Prove that alpha-beta search returns the correct minimax values in the following sense: Let s be the state of a game (and assume the game tree has a finite number of nodes). Let v be the actual minimax value of s so that

$$v = \textsc{Minimax}(s).$$

Let v' be the result of running alpha-beta search on s with some given values of α and β (where $-\infty \leq \alpha \leq \beta \leq +\infty$) so that

$$v' = \textsc{Alpha-Beta-Minimax}(s, \alpha, \beta).$$

Prove that the following statements are true:

- If $\alpha \leq v \leq \beta$ then $v' = v$.
- If $v \leq \alpha$ then $v' \leq \alpha$.
- If $v \geq \beta$ then $v' \geq \beta$.

In other words, if the true minimax value is between α and β, then alpha-beta search returns the correct value. On the other hand, if the true minimax value is outside this range, alpha-beta search may return a different value, but will at least accurately report that the true value is below α or above β. In particular, note that these statements imply that $v' = v$ if $\alpha = -\infty$ and $\beta = +\infty$.

Hint: Use induction. That is, if s is not a terminal state, then assume that the claim above holds for all of s's children (successor states), and use this assumption to prove that it also holds for s.

[Note: $\textsc{Minimax}(s)$ is the procedure presented in class for computing minimax values. It is the same as the functions $\textsc{Max-Value}(s)$ and $\textsc{Min-Value}(s)$ (for MAX states and MIN states respectively) which appear in Figure 5.3 of R&N (3rd ed.) and Figure 6.3 of R&N (2nd ed.). Likewise, $\textsc{Alpha-Beta-Minimax}(s, \alpha, \beta)$ is the procedure given in class for estimating minimax values using alpha-beta search. It is the same as the functions $\textsc{Max-Value}(s, \alpha, \beta)$ and $\textsc{Min-Value}(s, \alpha, \beta)$ given in Figure 5.7 of R&N (3rd ed.) and Figure 6.7 of R&N (2nd ed.).]