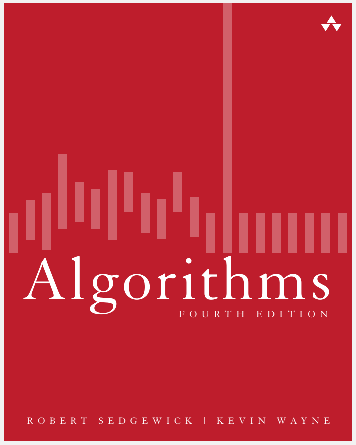
6.5 Reductions



- designing algorithms
- establishing lower bounds
- classifying problems
- intractability

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Bird's-eye view

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, ...
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, ...
quadratic	N^2	???
	...	
exponential	c^N	???

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

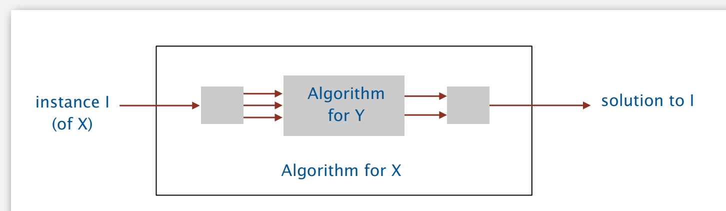
Desiderata. Classify problems according to computational requirements.

Desiderata'.
Suppose we could (could not) solve problem X efficiently.
What else could (could not) we solve efficiently?

" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. " — *Archimedes*

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

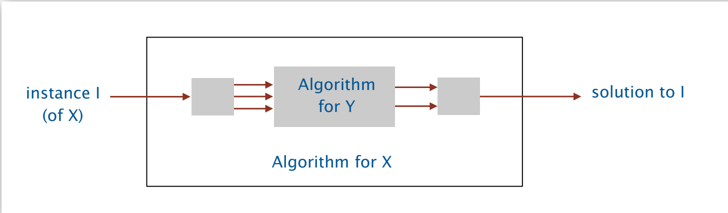


Cost of solving X = total cost of solving Y + cost of reduction.

- ↑ perhaps many calls to Y on problems of different sizes
- ↑ preprocessing and postprocessing

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



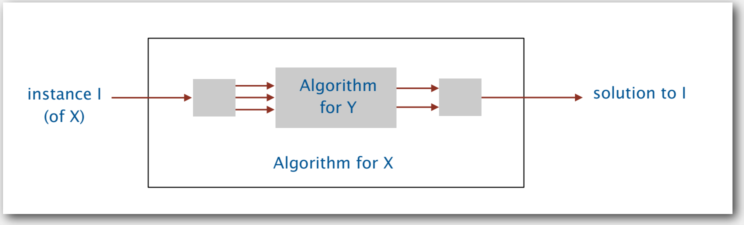
Ex 1. [element distinctness reduces to sorting]
To solve element distinctness on N integers:
- Sort N integers.
- Check adjacent pairs for equality.

cost of sorting
cost of reduction

Cost of solving element distinctness. $N \log N + N$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [3-collinear reduces to sorting]
To solve 3-collinear instance on N points in the plane:
- For each point, sort other points by polar angle.
  - check adjacent triples for collinearity

cost of sorting
cost of reduction

Cost of solving 3-collinear. $N^2 \log N + N^2$.

- designing algorithms
- establishing lower bounds
- classifying problems
- intractability

Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given algorithm for Y, can also solve X.

Ex.
- Element distinctness reduces to sorting.
- 3-collinear reduces to sorting.
- PERT reduces to topological sort. [see shortest paths lecture]
- h-v line intersection reduces to 1d range searching. [see geometry lecture]
- Burrows-Wheeler transform reduces to suffix sort. [see assignment 8]

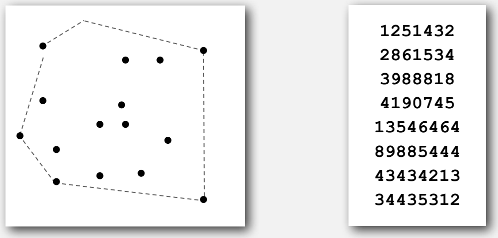
Mentality. Since I know how to solve Y, can I use that algorithm to solve X?

programmer's version: I have code for Y. Can I use it for X?

Convex hull reduces to sorting

Sorting. Given N distinct integers, rearrange them in ascending order.

Convex hull. Given N points in the plane, identify the extreme points of the convex hull (in counterclockwise order).



convex hull sorting

Proposition. Convex hull reduces to sorting.

Pf. Graham scan algorithm.

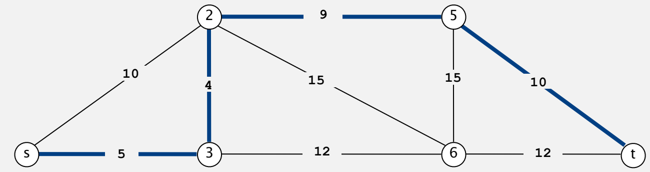
cost of sorting

cost of reduction

Cost of convex hull. $N \log N + N$.

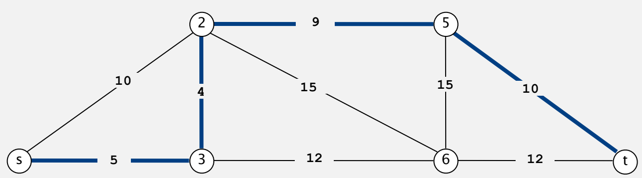
Shortest paths on graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.

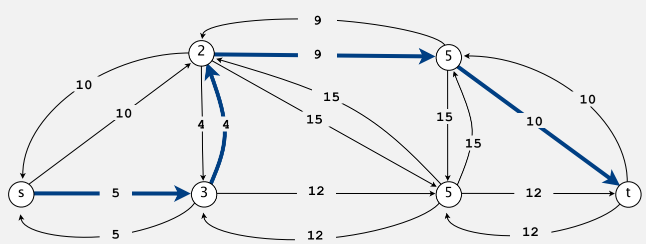


Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.

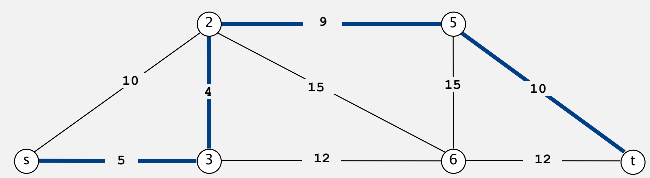


Pf. Replace each undirected edge by two directed edges.



Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.



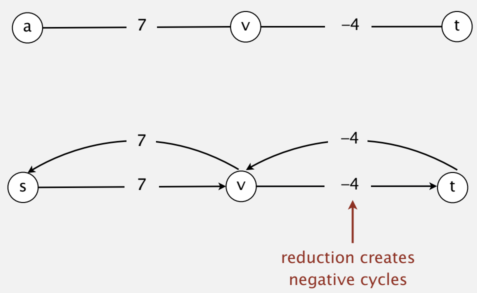
Cost of undirected shortest paths. $E \log V + E$.

cost of shortest paths in digraph

cost of reduction

Shortest paths with negative weights

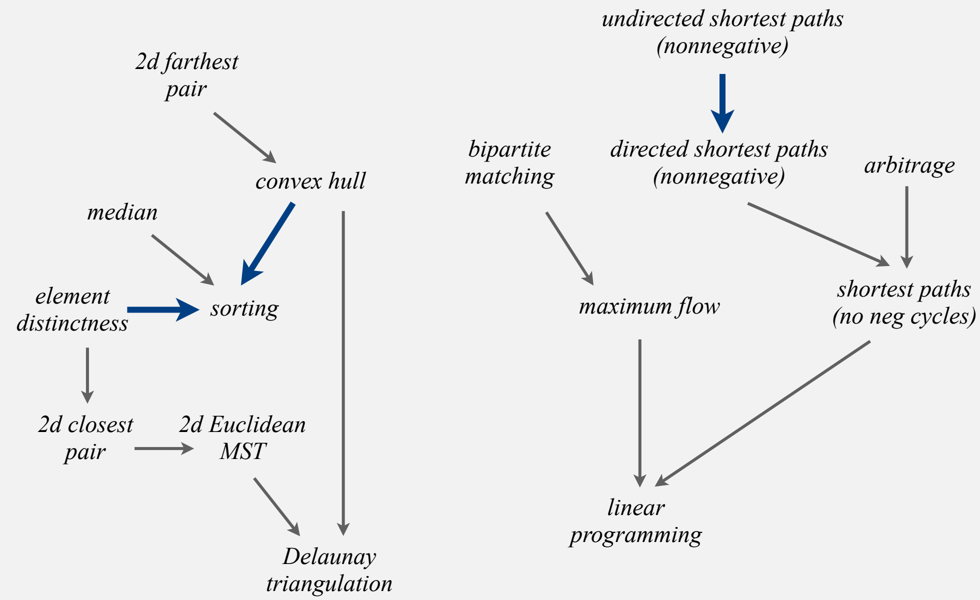
Caveat. Reduction is invalid for edge-weighted graphs with negative weights (even if no negative cycles).



Remark. Can still solve shortest paths problem in undirected graphs (if no negative cycles), but need more sophisticated techniques.

reduces to weighted non-bipartite matching (!)

Some reductions involving familiar problems



‣ designing algorithms
‣ **establishing lower bounds**
‣ classifying problems
‣ intractability

Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.
Ex. $\Omega(N \log N)$ lower bound for sorting.

```
1251432
2861534
3988818
4190745
13546464
89885444
43434213
```

argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.

Good news. Can spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to Y.

Ex. Almost all of the reductions we've seen so far. [Which one wasn't?]

Establish lower bound:

- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.

Mentality.

- If I could easily solve Y, then I could easily solve X.
- I can't easily solve X.
- Therefore, I can't easily solve Y.

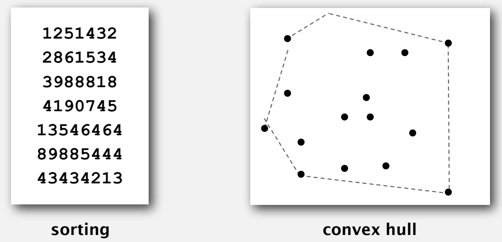
Lower bound for convex hull

Proposition. In quadratic decision tree model, any algorithm for sorting N integers requires $\Omega(N \log N)$ steps.

allows quadratic tests of the form: $x_i < x_j$ or $(x_j - x_i)(x_k - x_l) - (x_j)(x_j - x_i) < 0$

Proposition. Sorting linear-time reduces to convex hull.
Pf. [see next slide]

lower-bound mentality: if I can solve convex hull efficiently, I can sort efficiently



a quadratic test

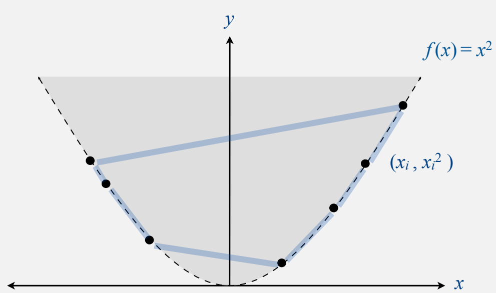
Implication. Any ccw-based convex hull algorithm requires $\Omega(N \log N)$ ccw's.

Sorting linear-time reduces to convex hull

Proposition. Sorting linear-time reduces to convex hull.

- Sorting instance: $x_1, x_2, \ldots, x_N$.
- Convex hull instance: $(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)$.

lower-bound mentality: if I can solve convex hull efficiently, I can sort efficiently



Pf.

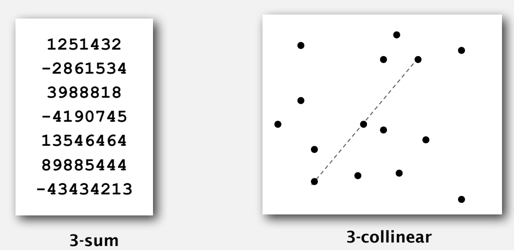
- Region $\{ x : x^2 \geq x \}$ is convex $\Rightarrow$ all points are on hull.
- Starting at point with most negative x, counterclockwise order of hull points yields integers in ascending order.

Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0?

3-COLLINEAR. Given N distinct points in the plane, are there 3 that all lie on the same line?

recall Assignment 3



Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0?

3-COLLINEAR. Given N distinct points in the plane, are there 3 that all lie on the same line?

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

Pf. [see next 2 slide]

Conjecture. Any algorithm for *3-SUM* requires $\Omega(N^2)$ steps.

Implication. No sub-quadratic algorithm for *3-COLLINEAR* likely.

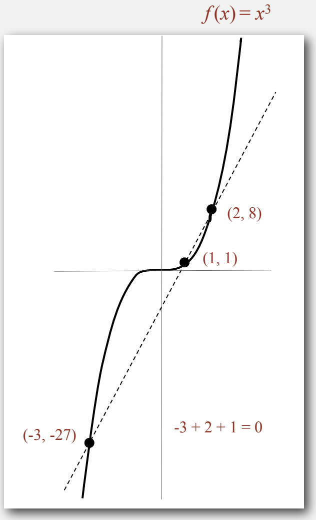
your $N^2 \log N$ algorithm was pretty good

3-SUM linear-time reduces to 3-COLLINEAR

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

- *3-SUM* instance: $x_1, x_2, \ldots, x_N$.
- *3-COLLINEAR* instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.



3-SUM linear-time reduces to 3-COLLINEAR

Proposition. *3-SUM* linear-time reduces to *3-COLLINEAR*.

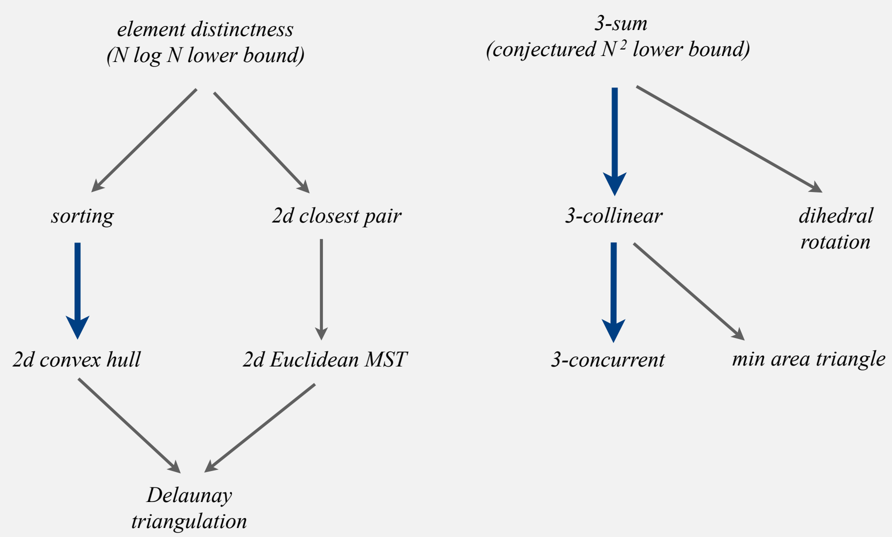
- *3-SUM* instance: $x_1, x_2, \ldots, x_N$.
- *3-COLLINEAR* instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.

Pf. Three distinct points (a, a^3), (b, b^3), and (c, c^3) are collinear iff:

$$
\begin{aligned}
0 &= \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} \\
&= a(b^3 - c^3) - b(a^3 - c^3) + c(a^3 - b^3) \\
&= (a - b)(b - c)(c - a)(a + b + c)
\end{aligned}
$$

More linear-time reductions and lower bounds



Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time convex hull algorithm exists?
A1. [hard way] Long futile search for a linear-time algorithm.
A2. [easy way] Linear-time reduction from sorting.

Q. How to convince yourself no sub-quadratic *3-COLLINEAR* algorithm likely.
A1. [hard way] Long futile search for a sub-quadratic algorithm.
A2. [easy way] Linear-time reduction from *3-SUM*.

- designing algorithms
- establishing lower bounds
- **classifying problems**
- intractability

Classifying problems: summary

Desiderata. Problem with algorithm that matches lower bound.
Ex. Sorting, convex hull, and closest pair have complexity $N \log N$.

Desiderata'. Prove that two problems X and Y have the same complexity.

- First, show that problem X linear-time reduces to Y.
- Second, show that Y linear-time reduces to X.

Conclude that X and Y have the same complexity.

even if we don't know what it is!

Primality testing

PRIME. Given an integer x (represented in binary), is x prime?
COMPOSITE. Given an integer x, does x have a nontrivial factor?

Proposition. *PRIME* linear-time reduces to *COMPOSITE*.

```
public static boolean isPrime(BigInteger x)
{
   if (isComposite(x)) return false;
   else                return true;
}
```

147573952589676412931
prime

147573952589676412927
composite

Primality testing

PRIME. Given an integer x (represented in binary), is x prime?
COMPOSITE. Given an integer x, does x have a nontrivial factor?

Proposition. *COMPOSITE* linear-time reduces to *PRIME*.

```
public static boolean isComposite(BigInteger x)
{
   if (isPrime(x)) return false;
   else            return true;
}
```

147573952589676412931
prime

147573952589676412927
composite

Caveat

PRIME. Given an integer x (represented in binary), is x prime?
COMPOSITE. Given an integer x, does x have a nontrivial factor?

Proposition. *PRIME* linear-time reduces to *COMPOSITE*.
Proposition. *COMPOSITE* linear-time reduces to *PRIME*.
Conclusion. *PRIME* and *COMPOSITE* have the same complexity.

best known deterministic algorithm is about N^6 for N-bit integer

A possible real-world scenario.

- System designer specs the APIs for project.
- Alice implements `isComposite()` using `isPrime()`.
- Bob implements `isPrime()` using `isComposite()`.
- Infinite reduction loop!
- Who's fault?

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

								1	1	0	1	0	1	0	1
							×	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Q. Is brute-force algorithm optimal?

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.
Karatsuba-Ofman (1962) $N^{1.585}$ bit operations.
Toom (1963) $N^{1+\varepsilon}$ bit operations.
Schönhage-Strassen (1971). $N \log N \log \log N$ bit operations.
Fürer (2007). $N \log N \, 2^{\log^* N}$ bit operations.

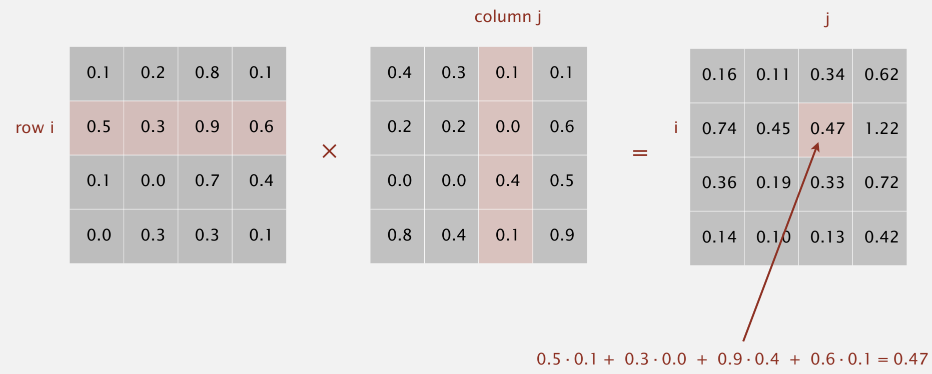
problem	arithmetic	order of growth
integer multiplication	a × b	M(N)
integer division	a / b, a mod b	M(N)
integer square	a ^{2}	M(N)
integer square root	⌊ √a ⌋	M(N)

integer arithmetic problems with the same complexity

Linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.



Q. Is brute-force algorithm optimal?

Linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.
Brute force. N^3 flops.
Strassen (1969). $N^{2.81}$ flops.
Coppersmith-Winograd (1987). $N^{2.376}$ flops.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	MM(N)
matrix inversion	A^{-1}	MM(N)
determinant	$\lvert A \rvert$	MM(N)
system of linear equations	$Ax = b$	MM(N)
LU decomposition	$A = L U$	MM(N)
least squares	$\min \lVert Ax - b \rVert_2$	MM(N)

numerical linear algebra problems with the same complexity

- designing algorithms
- establishing lower bounds
- classifying problems
- **intractability**

Bird's-eye view

Def. A problem is intractable if it can't be solved in polynomial time.
Desiderata. Prove that a problem is intractable.

Two problems that provably require exponential time.

- Given a constant-size program, does it halt in at most K steps? (input size = c + lg K)
- Given N-by-N checkers board position, can the first player force a win? (using forced capture rule)

Frustrating news. Few successes.

3-satisfiability

Literal. A boolean variable or its negation. x_i or $\neg x_i$

Clause. An *or* of 3 distinct literals. $C_1 = (\neg x_1 \vee x_2 \vee x_3)$

Conjunctive normal form. An *and* of clauses. $\Phi = (C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5)$

3-SAT. Given a CNF formula Φ consisting of k clauses over n literals, does it have a satisfying truth assignment?

$$\Phi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_2 \vee x_3 \vee x_4)$$

yes instance

x_1	x_2	x_3	x_4
T	T	F	T

$$(\neg T \vee T \vee F) \wedge (T \vee \neg T \vee F) \wedge (\neg T \vee \neg T \vee \neg F) \wedge (\neg T \vee \neg T \vee T) \wedge (\neg T \vee F \vee T)$$

Applications. Circuit design, program correctness, ...

3-satisfiability is conjectured to be intractable

Q. How to solve an instance of *3-SAT* with n variables?

A. Exhaustive search: try all 2^n truth assignments.

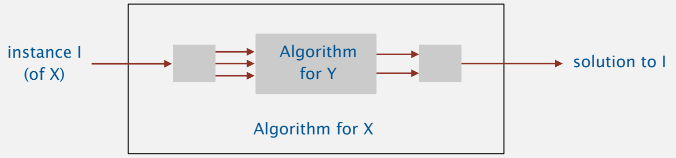
Q. Can we do anything substantially more clever?

Conjecture (P ≠ NP). *3-SAT* is intractable (no poly-time algorithm).

Polynomial-time reductions

Problem X poly-time (Cook) reduces to problem Y if X can be solved with:

- Polynomial number of standard computational steps.
- Polynomial number of calls to Y.



Establish intractability. If *3-SAT* poly-time reduces to Y, then Y is intractable. (assuming *3-SAT* is intractable)

Mentality.

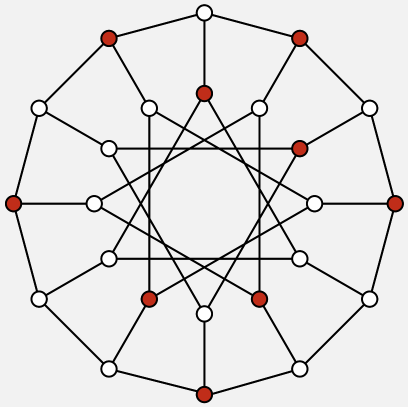
- If I could solve Y in poly-time, then I could also solve *3-SAT* in poly-time.
- *3-SAT* is believed to be intractable.
- Therefore, so is Y.

Independent set

An independent set is a set of vertices, no two of which are adjacent.

IND-SET. Given a graph G and an integer k, find an independent set of size k.

k = 9



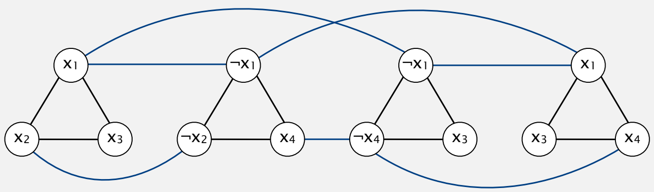
Applications. Scheduling, computer vision, clustering, ...

3-satisfiability reduces to independent set

Proposition. *3-SAT* poly-time reduces to *IND-SET*.

Pf. Given an instance Φ of *3-SAT*, create an instance G of *IND-SET*:
- For each clause in Φ, create 3 vertices in a triangle.
- Add an edge between each literal and its negation.

k = 4



$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

3-satisfiability reduces to independent set

Proposition. *3-SAT* poly-time reduces to *IND-SET*.

Pf. Given an instance Φ of *3-SAT*, create an instance G of *IND-SET*:
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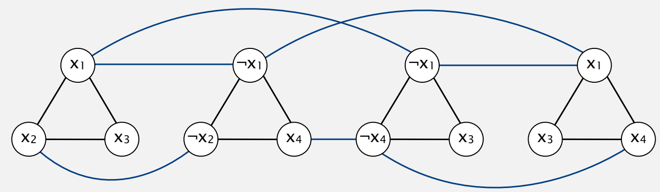

$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

- G has independent set of size k $\Rightarrow$ Φ satisfiable.

set literals corresponding to vertices in independent set to true;
set remaining literals in consistent manner

3-satisfiability reduces to independent set

Proposition. *3-SAT* poly-time reduces to *IND-SET*.

Pf. Given an instance Φ of *3-SAT*, create an instance G of *IND-SET*:
- For each clause in Φ, create 3 vertices in a triangle.
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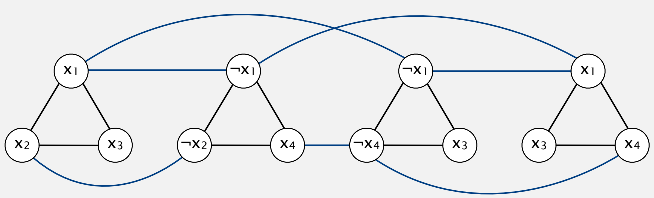


$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

- G has independent set of size k $\Rightarrow$ Φ satisfiable.
- Φ satisfiable $\Rightarrow$ G has independent set of size k.

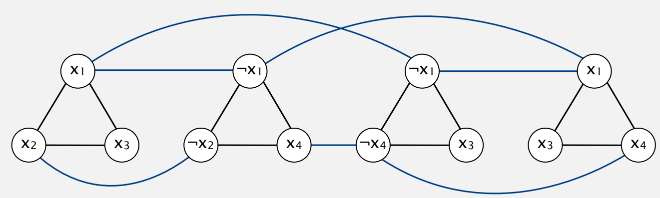
for each clause, take vertex corresponding to one true literal

3-satisfiability reduces to independent set

Proposition. *3-SAT* poly-time reduces to *IND-SET*. ← lower-bound mentality: if I could solve IND-SET efficiently, I could solve 3-SAT efficiently

Implication. Assuming *3-SAT* is intractable, so is *IND-SET*.

k = 4



$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

Integer linear programming

ILP. Given a system of linear inequalities, find an integral solution.

$$3x_1 + 5x_2 + 2x_3 + x_4 + 4x_5 \geq 10$$
$$5x_1 + 2x_2 + 4x_4 + 1x_5 \leq 7$$
$$x_1 + x_3 + 2x_4 \leq 2$$
$$3x_1 + 4x_3 + 7x_4 \leq 7$$
$$x_1 + x_4 \leq 1$$
$$x_1 + x_3 + x_5 \leq 1$$
$$\text{all } x_i = \{0, 1\}$$

linear inequalities

integer variables

Context. Cornerstone problem in operations research.

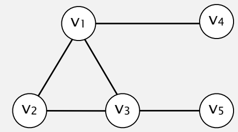
Remark. Finding a real-valued solution is tractable (linear programming).

Independent set reduces to integer linear programming

Proposition. *IND-SET* poly-time reduces to *ILP*.

Pf. Given an instance G, k of *IND-SET*, create an instance of *ILP* as follows:

Intuition. $x_i = 1$ if and only if vertex v_i is in independent set.



is there an independent set of size 3 ?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$
$$x_1 + x_2 \leq 1$$
$$x_2 + x_3 \leq 1$$
$$x_1 + x_3 \leq 1$$
$$x_1 + x_4 \leq 1$$
$$x_3 + x_5 \leq 1$$
$$\text{all } x_i = \{0, 1\}$$

number of vertices selected

at most one vertex selected from each edge

binary variables

is there a feasible solution?

3-satisfiability reduces to integer linear programming

Proposition. *3-SAT* poly-time reduces to *IND-SET*.

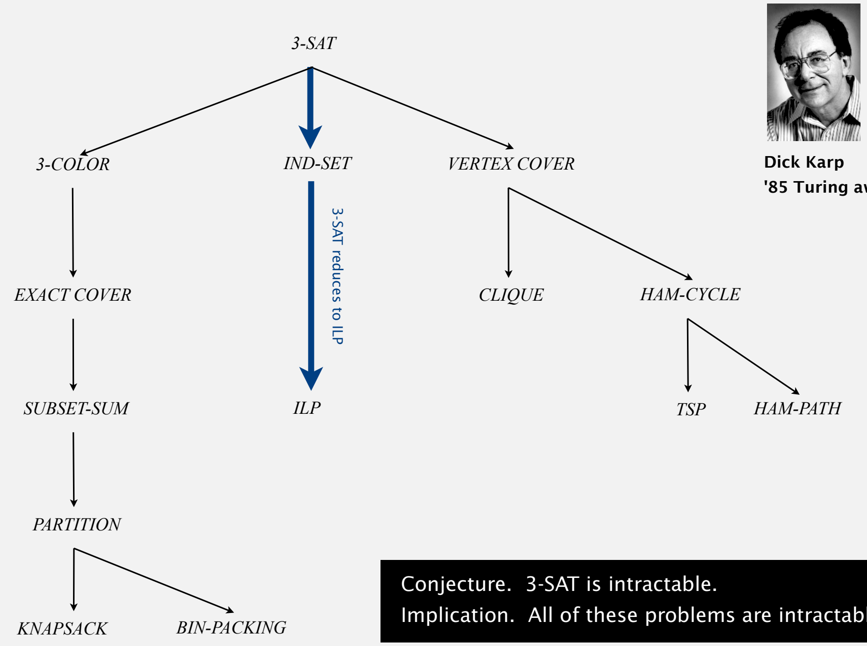
Proposition. *IND-SET* poly-time reduces to *ILP*.

Transitivity. If X poly-time reduces to Y and Y poly-time reduces to Z, then X poly-time reduces to Z.

Implication. Assuming *3-SAT* is intractable, so is *ILP*.

lower-bound mentality:
if I could solve ILP efficiently,
I could solve IND-SET efficiently;
if I could solve IND-SET efficiently,
I could solve 3-SAT efficiently

More poly-time reductions from 3-satisfiability



Dick Karp
'85 Turing award

Conjecture. 3-SAT is intractable.
Implication. All of these problems are intractable.

Implications of poly-time reductions from 3-satisfiability

Establishing intractability through poly-time reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself that a new problem is (probably) intractable?
A1. [hard way] Long futile search for an efficient algorithm (as for *3-SAT*).
A2. [easy way] Reduction from *3-SAT*.

Caveat. Intricate reductions are common.

Search problems

Search problem. Problem where you can check a solution in poly-time.

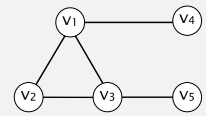
Ex 1. *3-SAT*.

$$\Phi \;=\; (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_3 \vee x_4)$$

x_1 = true, x_2 = true, x_3 = true, x_4 = true

Ex 2. *IND-SET*.

$k = 3$



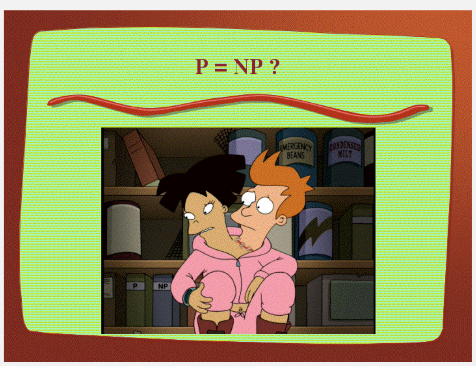
{ v_2 , v_4, v_5 }

P vs. NP

P. Set of search problems solvable in poly-time.
Importance. What scientists and engineers can compute feasibly.

NP. Set of search problems.
Importance. What scientists and engineers aspire to compute feasibly.

Fundamental question.



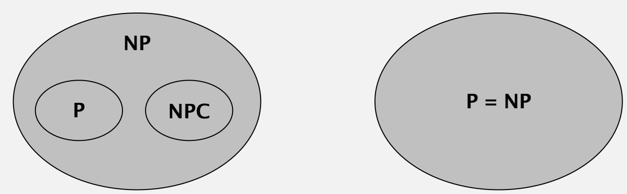
Consensus opinion. No.

Cook's theorem

An *NP* problem is *NP-complete* if all problems in *NP* poly-time to reduce to it.

Cook's theorem. *3-SAT* is *NP-complete*.
Corollary. *3-SAT* is tractable if and only if $P = NP$.

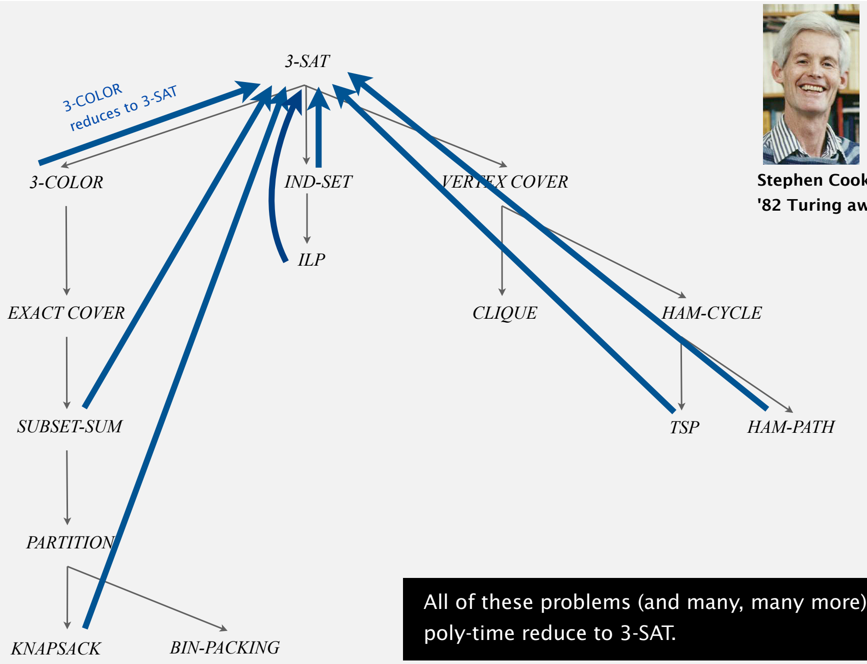
Two worlds.



P ≠ NP

P = NP

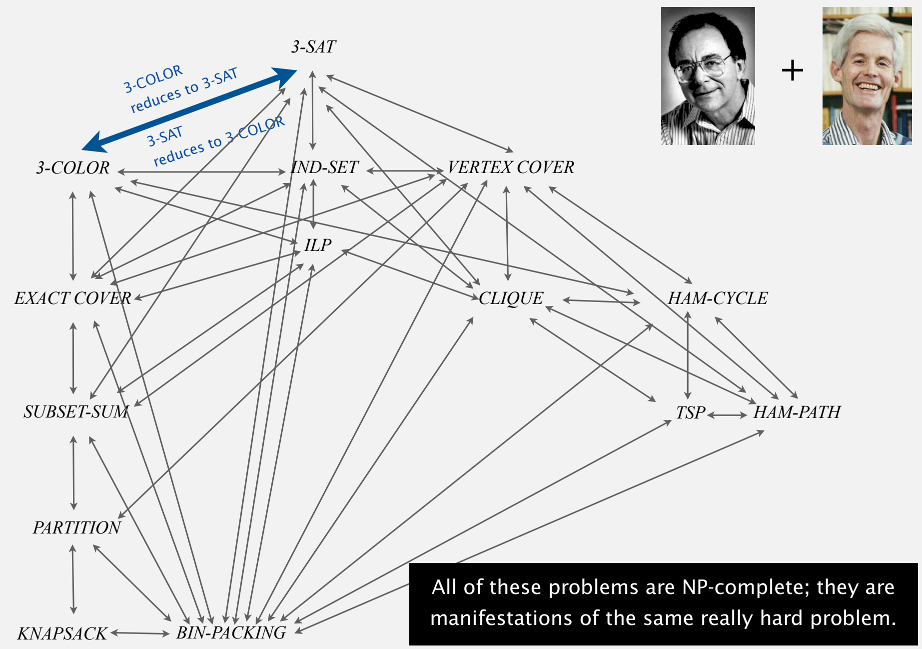
Implications of Cook's theorem



Stephen Cook
'82 Turing award

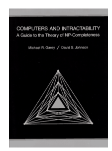
All of these problems (and many, many more) poly-time reduce to 3-SAT.

Implications of Karp + Cook



All of these problems are NP-complete; they are manifestations of the same really hard problem.

Implications of NP-completeness



"I can't find an efficient algorithm, but neither can all these famous people."

Birds-eye view: review

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, ...
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, ...
quadratic	N^2	???
	...	
exponential	c^N	???

Frustrating news. Huge number of problems have defied classification.

Birds-eye view: revised

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, ...
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, ...
M(N)	?	integer multiplication, division, square root, ...
3-SUM complete	probably N^2	3-SUM, 3-COLLINEAR, 3-CONCURRENT, ...
MM(N)	?	matrix multiplication, Ax = b, least square, determinant, ...
	...	
NP-complete	probably not N^b	3-SAT, IND-SET, ILP, ...

Good news. Can put many problems into equivalence classes.

Summary

Reductions are important in theory to:

- Establish tractability.
- Establish intractability.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, Delaunay triangulation
  - minimum spanning tree, shortest path, max flow, linear programming
- Determine difficulty of your problem and choose the right tool.
  - use exact algorithm for tractable problems
  - use heuristics for intractable problems