Geometric Search

- range search
- space partitioning trees
- intersection search

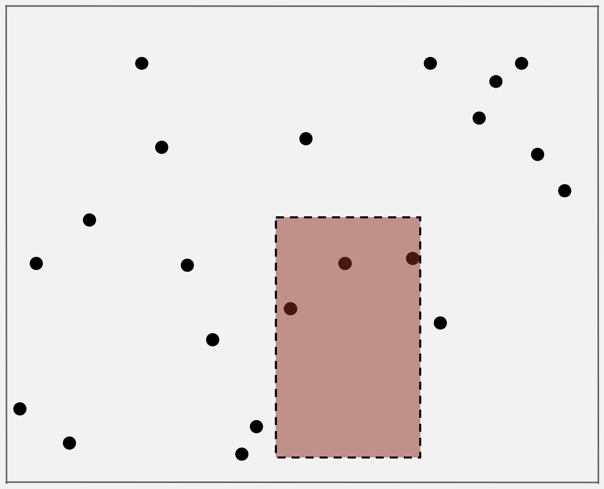
Overview

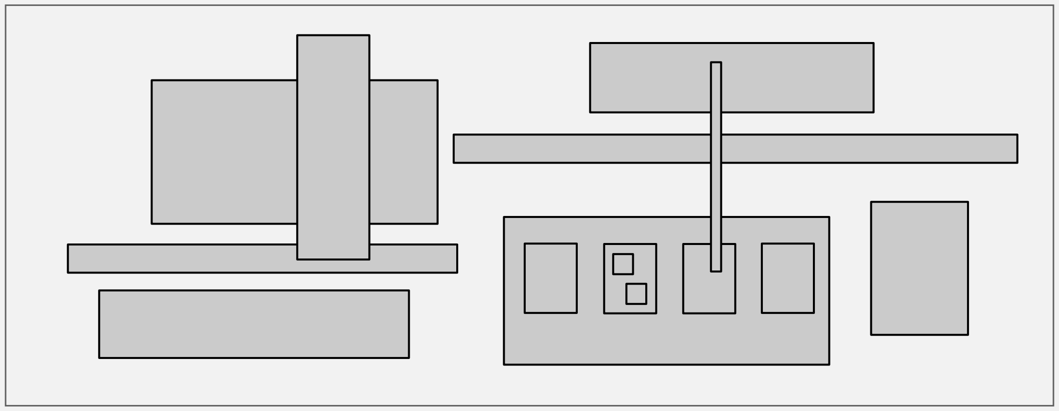
Geometric objects. Points, lines, intervals, circles, rectangles, polygons, ...

This lecture. Intersections among objects.

Example problems.

- 1d range search in set of N numbers.
- 2d orthogonal range search in set of N points.
- Find all intersections among N orthogonal line segments.
- Find all intersections among N orthogonal rectangles.


2d orthogonal range search


orthogonal rectangle intersection

1d range search

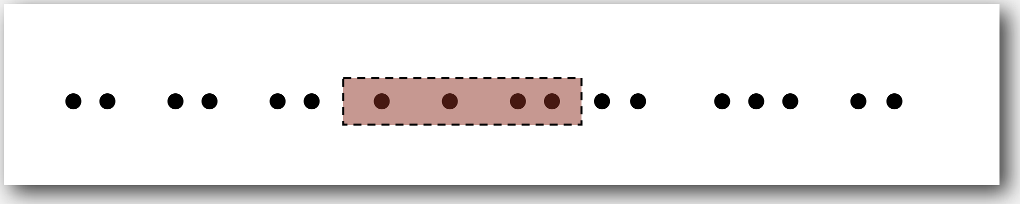
Extension of ordered symbol table.

- Insert key-value pair.
- Search for key k.
- Range search: find all keys between k_1 and k_2.
- Range count: number of keys between k_1 and k_2.

Application. Database queries.

Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given interval.



insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
count G to K	2
search G to K	H I

1d range search: implementations

Ordered array. Slow insert, binary search for k_1 and k_2 to find range.

Hash table. No reasonable algorithm (key order lost in hash).

Red-black BST. All operations fast.

Parameters. N = number of keys; R = number of keys that match.

running time is output sensitive
(number of matching keys can be N)

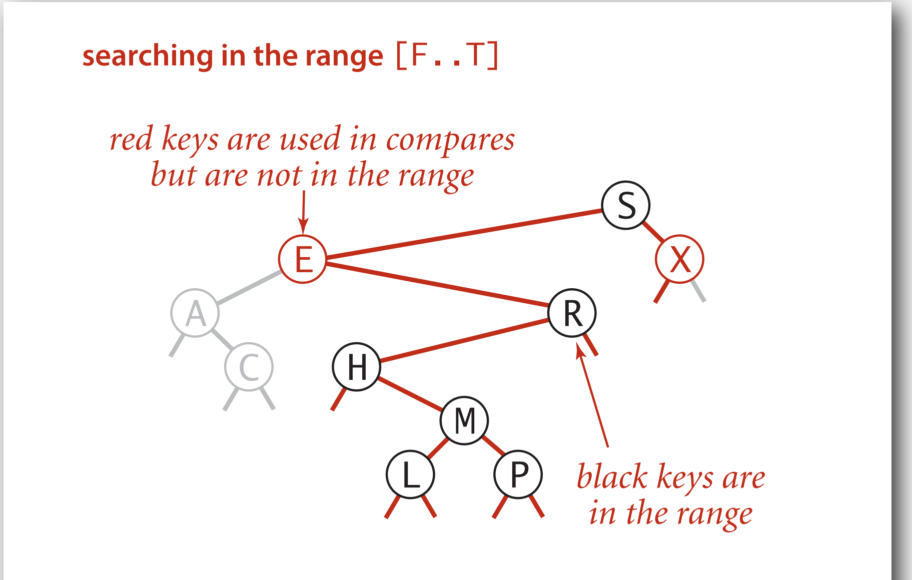
data structure	insert	range count	range search
ordered array	N	log N	R + log N
hash table	1	N	N
red-black BST	log N	log N	R + log N

order of growth of running time for 1d range search

1d range search: BST implementation

Range search. Find all keys between k_1 and k_2.

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).



searching in the range [F..T]
red keys are used in compares but are not in the range
black keys are in the range
S
E
X
A
R
C
H
M
L
P

Range search in a BST

Proposition. Running time is proportional to $R + \log N$ (assuming BST is balanced).

2d orthogonal range search

Extension of ordered symbol-table to 2d keys.

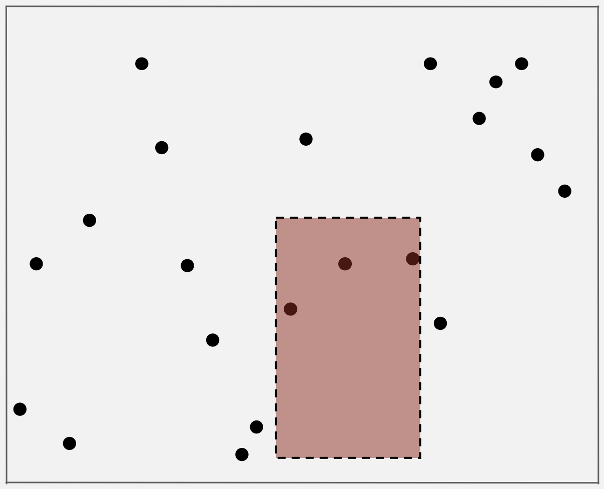
- Insert a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Applications. Networking, circuit design, databases.

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given h–v rectangle.

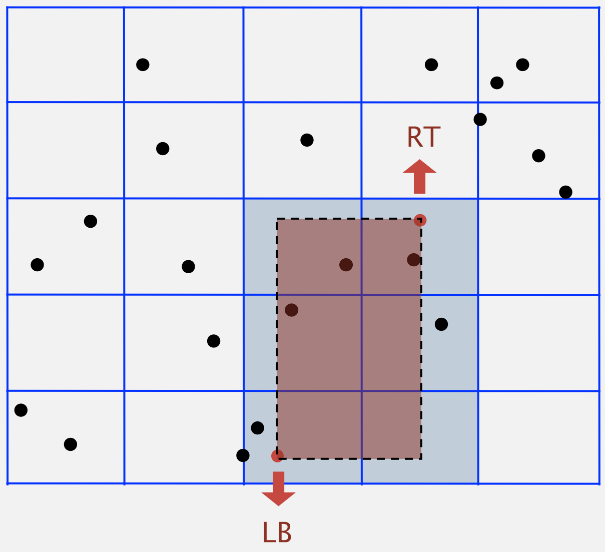
$\uparrow$ rectangle is axis-aligned



2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into M-by-M grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only those squares that intersect 2d range query.



2d orthogonal range search: grid implementation costs

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

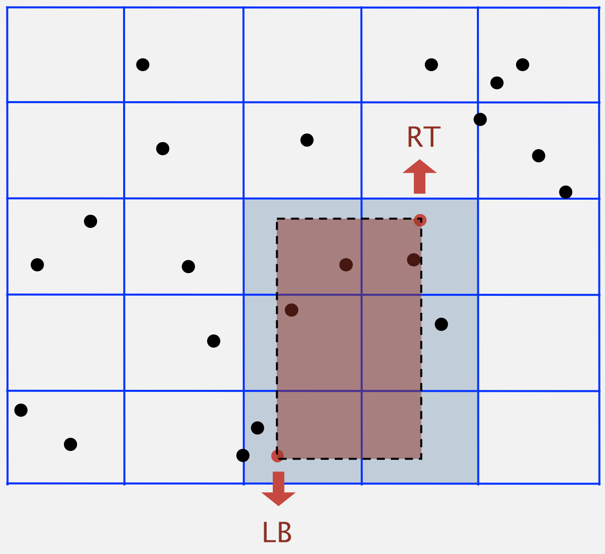
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]

- Initialize data structure: N.
- Insert point: 1.
- Range search: 1 per point in range.

choose M ~ √N

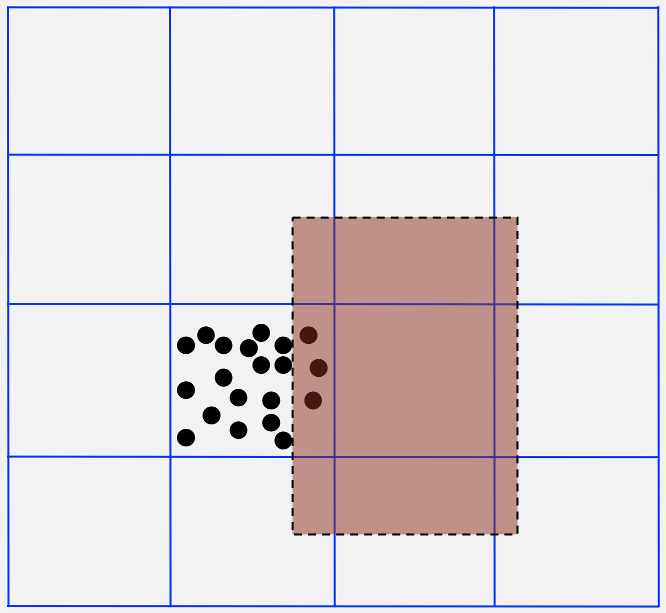


Clustering

Grid implementation. Fast, simple solution for well-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that gracefully adapts to data.

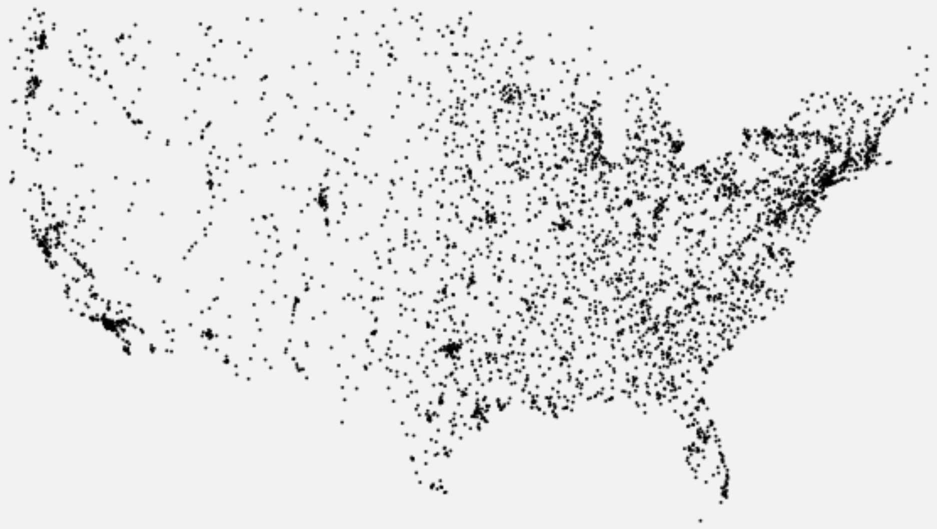


Clustering

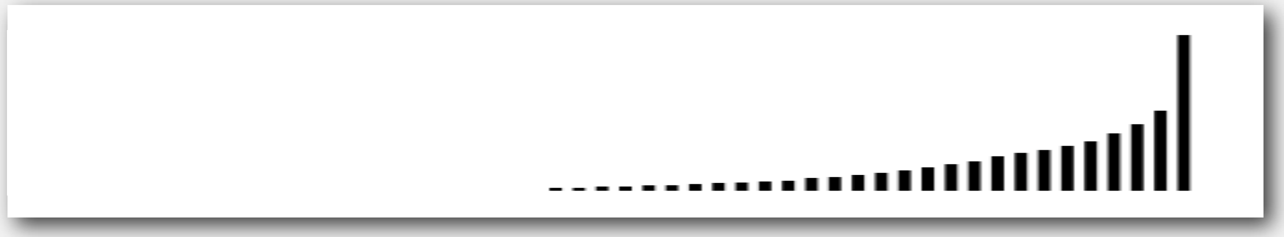
Grid implementation. Fast, simple solution for well-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.



13,000 points, 1000 grid squares



half the squares are empty

half the points are in 10% of the squares

- range search

space partitioning trees

- intersection search

Space-partitioning trees

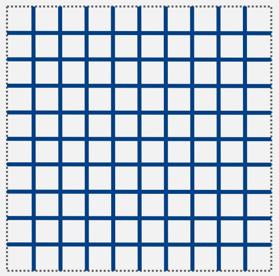
Use a tree to represent a recursive subdivision of 2d space.

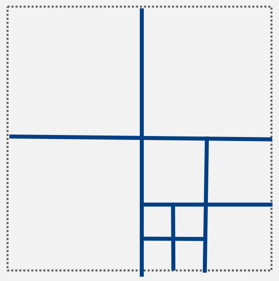
Grid. Divide space uniformly into squares.

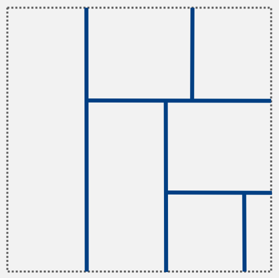
Quadtree. Recursively divide space into four quadrants.

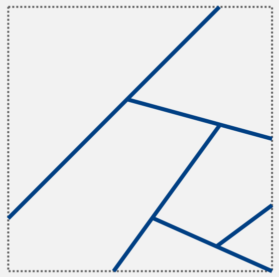
2d tree. Recursively divide space into two halfplanes.

BSP tree. Recursively divide space into two regions.


Grid


Quadtree

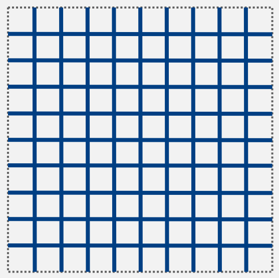

2d tree

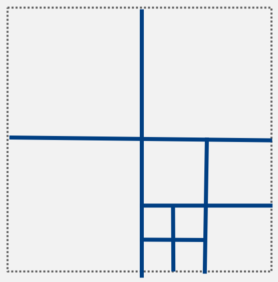

BSP tree

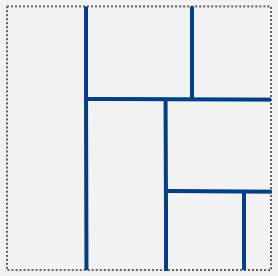
Space-partitioning trees: applications

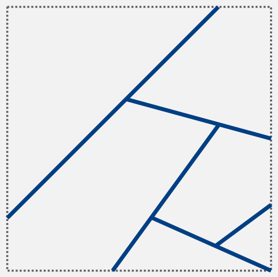
Applications.

- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.


Grid


Quadtree

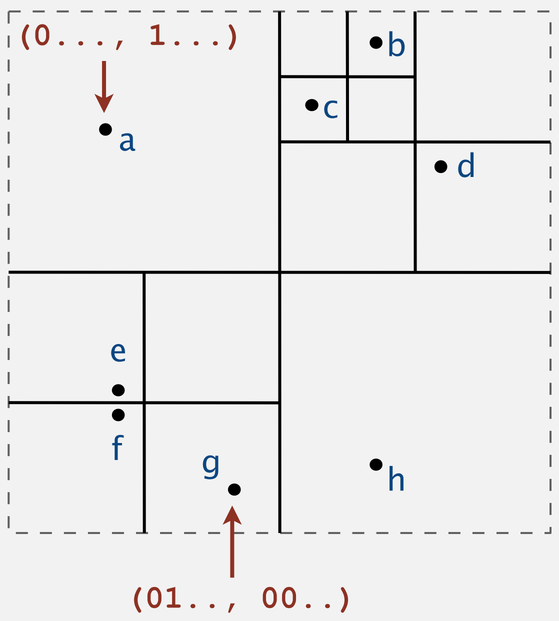

2d tree

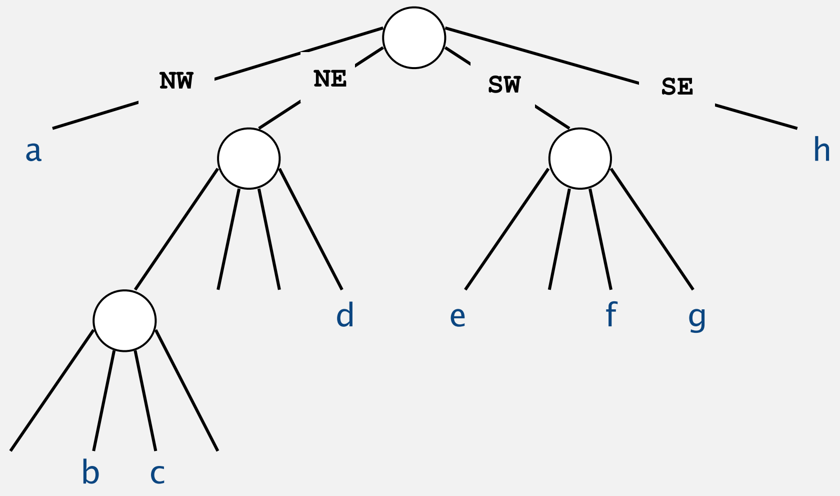

BSP tree

Quadtree

Idea. Recursively divide space into 4 quadrants.

Implementation. 4-way tree (actually a trie).



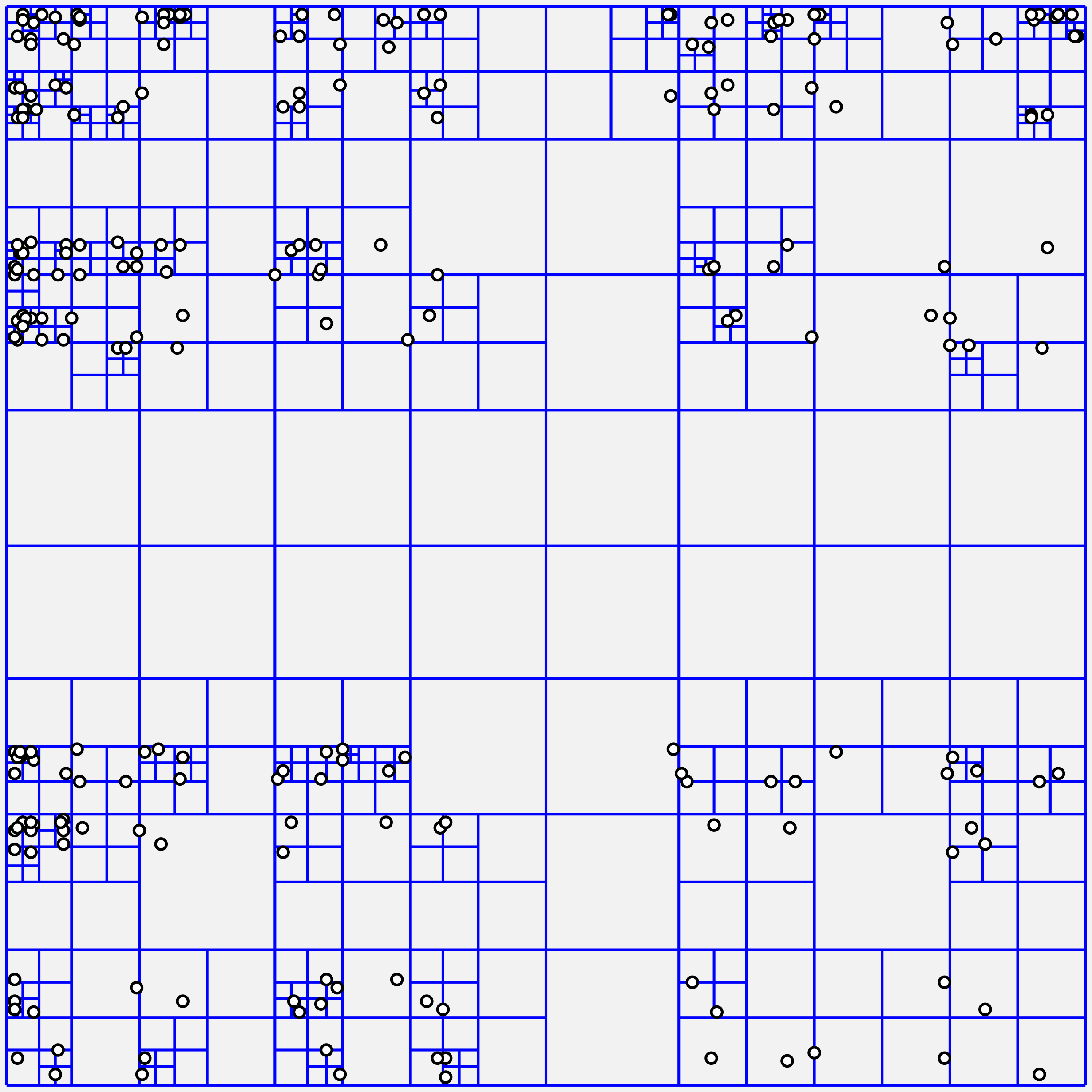


```
public class QuadTree
{
   private Quad quad;
   private Value val;
   private QuadTree NW, NE, SW, SE;
}
```

Benefit. Good performance in the presence of clustering.

Drawback. Arbitrary depth!

Quadtree: larger example

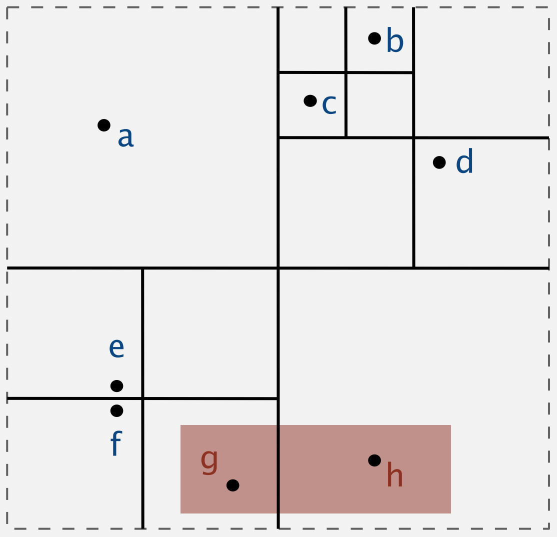


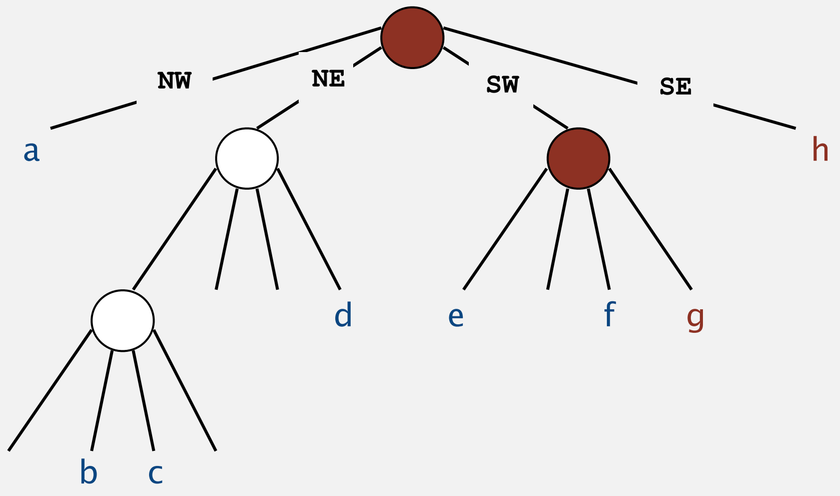
http://en.wikipedia.org/wiki/Image:Point_quadtree.svg

Quadtree: 2d orthogonal range search

Range search. Find all keys in a given 2d range.

- Recursively find all keys in NE quadrant (if any could fall in range).
- Recursively find all keys in NW quadrant (if any could fall in range).
- Recursively find all keys in SE quadrant (if any could fall in range).
- Recursively find all keys in SW quadrant (if any could fall in range).

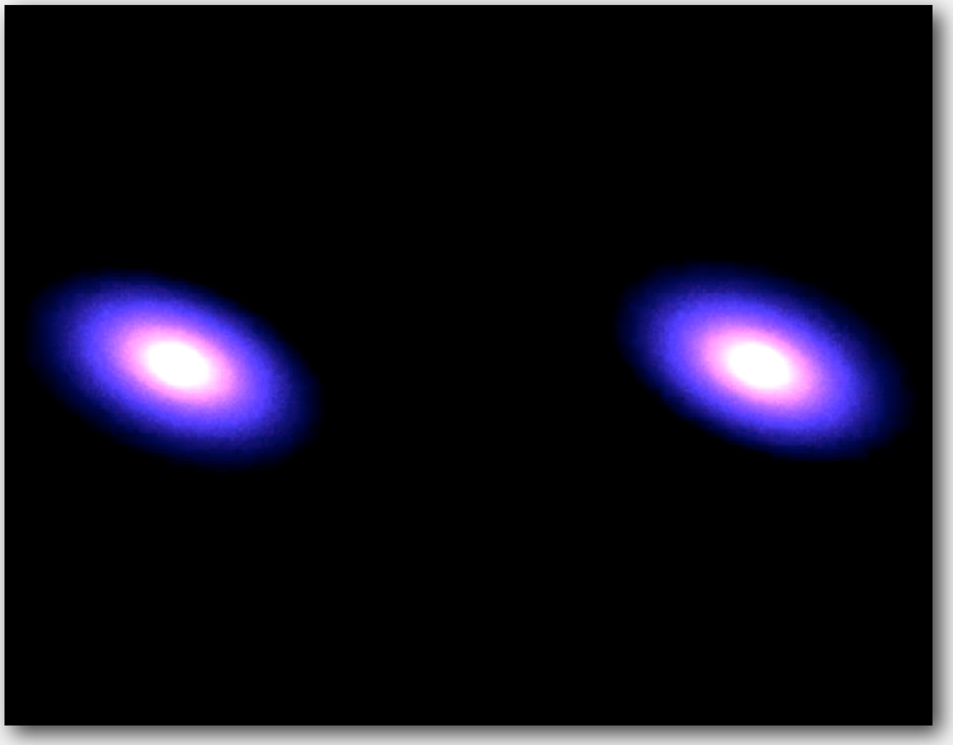




Typical running time. $R + \log N$.

N-body simulation

Goal. Simulate the motion of N particles, mutually affected by gravity.



http://www.youtube.com/watch?v=ua7YlN4eL_w

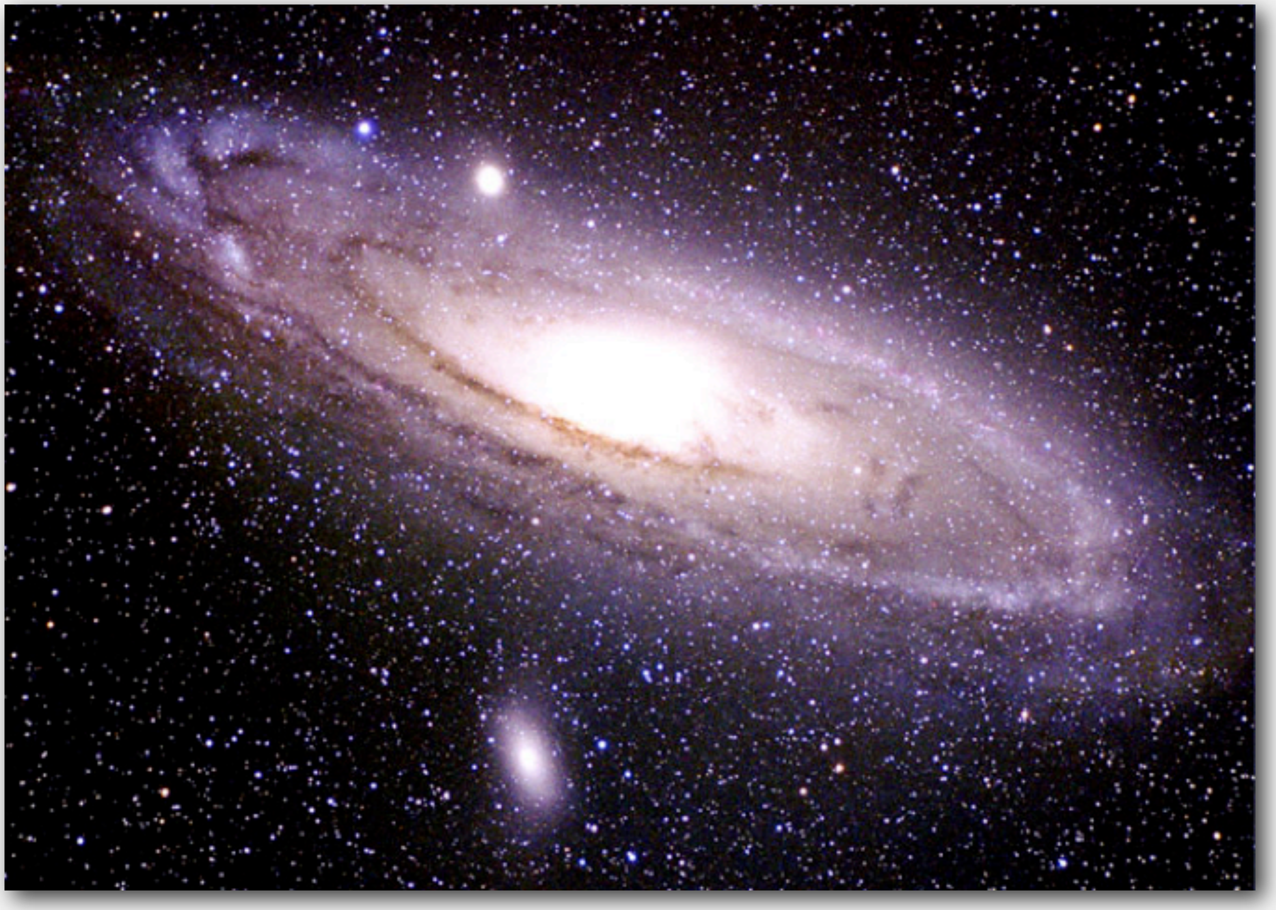
Brute force. For each pair of particles, compute force.

$$F = \frac{G\, m_1\, m_2}{r^2}$$

Barnes-Hut algorithm for N-body simulation

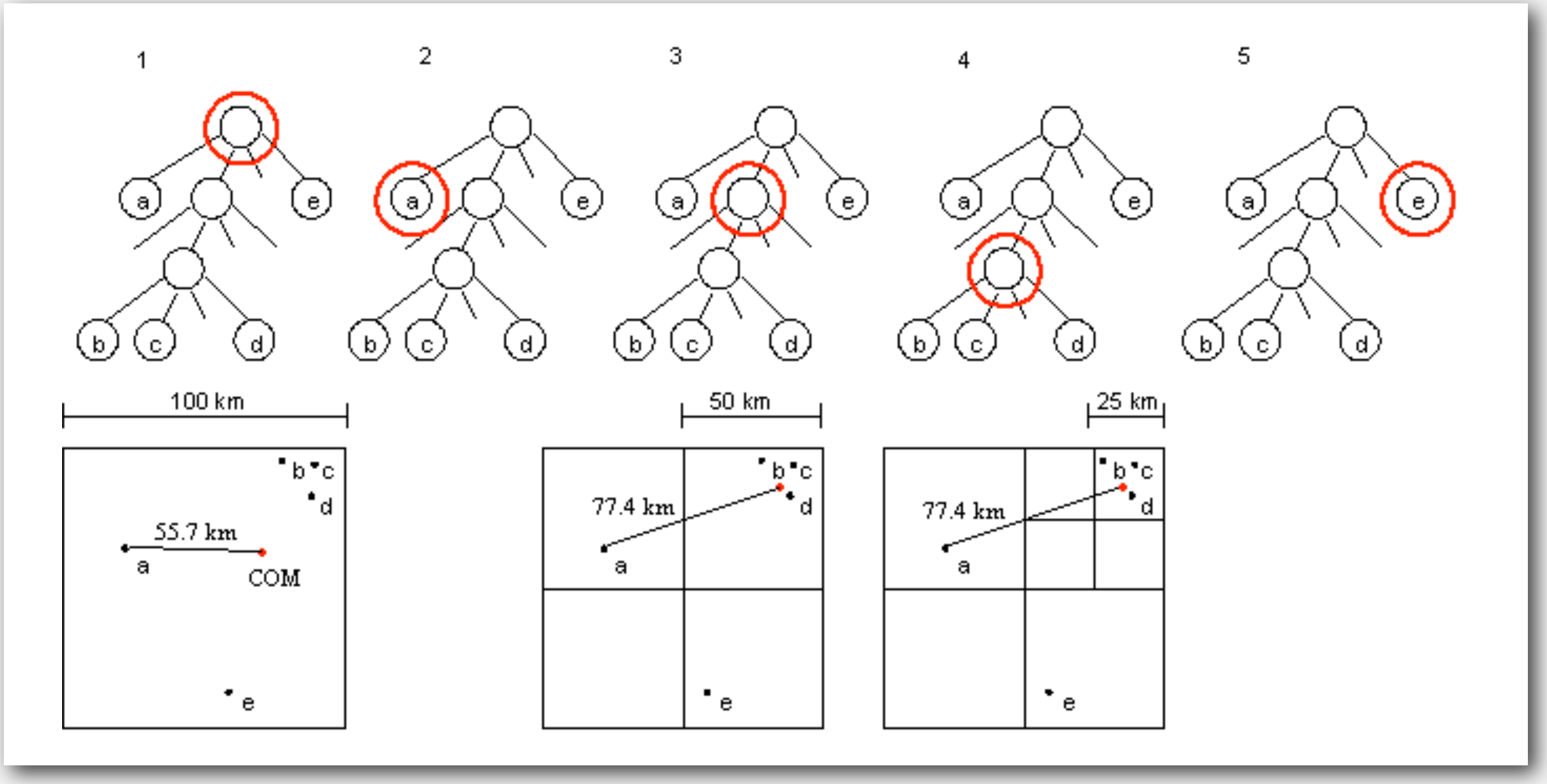
Key idea. Suppose particle is far, far away from cluster of particles.

- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.



Barnes-Hut algorithm for N-body simulation

- Build quadtree with N particles as external nodes.
- Store center-of-mass of subtree in each internal node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to quad is sufficiently large.



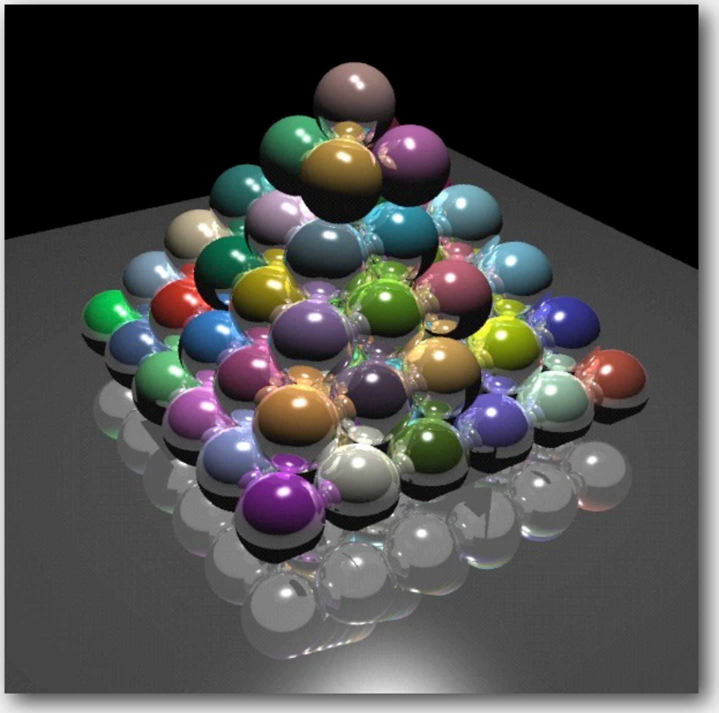
Curse of dimensionality

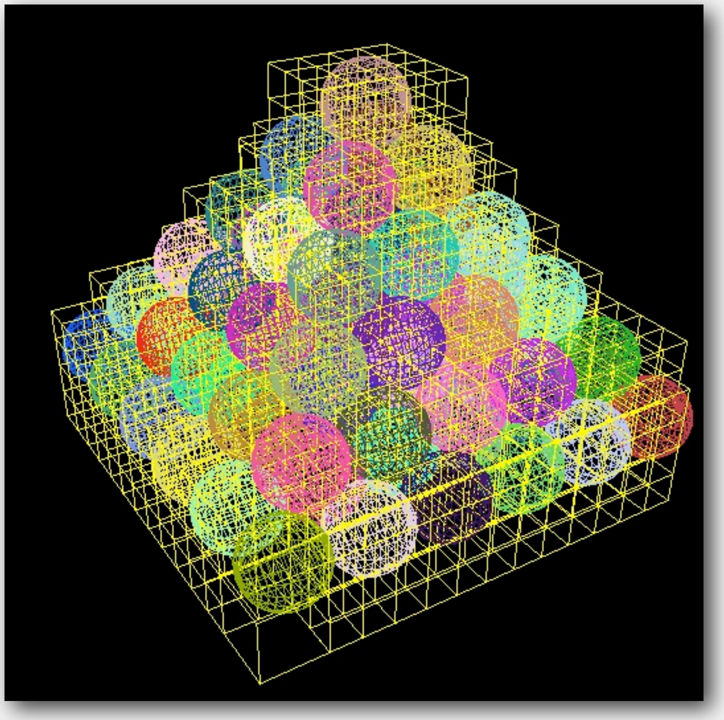
kd range search. Orthogonal range search in k-dimensions.

Main application. Multi-dimensional databases.

3d space. Octrees: recursively subdivide 3d space into 8 octants.

100d space. Centrees: recursively subdivide 100d space into 2^{100} centrants???



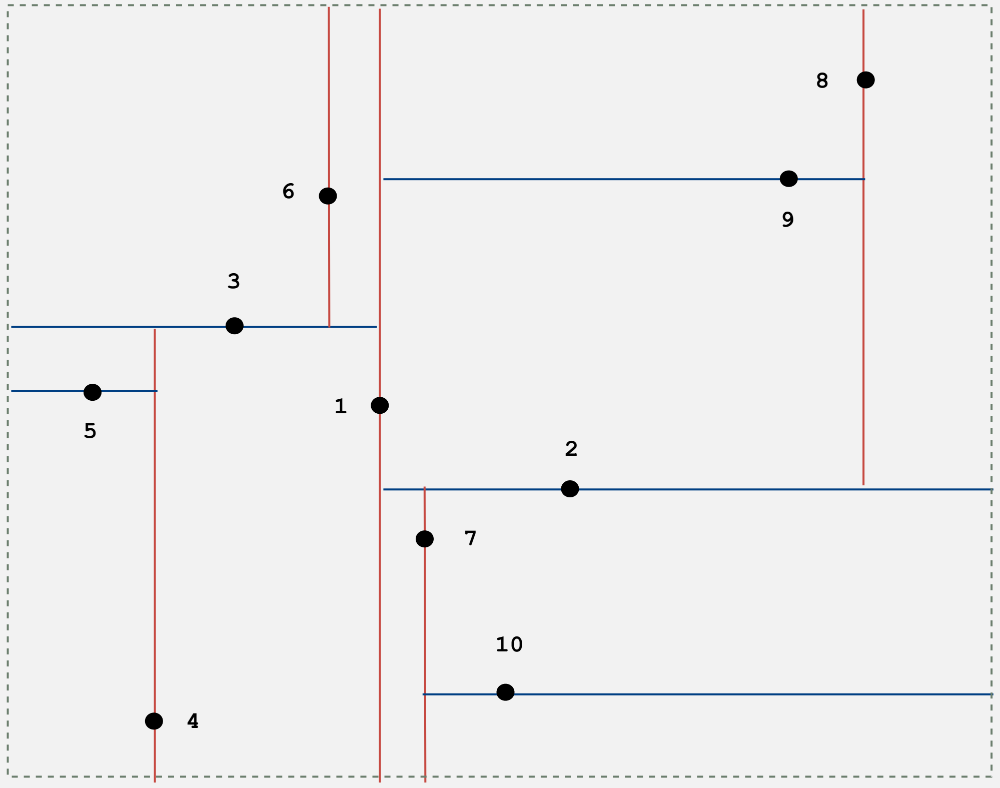


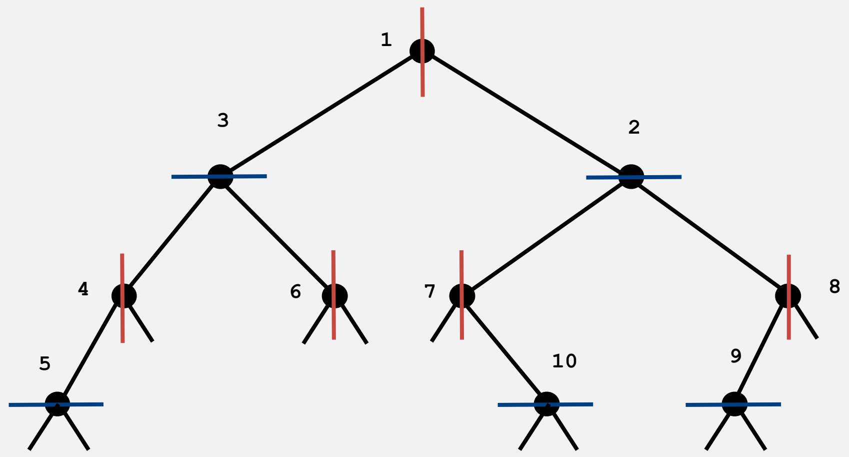
Raytracing with octrees

http://graphics.cs.ucdavis.edu/~gregorsk/graphics/275.html

2d tree

Recursively partition plane into two halfplanes.

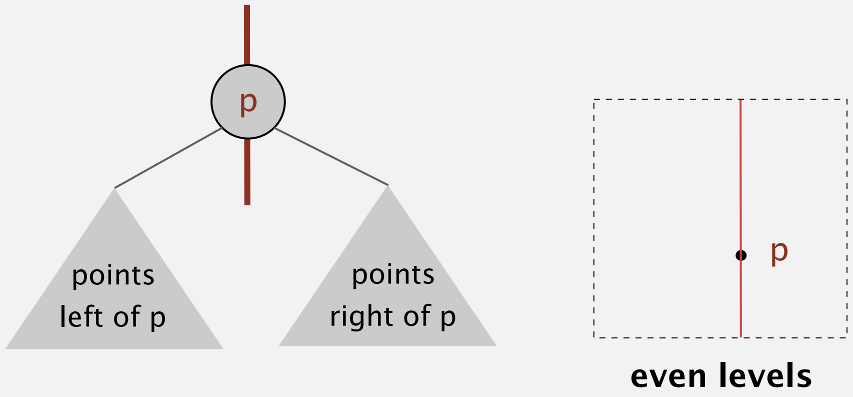


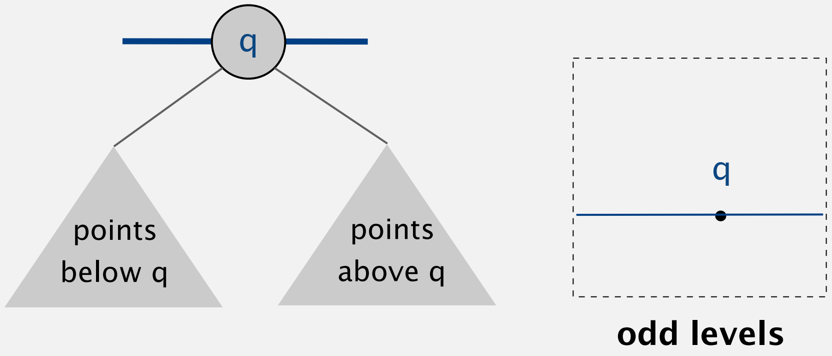


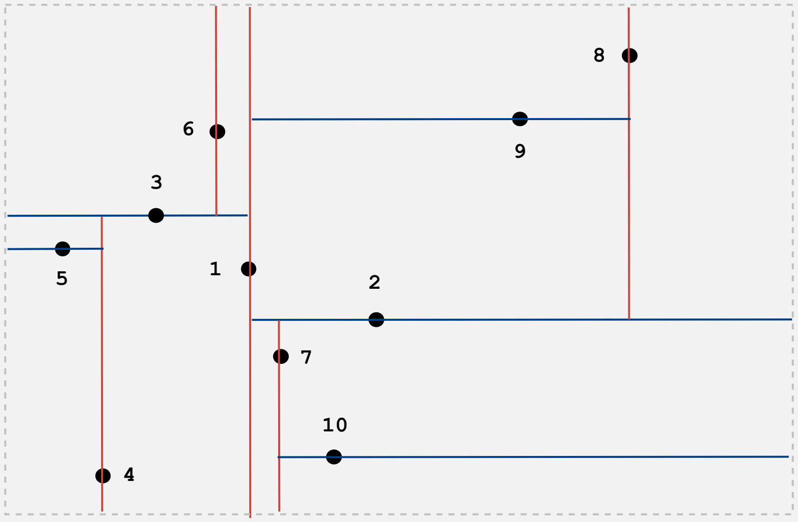
2d tree implementation

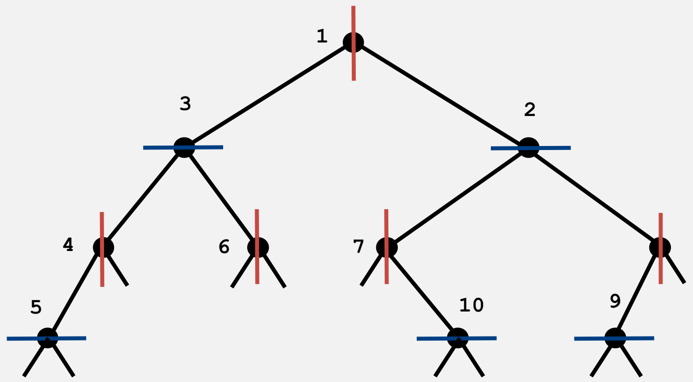
Data structure. BST, but alternate using x- and y-coordinates as key.

- Search gives rectangle containing point.
- Insert further subdivides the plane.









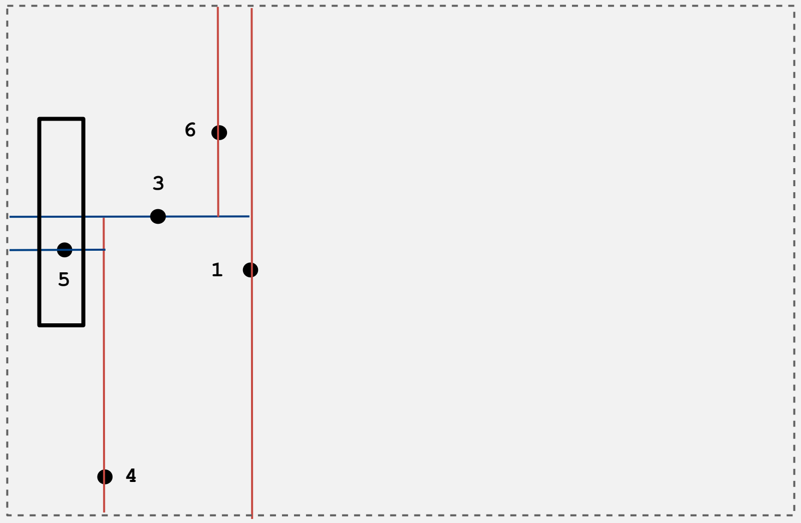
2d tree: 2d orthogonal range search

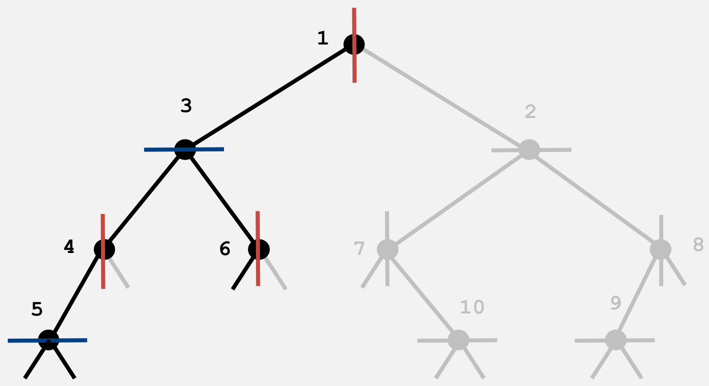
Range search. Find all points in a query axis-aligned rectangle.

- Check if point in node lies in given rectangle.
- Recursively search left/top subdivision (if any could fall in rectangle).
- Recursively search right/bottom subdivision (if any could fall in rectangle).

Typical case. $R + \log N$.

Worst case (assuming tree is balanced). $R + \sqrt{N}$.





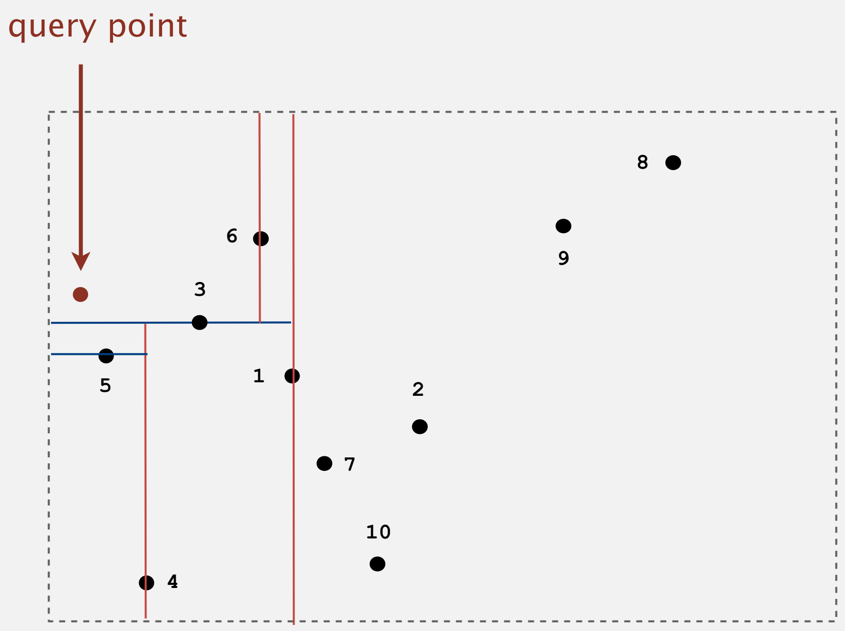
2d tree: nearest neighbor search

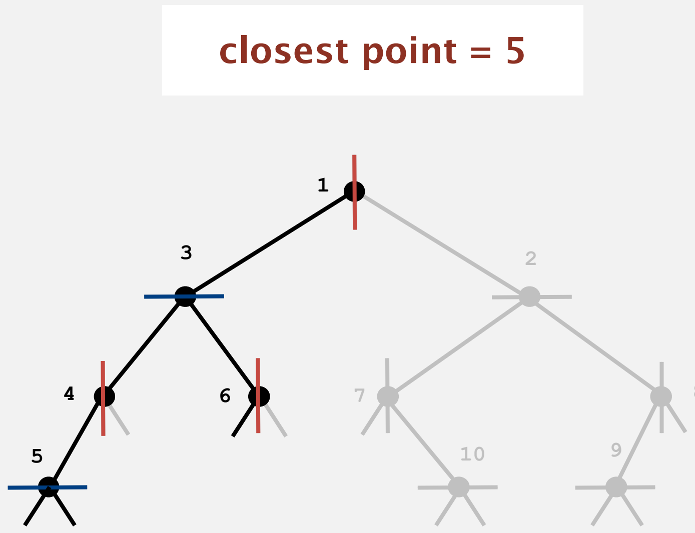
Nearest neighbor search. Given a query point, find the closest point.

- Check distance from point in node to query point.
- Recursively search left/top subdivision (if it could contain a closer point).
- Recursively search right/bottom subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

Typical case. $\log N$.

Worst case (even if tree is balanced). N.

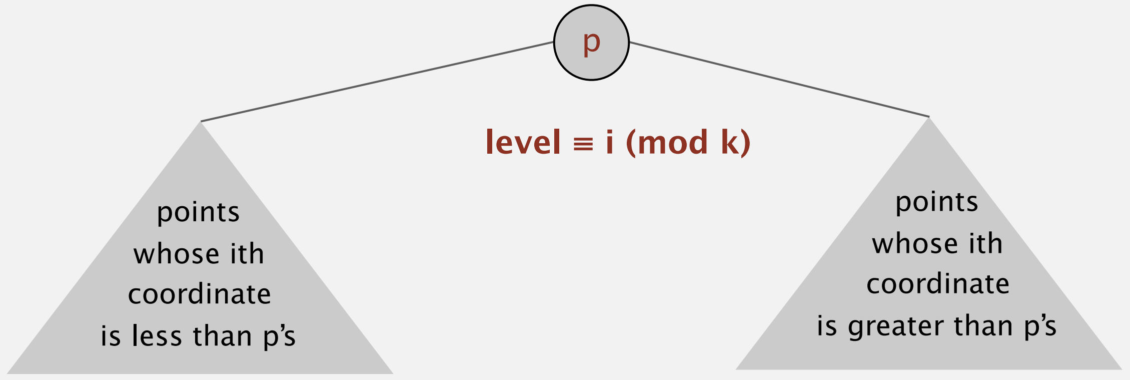


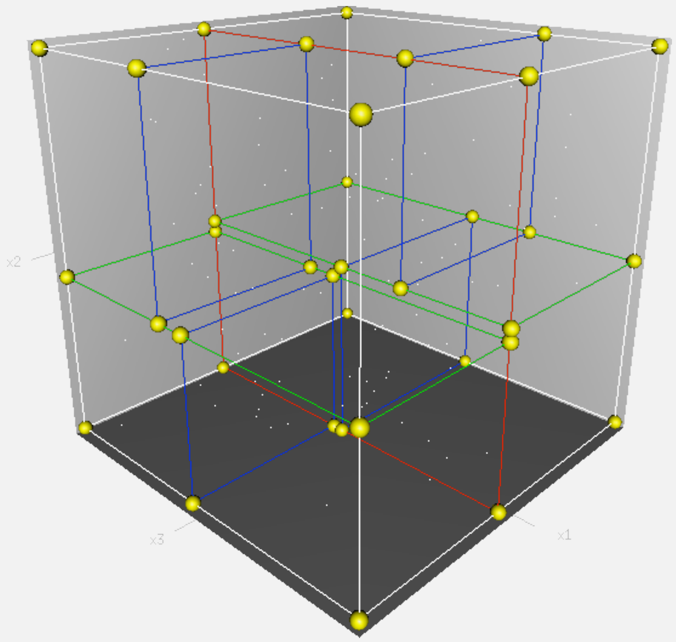


Kd tree

Kd tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.


p
level ≡ i (mod k)
points whose ith coordinate is less than p's
points whose ith coordinate is greater than p's



Efficient, simple data structure for processing k-dimensional data.

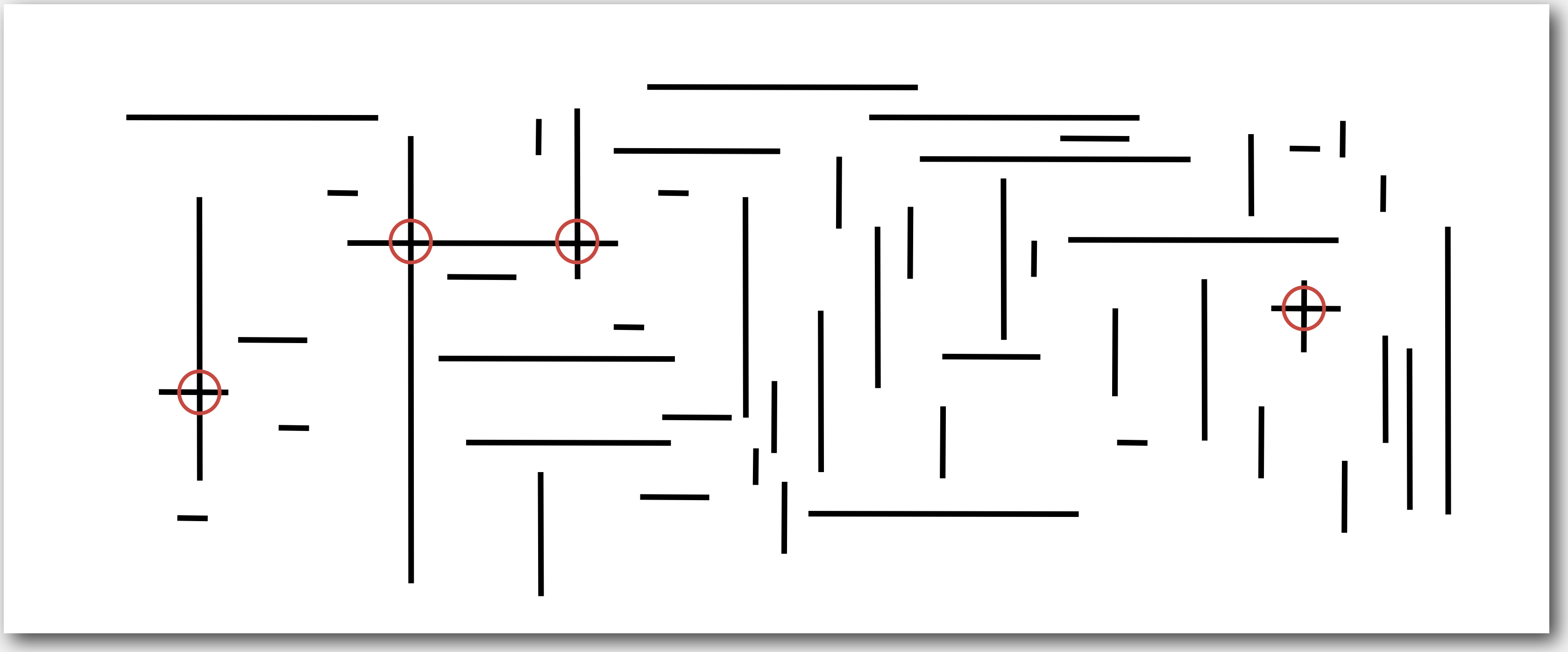
- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad (Jon Bentley) in an algorithms class!

Search for intersections

Problem. Find all intersecting pairs among N geometric objects.

Applications. CAD, games, movies, virtual reality, ...

Simple version. 2d, all objects are horizontal or vertical line segments.

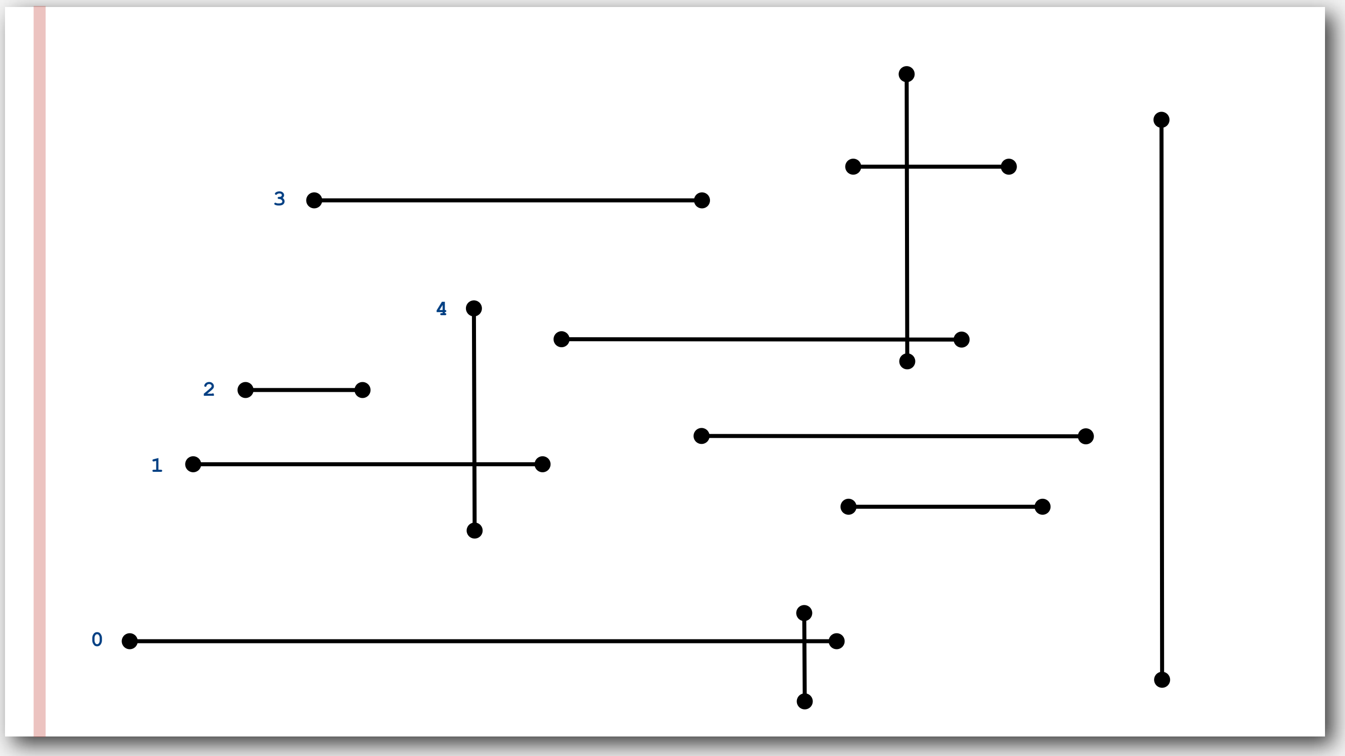


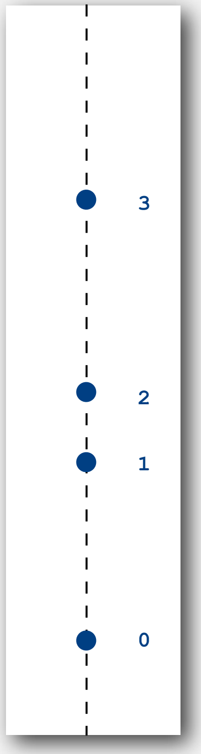
Brute force. Test all $\Theta(N^2)$ pairs of line segments for intersection.

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into ST.



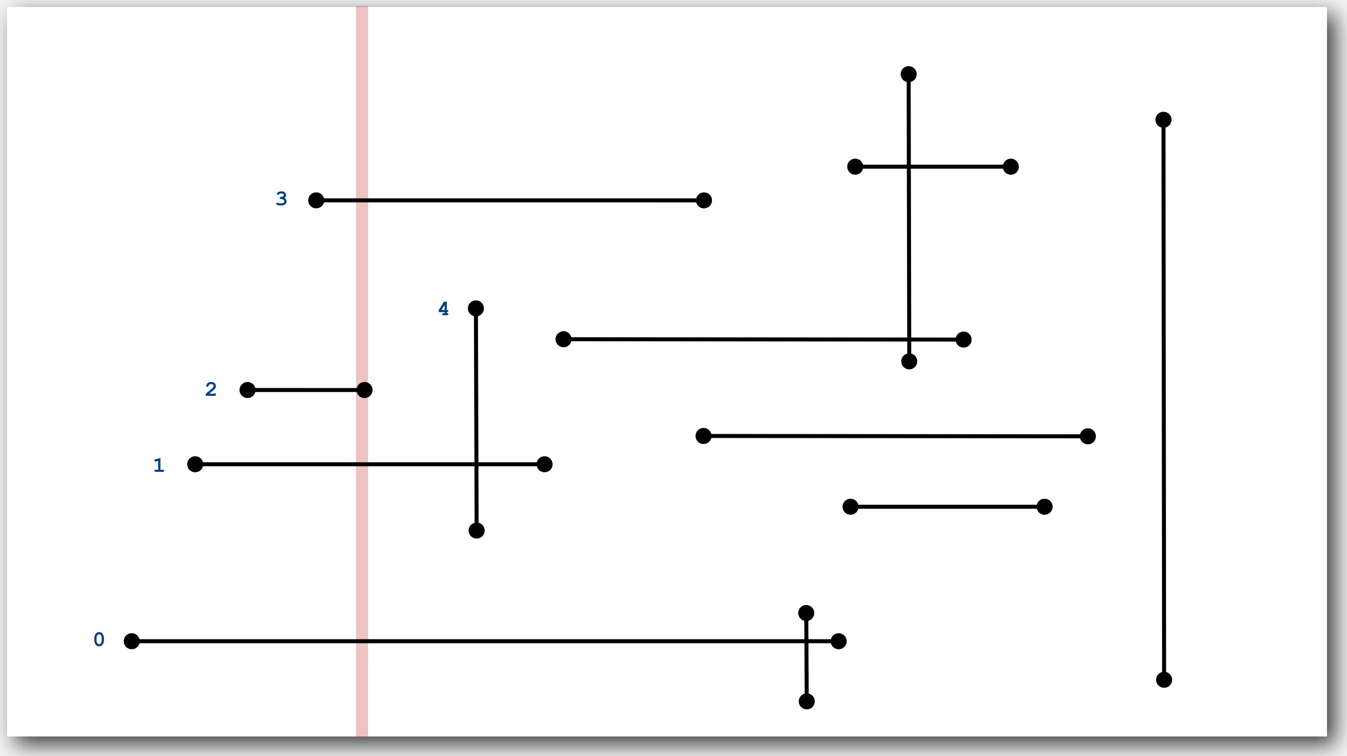


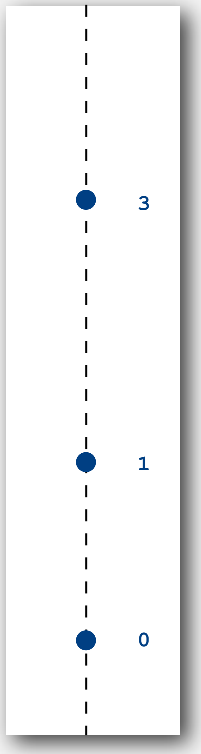
y-coordinates

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into ST.
- h-segment (right endpoint): remove y-coordinate from ST.



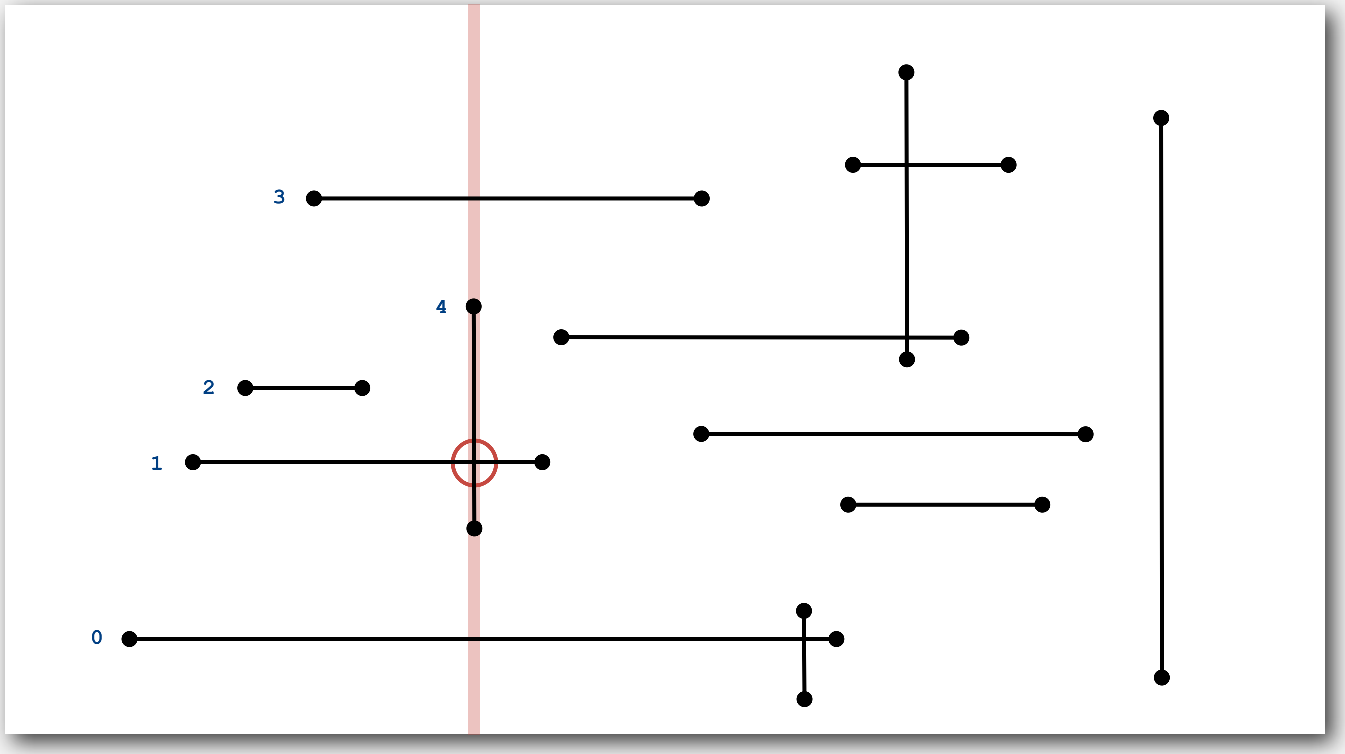


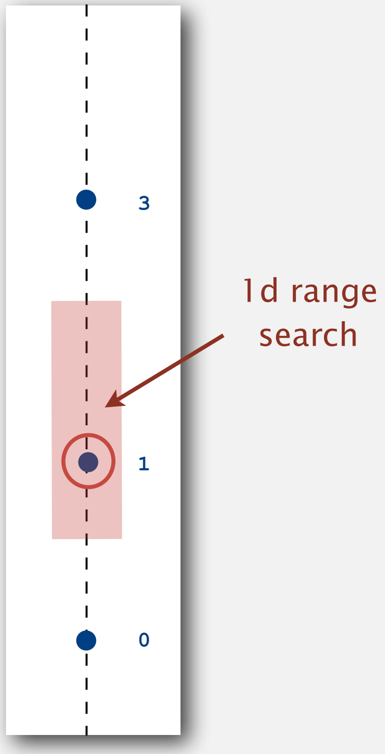
y-coordinates

Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into ST.
- h-segment (right endpoint): remove y-coordinate from ST.
- v-segment: range search for interval of y-endpoints.





y-coordinates

Orthogonal line segment intersection search: sweep-line algorithm

Sweep line reduces 2d orthogonal line segment intersection to 1d range search.

Proposition. The sweep-line algorithm takes time proportional to $N \log N + R$ to find all R intersections among N orthogonal segments.

• Put x-coordinates on a PQ (or sort).	$N \log N$
• Insert y-coordinates into ST.	$N \log N$
• Delete y-coordinates from ST.	$N \log N$
• Range searches.	$N \log N \; + R$

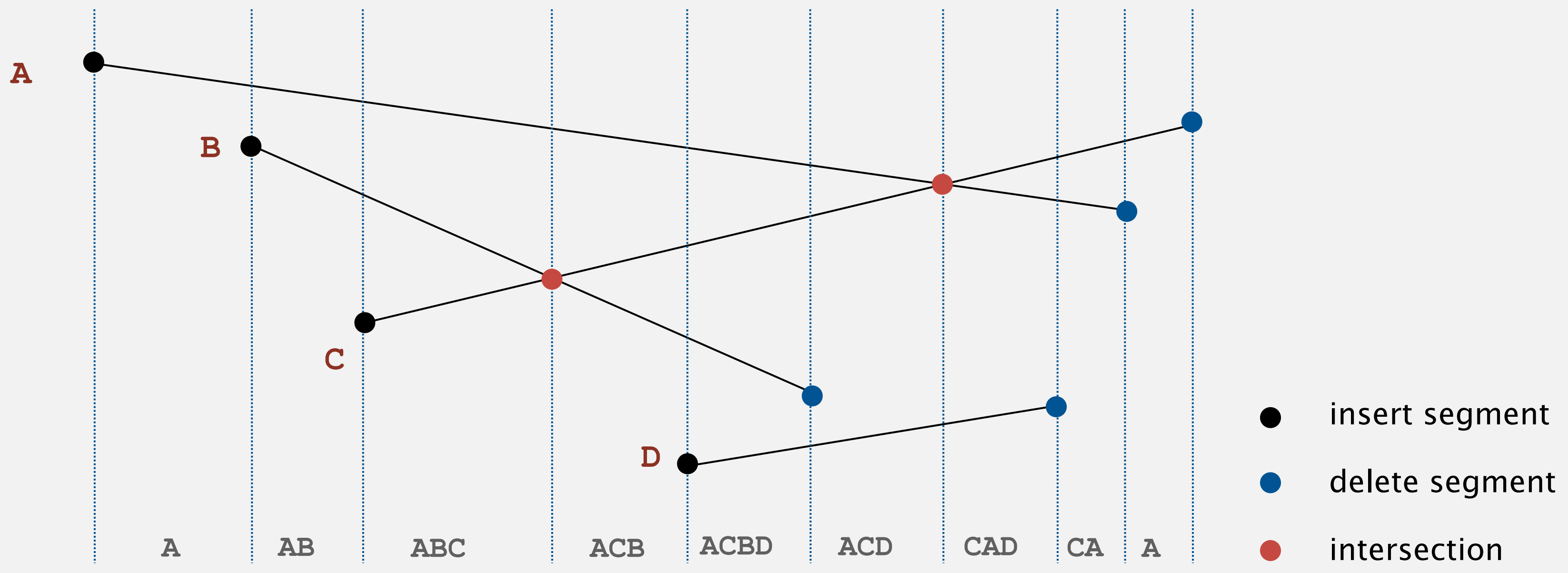
Efficiency relies on judicious use of data structures.

Remark. Sweep-line solution extends to 3d and more general shapes.

General line segment intersection search

Extend sweep-line algorithm.

- Maintain segments that intersect sweep line ordered by y-coordinate.
- Intersections can only occur between adjacent segments.
- Add/delete line segment $\Rightarrow$ one new pair of adjacent segments.
- Intersection $\Rightarrow$ swap adjacent segments.



order of segments that intersect sweep line

Line segment intersection: implementation

Efficient implementation of sweep line algorithm.

- Maintain PQ of important x-coordinates: endpoints and intersections.
- Maintain set of segments intersecting sweep line, sorted by x.
- Time proportional to $R \log N + N \log N$.

$\uparrow$ to support "next largest" and "next smallest" queries

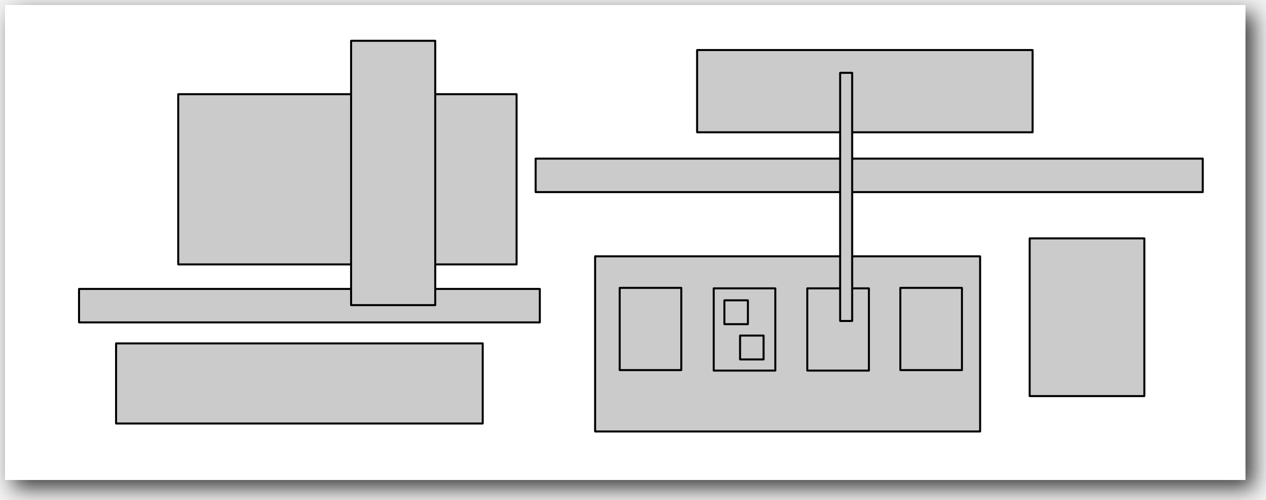
Implementation issues.

- Degeneracy.
- Floating-point precision.
- Must use PQ, not presort (intersection events are unknown ahead of time).

Orthogonal rectangle intersection search

Goal. Find all intersections among a set of N orthogonal rectangles.

Non-degeneracy assumption. All x- and y-coordinates are distinct.



Application. Design-rule checking in VLSI circuits.

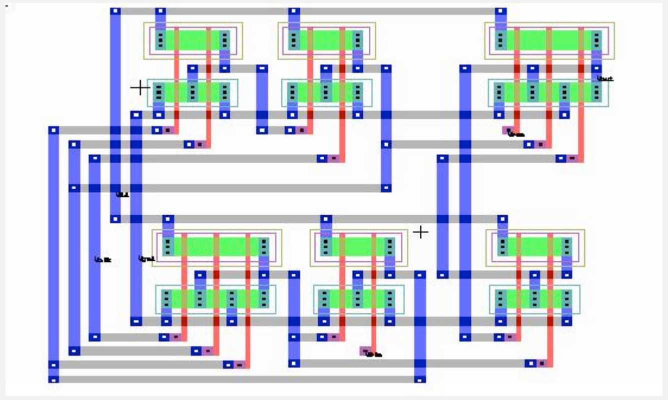
Microprocessors and geometry

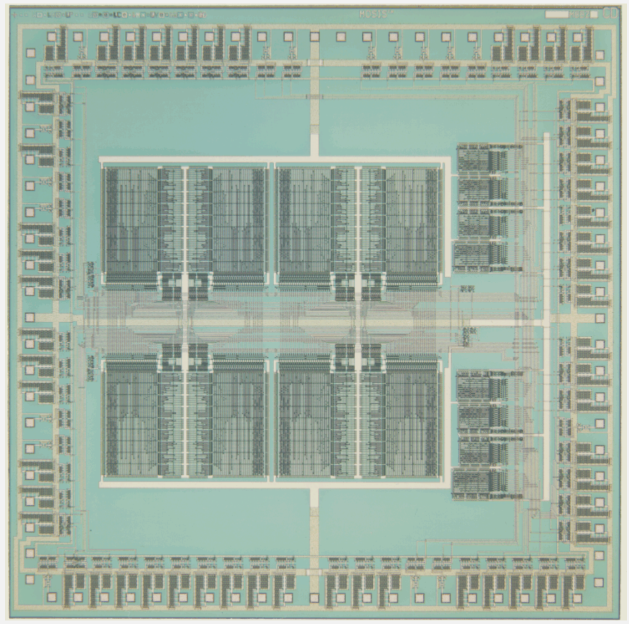
Early 1970s. microprocessor design became a geometric problem.

- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.

- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = orthogonal rectangle intersection search.





Algorithms and Moore's law

"Moore's law." Processing power doubles every 18 months.

- 197x: need to check N rectangles.
- 197$(x+1.5)$: need to check $2N$ rectangles on a 2x-faster computer.

Bootstrapping. We get to use the faster computer for bigger circuits.

But bootstrapping is not enough if using a quadratic algorithm:

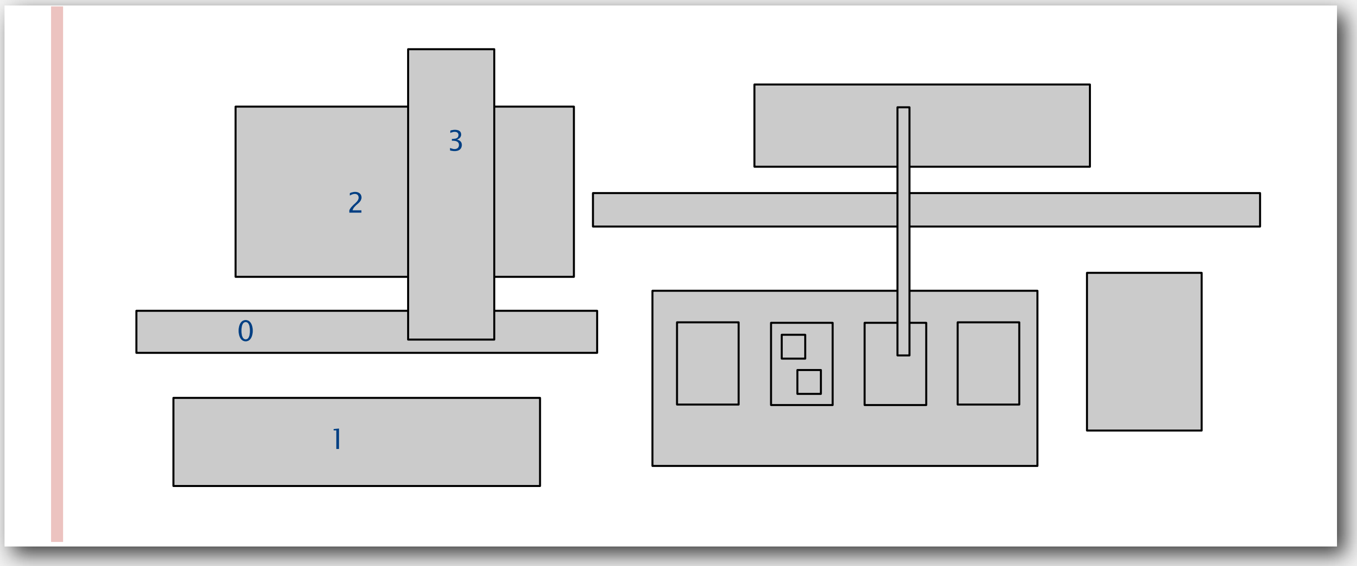
- 197x: takes M days.
- 197$(x+1.5)$: takes $\underset{\substack{\uparrow \\ \text{quadratic} \\ \text{algorithm}}}{(4M)} / 2 = \underset{\substack{\uparrow \\ \text{2x-faster} \\ \text{computer}}}{2M}$ days. (!)

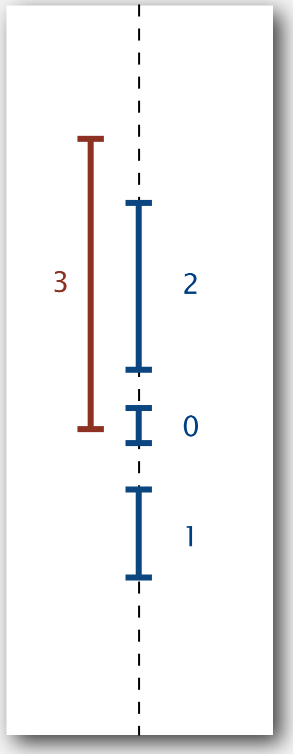
Bottom line. Linearithmic CAD algorithm is necessary to sustain Moore's Law.

Orthogonal rectangle intersection search

Move a vertical "sweep line" from left to right.

- Sweep line: sort rectangles by x-coordinates and process in this order, stopping on left and right endpoints.
- Maintain set of y-intervals intersecting sweep line.
- Left endpoint: search set for intersecting y-intervals; insert y-interval.
- Right endpoint: delete y-interval.



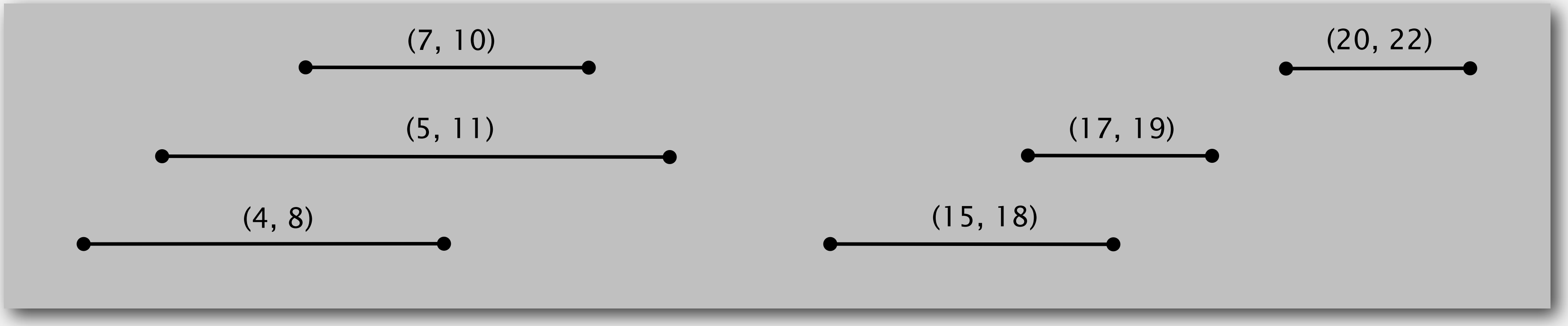


y-coordinates

Interval search trees

```
public class IntervalST<Key extends Comparable<Key>, Value>
```

	`IntervalST()`	*create interval search tree*
`void`	`put(Key lo, Key hi, Value val)`	*put interval-value pair into ST*
`Value`	`get(Key lo, Key hi)`	*return value paired with given interval*
`void`	`remove(Key lo, Key hi)`	*remove the given interval*
`Iterable<Value>`	`intersects(Key lo, Key hi)`	*return all intervals that intersect the given interval*



Interval search trees

`public class IntervalST<Key extends Comparable<Key>, Value>`

	`IntervalST()`	*create interval search tree*
`void`	`put(Key lo, Key hi, Value val)`	*put interval-value pair into ST*
`Value`	`get(Key lo, Key hi)`	*return value paired with given interval*
`void`	`remove(Key lo, Key hi)`	*remove the given interval*
`Iterable<Value>`	`intersects(Key lo, Key hi)`	*return all intervals that intersect the given interval*

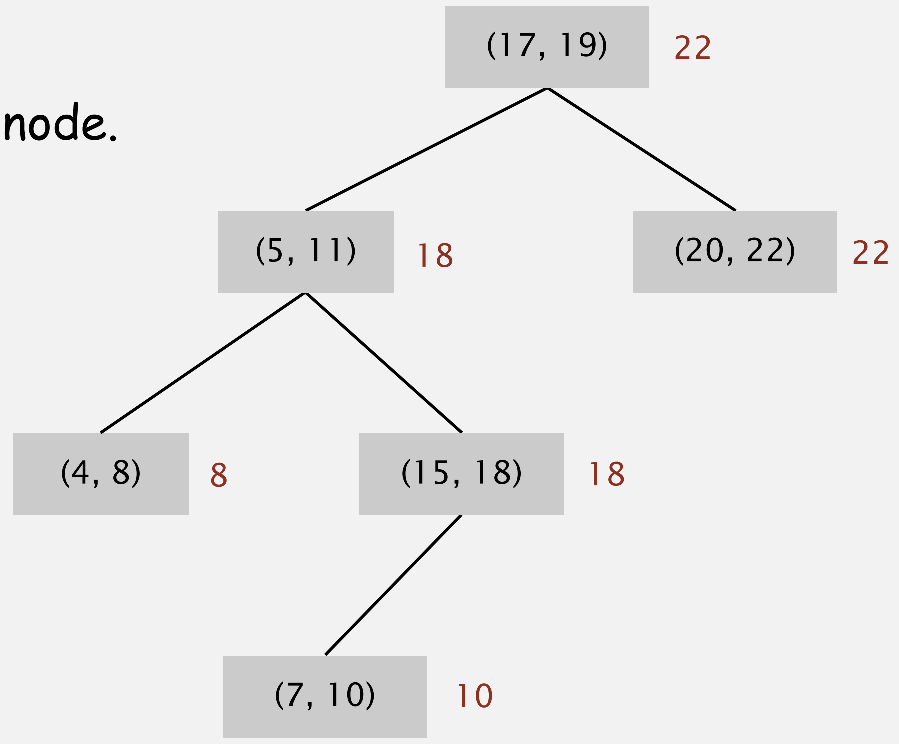
Non-degeneracy assumption. No two intervals have the same left endpoint.

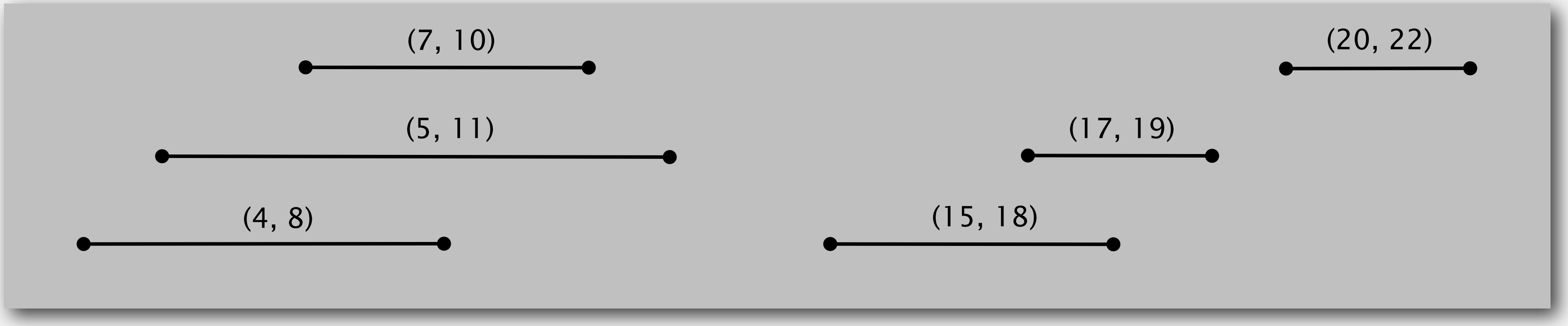
Interval search trees

Create BST, where each node stores an interval `(lo, hi)`.

- Use left endpoint as BST key.
- Store max endpoint in subtree rooted at node.

Suffices to implement all ops efficiently!



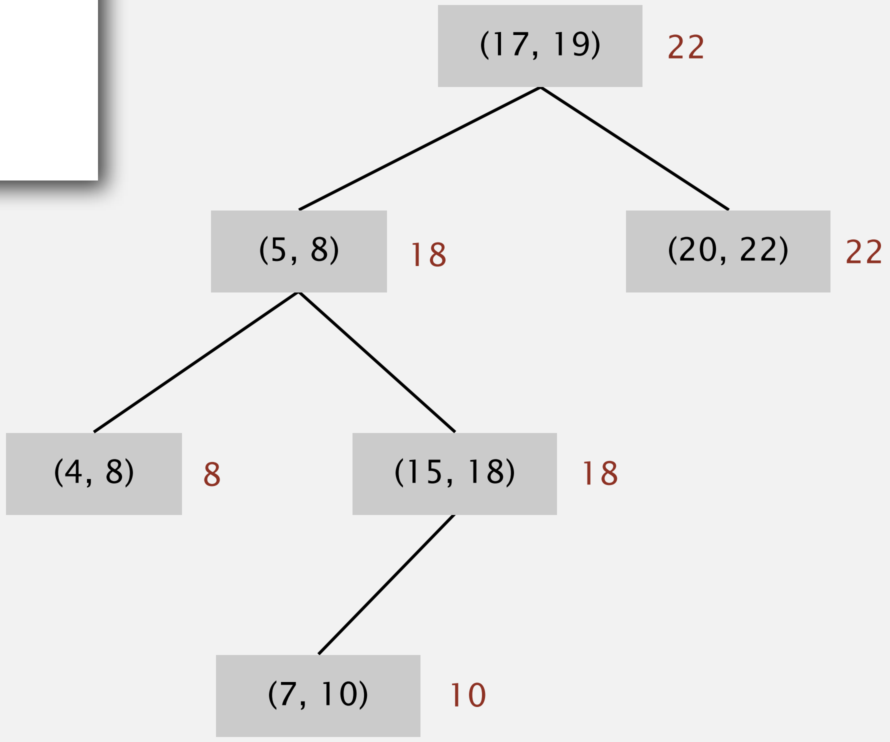


Finding an intersecting interval

To search for any interval that intersects query interval (lo, hi) :

```
Node x = root;
while (x != null)
{
   if (x.interval.intersects(lo, hi))
      return x.interval;
   else if (x.left == null)   x = x.right;
   else if (x.left.max < lo)  x = x.right;
   else                       x = x.left;
}
return null;
```

Ex. Search for $(9, 10)$.



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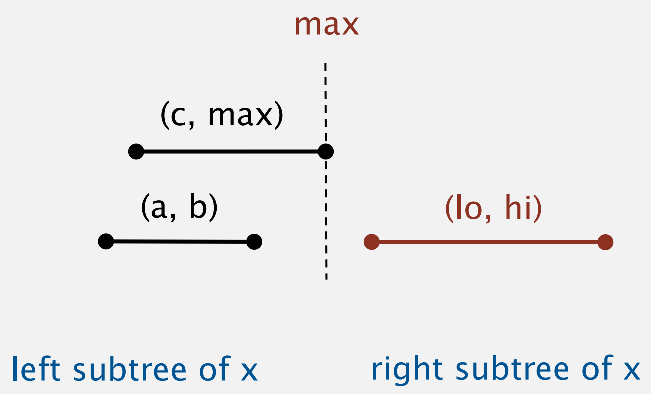

Case 1. If search goes right, then no intersection in left.

Pf.

- `(x.left == null)` $\Rightarrow$ trivial.
- `(x.left.max < lo)` $\Rightarrow$ for any interval (a, b) in left subtree of x, we have $b \le max < lo$.

definition of max

reason for going right

Finding an intersecting interval

To search for any interval that intersects query interval (lo, hi) :

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Node x = root;
while (x != null)
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   if (x.interval.intersects(lo, hi))
      return x.interval;
   else if (x.left == null)   x = x.right;
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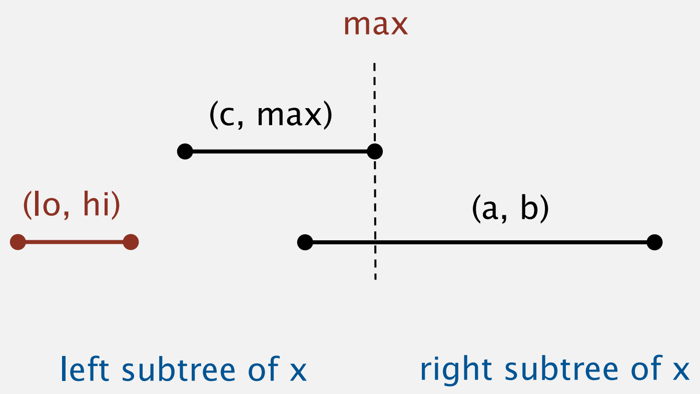
}
return null;
```



Case 2. If search goes left, then there is either an intersection in left subtree or no intersections in either.

Pf. Suppose no intersection in left. Then for any interval (a, b) in right subtree of x, $hi < c \leq a \Rightarrow$ no intersection in right.

no intersections in left subtree

intervals sorted by left endpoint

Interval search tree: analysis

Implementation. Use a red-black BST to guarantee performance.

$\nwarrow$ can maintain auxiliary information using log N extra work per op

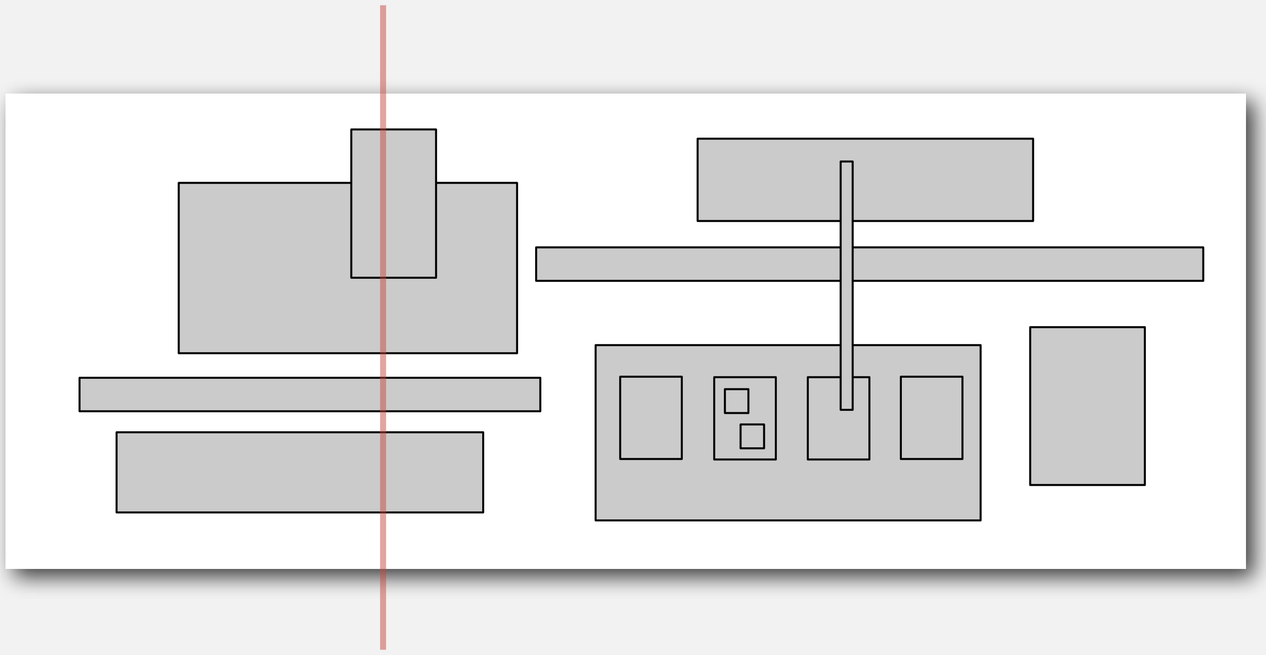
operation	brute	interval search tree	best in theory
insert interval	1	log N	log N
find interval	N	log N	log N
delete interval	N	log N	log N
find any interval that intersects (lo, hi)	N	log N	log N
find all intervals that intersects (lo, hi)	N	R log N	R + log N

order of growth of running time for N intervals

Rectangle intersection sweep-line algorithm: review

Move a vertical "sweep line" from left to right.

- Sweep line: sort rectangles by x-coordinates and process in this order, stopping on left and right endpoints.
- Maintain set of rectangles that intersect the sweep line in an interval search tree (using y-intervals of rectangle).
- Left endpoint: interval search for y-interval of rectangle; insert y-interval.
- Right endpoint: delete y-interval.



Rectangle intersection search: costs summary

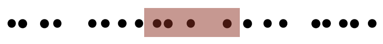
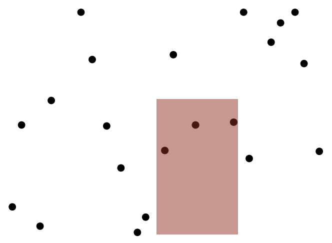
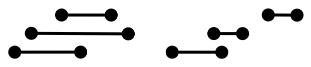
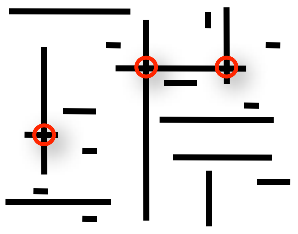
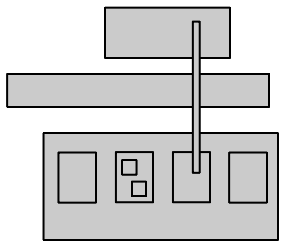
Sweep line reduces 2d orthogonal rectangle intersection search to 1d interval search.

Proposition. Sweep line algorithm takes time proportional to $N \log N + R$ to find R intersections among a set of N rectangles.

• Put x-coordinates on a PQ (or sort).	$N \log N$
• Insert y-intervals into ST.	$N \log N$
• Delete y-intervals from ST.	$N \log N$
• Interval searches for y-intervals.	$N \log N + R$

Efficiency relies on judicious use of data structures.

Geometric search summary: algorithms of the day

problem	example	solution
1d range search		BST
kd orthogonal range search		kd tree
1d interval search		interval search tree
2d orthogonal line segment intersection		sweep line reduces to 1D range search
2d orthogonal rectangle intersection		sweep line reduces to 1D interval search