \;\texttt{-->}\; v3}$$

Almost isomorphic:

- one rule per pattern-matching clause
- recursive call to eval whenever there is a --> premise in a rule
- what's the main difference?

Comparison: Code vs. Rules

complete eval code:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
         | Fun_e (x,e) -> eval (Let_e (x,e2,e))
         | _ -> raise TypeError)
   | LetRec_e (x,e1,e2) ->
      (Rec_e (f,x,e)) as f_val ->
        let v = eval e2 in
        substitute f_val f (substitute v x e)
```

complete set of rules:

$$\frac{i \in Z}{i \text{ --> } i}$$

$$\frac{e1 \text{ --> } v1 \qquad e2 \text{ --> } v2 \qquad eval\_op\ (v1, op, v2) == v}{e1\ op\ e2 \text{ --> } v}$$

$$\frac{e1 \text{ --> } v1 \qquad e2\ [v1/x] \text{ --> } v2}{let\ x = e1\ in\ e2 \text{ --> } v2}$$

$$\frac{}{\lambda x.e \text{ --> } \lambda x.e}$$

$$\frac{e1 \text{ --> } \lambda x.e \qquad e2 \text{ --> } v2 \qquad e[v2/x] \text{ --> } v}{e1\ e2 \text{ --> } v}$$

$$\frac{e1 \text{ --> } rec\ f\ x = e \qquad e2 \text{ --> } v2 \qquad e[rec\ f\ x = e/f][v2/x] \text{ --> } v3}{e1\ e2 \text{ --> } v3}$$

- There's no formal rule for handling free variables
- No rule for evaluating function calls when a non-function in the caller position
- In general, *no rule when further evaluation is impossible*
  - the rules express the *legal evaluations* and say nothing about what to do in error situations
  - the code handles the error situations by raising exceptions
  - type theorists prove that well-typed programs don't run into undefined cases

Summary

- We can reason about OCaml programs using a *substitution model*.
  - integers, bools, strings, chars, and *functions* are values
  - value rule: values evaluate to themselves
  - let rule: "let x = e1 in e2" : substitute e1's value for x into e2
  - fun call rule: "(fun x -> e2) e1": substitute e1's value for x into e2
  - rec call rule: "(rec x = e1) e2" : like fun call rule, but also substitute recursive function for name of function
    - To unwind: substitute (rec x = e1) for x in e1

- We can make the evaluation model precise by building an interpreter and using that interpreter as a specification of the language semantics.

- We can also specify the evaluation model using a set of *inference rules*
  - more on this in COS 510

Some Final Words

- The substitution model is only a model.
  - it does not accurately model all of OCaml's features
    - I/O, exceptions, mutation, concurrency, …
    - we can build models of these things, but they aren't as simple.
    - even substitution is tricky to formalize!
- It's useful for reasoning about higher-order functions, correctness of algorithms, and optimizations.
  - we can use it to formally prove that, for instance:
    - map f (map g xs) == map (comp f g) xs
    - proof: by induction on the length of the list xs, using the definitions of the substitution model.
  - we often model complicated systems (e.g., protocols) using a small functional language and substitution-based evaluation.
- It is *not* useful for reasoning about execution time or space
  - more complex models needed there

Some Final Words

- The substitution model is only a model.
  - it does not accurately model all of OCaml's features
    - I/O, exceptions, mutation, concurrency, …
    - we can build models of these things, but they aren't as simple.
    - even substitution was tricky to formalize!
- It's useful for reasoning about higher-ord… correctness of algorithms, and optimizati…
  - we can us… …prove that, for i…
    - …
    - … of the li… …odel.
  - we … …stems (e.g … …g a small fu… …and substitution… …n.
- It is *not* useful for reasoning about exec… …e
  - more complex models needed there

You can say that again! I got it wrong the first time I tried, in 1932. Fixed the bug by 1934, though.

Alonzo Church, 1903-1995
Princeton Professor, 1929-1967

Church's mistake

substitute:

```
fun xs -> map (+) xs
```

for f in:

```
fun ys ->
  let map xs = 0::xs in
  f (map ys)
```

and if you don't watch out, you will get:

```
fun ys ->
  let map xs = 0::xs in
  (fun xs -> map (+) xs) (map ys)
```

Church's mistake

substitute:

```
fun xs -> map (+) xs
```

for f in:

```
fun ys ->
  let map xs = 0::xs in
  f (map ys)
```

and if you don't watch out, you will get:

```
fun ys ->
  let map xs = 0::xs in
  (fun xs -> map (+) xs) (map ys)
```

the problem was that the value you substituted in had a *free variable* (map) in it that was *captured*.

Church's mistake

substitute:

```
fun xs -> map (+) xs
```

for f in:

```
fun ys ->
  let map xs = 0::xs in
  f (map ys)
```

to do it right, you need to rename some variables:

```
fun ys ->
  let z xs = 0::xs in
  (fun xs -> map (+) xs) (z ys)
```

NOW WE ARE REALLY DONE!