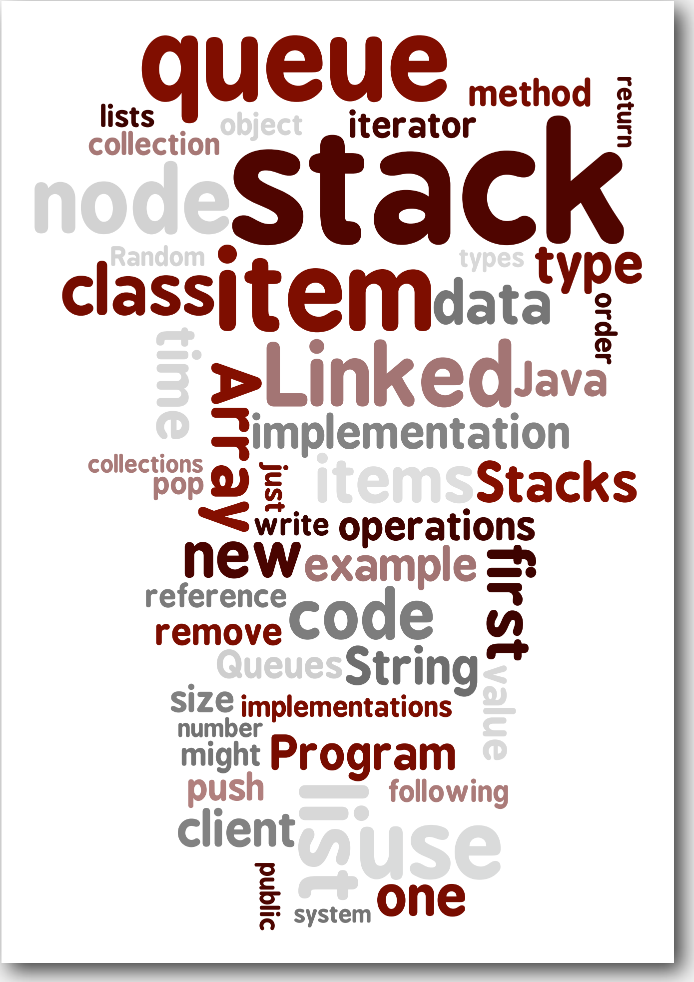
1.3 Stacks and Queues



- stacks
- dynamic resizing
- queues
- generics
- iterators
- applications

Algorithms, 4th Edition · *Robert Sedgewick and Kevin Wayne* ·

· *September 22, 2010 8:33:05 PM*

Stacks and queues

Fundamental data types.

- Values: sets of objects
- Operations: insert, remove, test if empty.
- Intent is clear when we insert.
- Which item do we remove?

LIFO = "last in first out"

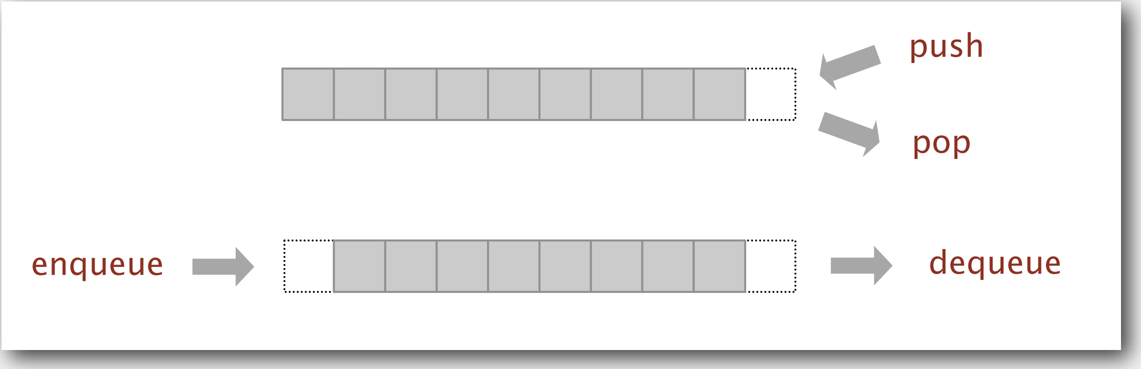
Stack. Remove the item most recently added.

Analogy. Cafeteria trays, Web surfing.

FIFO = "first in first out"

Queue. Remove the item least recently added.

Analogy. Registrar's line.



Client, implementation, interface

Separate interface and implementation.

Ex: stack, queue, priority queue, symbol table, union-find, ….

Benefits.

- Client can't know details of implementation ⇒ client has many implementation from which to choose.
- Implementation can't know details of client needs ⇒ many clients can re-use the same implementation.
- **Design:** creates modular, reusable libraries.
- **Performance:** use optimized implementation where it matters.

Client: program using operations defined in interface.
Implementation: actual code implementing operations.
Interface: description of data type, basic operations.

- stacks
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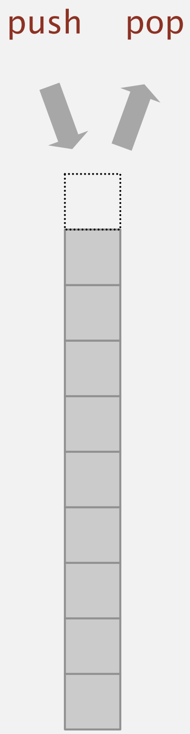
Stack API

Warmup. Stack of strings objects.

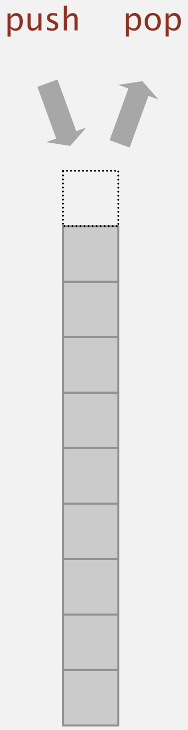
public class StackOfStrings		
	StackOfStrings()	*create an empty stack*
void	**push(String s)**	*insert a new item onto stack*
String	**pop()**	*remove and return the item most recently added*
boolean	**isEmpty()**	*is the stack empty?*
int	size()	*number of items on the stack*



Challenge. Reverse sequence of strings from standard input.

Stack test client

```
public static void main(String[] args)
{
   StackOfStrings stack = new StackOfStrings();
   while (!StdIn.isEmpty())
   {
      String item = StdIn.readString();
      if (item.equals("-")) StdOut.print(stack.pop());
      else                  stack.push(item);
   }
}
```



```
% more tobe.txt
to be or not to - be - - that - - - is

% java StackOfStrings < tobe.txt
to be not that or be
```

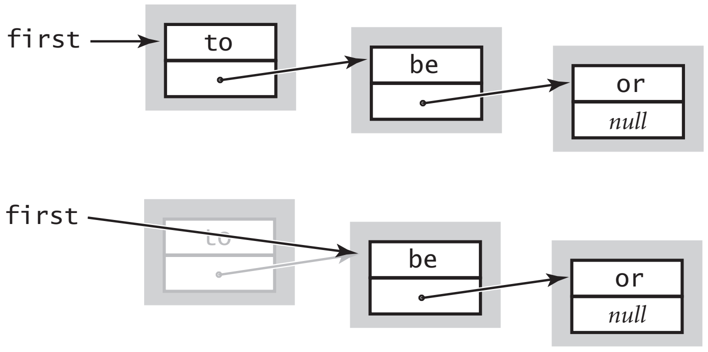
Stack pop: linked-list implementation

save item to return

```
String item = first.item;
```

save item to return

```
first = first.next;
```



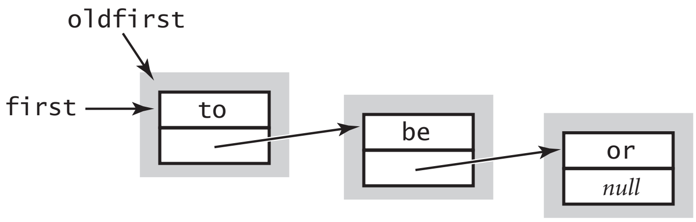
return saved item

```
return item;
```

Stack push: linked-list implementation

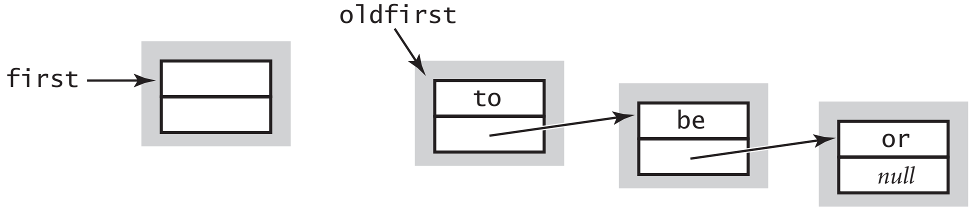
save a link to the list

```
Node oldfirst = first;
```



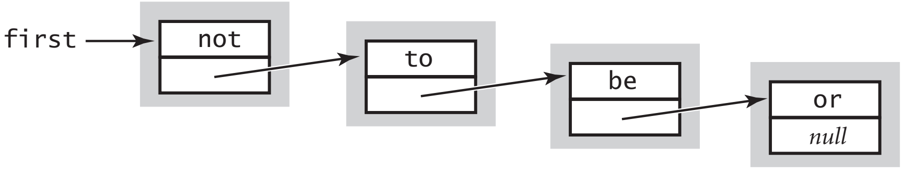
create a new node for the beginning

```
first = new Node();
```



set the instance variables in the new node

```
first.item = "not";
first.next = oldfirst;
```



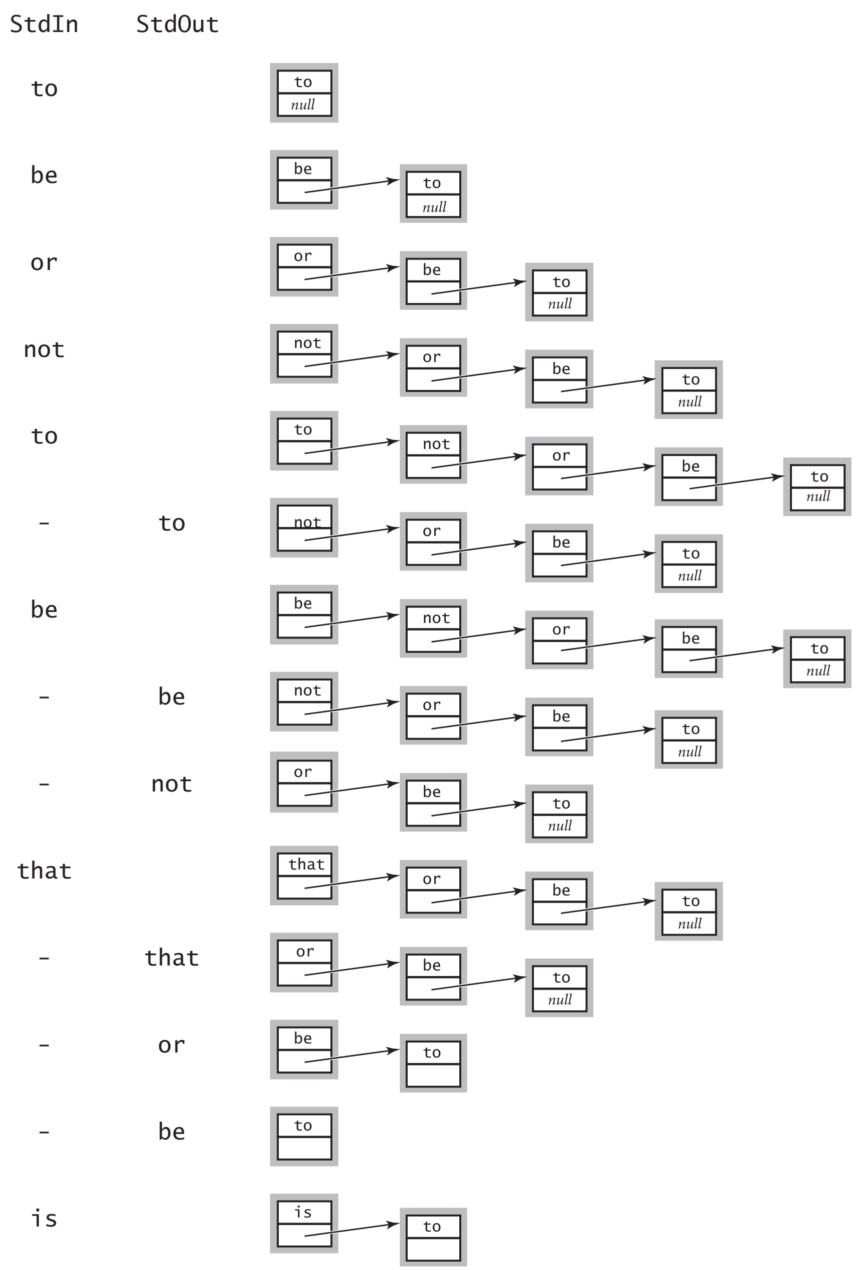
Stack: linked-list implementation in Java

```
public class StackOfStrings
{
   private Node first = null;

   private class Node
   {
      String item;
      Node next;
   }

   public boolean isEmpty()
   {  return first == null;   }

   public void push(String item)
   {
      Node oldfirst = first;
      first = new Node();
      first.item = item;
      first.next = oldfirst;
   }

   public String pop()
   {
      if (isEmpty()) throw new RuntimeException();
      String item = first.item;
      first = first.next;
      return item;
   }
}
```

inner class

stack underflow

Stack: linked-list trace



Stack: array implementation

Array implementation of a stack.

- Use array `s[]` to store `N` items on stack.
- `push()`: add new item at `s[N]`.
- `pop()`: remove item from `s[N-1]`.

s[]	to	be	or	not	to	be	*null*	*null*	*null*	*null*
	0	1	2	3	4	5	6	7	8	9
							N			

`capacity = 10`

Defect. Stack overflows when `N` exceeds capacity. [stay tuned]

Stack: array implementation

```
public class StackOfStrings
{
   private String[] s;
   private int N = 0;

   public StackOfStrings(int capacity)
   {  s = new String[capacity];  }

   public boolean isEmpty()
   {  return N == 0;  }

   public void push(String item)
   {  s[N++] = item;  }

   public String pop()
   {  return s[--N];  }
}
```

a cheat (stay tuned)

decrement N;
then use to index into array

```
public String pop()
{
   String item = s[--N];
   s[N] = null;
   return item;
}
```

this version avoids "loitering":
garbage collector reclaims memory
only if no outstanding references

- stacks
- **dynamic resizing**
- queues
- generics
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Stack: dynamic-array implementation

Problem. Requiring client to provide capacity does not implement API!

Q. How to grow and shrink array?

First try.

- `push()`: increase size of `s[]` by 1.
- `pop()`: decrease size of `s[]` by 1.

Too expensive.

- Need to copy all item to a new array.
- Inserting first N items takes time proportional to $1 + 2 + \ldots + N \sim N^2 / 2$.

infeasible for large N

Challenge. Ensure that array resizing happens infrequently.

Stack: dynamic-array implementation

"repeated doubling"

Q. How to grow array?

A. If array is full, create a new array of twice the size, and copy items.

```
public StackOfStrings() {  s = new String[1];  }

public void push(String item)
{
   if (N == s.length) resize(2 * s.length);
   s[N++] = item;
}

private void resize(int capacity)
{
   String[] copy = new String[capacity];
   for (int i = 0; i < N; i++)
      copy[i] = s[i];
   s = copy;
}
```

cost of array resizing is now
$2 + 4 + 8 + \ldots + N \sim 2N$

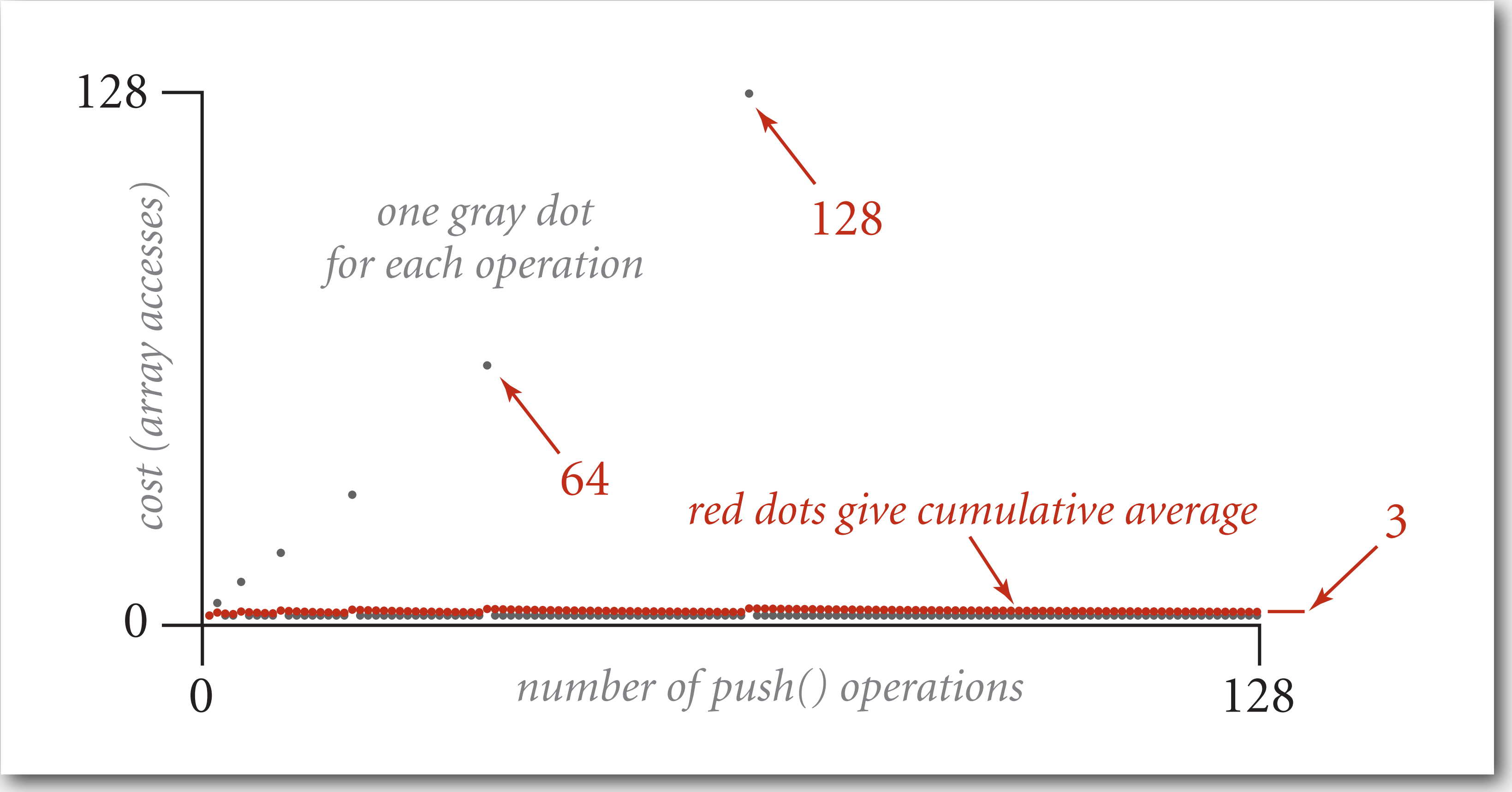
Consequence. Inserting first N items takes time proportional to N (not N^2).

Stack: amortized cost of adding to a stack

Cost of inserting first N items. $N \; + \; (2 + 4 + 8 + \ldots + N) \;\; \sim \;\; 3N.$

1 array accesses per push

k array accesses to double to size k



Stack: dynamic-array implementation

Q. How to shrink array?

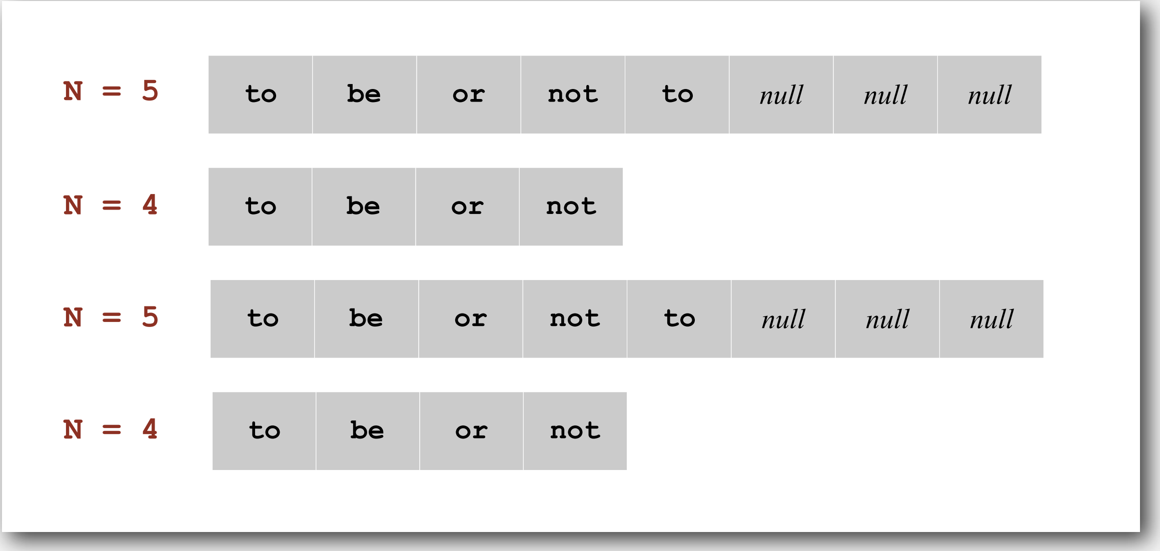
First try.

- `push()`: double size of `s[]` when array is full.
- `pop()`: halve size of `s[]` when array is one-half full.

"thrashing"

Too expensive.

- Consider push-pop-push-pop-… sequence when array is full.
- Takes time proportional to N per operation in worst case.



N = 5	to	be	or	not	to	*null*	*null*	*null*
N = 4	to	be	or	not				
N = 5	to	be	or	not	to	*null*	*null*	*null*
N = 4	to	be	or	not				

Stack: dynamic-array implementation

Q. How to shrink array?

Efficient solution.

- `push()`: double size of `s[]` when array is full.
- `pop()`: halve size of `s[]` when array is one-quarter full.

```
public String pop()
{
   String item = s[--N];
   s[N] = null;
   if (N > 0 && N == s.length/4) resize(s.length / 2);
   return item;
}
```

Invariant. Array is between 25% and 100% full.

Stack: dynamic-array implementation trace

StdIn	StdOut	N	a.length	a							
				0	1	2	3	4	5	6	7
		0	1	*null*							
to		1	1	to							
be		2	2	to	be						
or		3	4	to	be	or	*null*				
not		4	4	to	be	or	not				
to		5	8	to	be	or	not	to	*null*	*null*	*null*
-	to	4	8	to	be	or	not	*null*	*null*	*null*	*null*
be		5	8	to	be	or	not	be	*null*	*null*	*null*
-	be	4	8	to	be	or	not	*null*	*null*	*null*	*null*
-	not	3	8	to	be	or	*null*	*null*	*null*	*null*	*null*
that		4	8	to	be	or	that	*null*	*null*	*null*	*null*
-	that	3	8	to	be	or	*null*	*null*	*null*	*null*	*null*
-	or	2	4	to	be	*null*	*null*				
-	be	1	2	to	*null*						
is		2	2	to	is						

Stack: dynamic-array implementation performance

Amortized analysis. Average running time per operation over a worst-case sequence of operations.

Proposition. Starting from empty stack (with dynamic resizing), any sequence of M push and pop operations takes time proportional to M.

	best	worst	amortized
construct	1	1	1
push	1	N	1
pop	1	N	1
size	1	1	1

doubling and shrinking

running time for doubling stack with N items

Stack implementations: memory usage

Linked-list implementation. $\sim 16\,N$ bytes.

```
private class Node
{
   String item;
   Node next;
}
```

8 bytes (object overhead)

4 bytes (reference to String)

4 bytes (reference to Node)

---

16 bytes per stack item

Dynamic-array implementation. Between $\sim 4\,N$ (100% full) and $\sim 16\,N$ (25% full).

```
public class DoublingStackOfStrings
{
   private String[] s;
   private int N = 0;
   …
}
```

4 bytes × array size

4 bytes

Remark. Analysis includes memory for the stack (but not the strings themselves).

Stack implementations: dynamic array vs. linked List

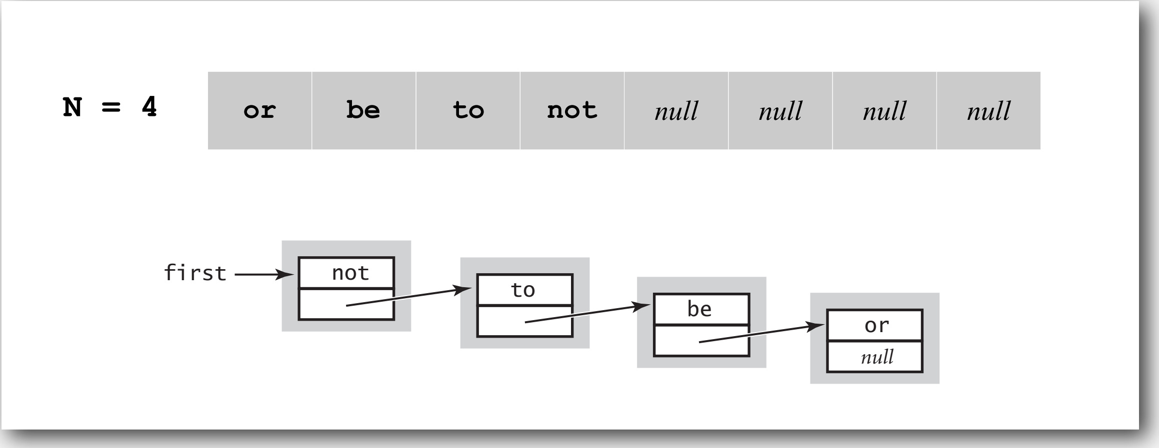
Tradeoffs. Can implement a stack with either dynamic array or linked list; client can use interchangeably. Which one is better?

Linked-list implementation.

- Every operation takes constant time in the worst case.
- Uses extra time and space to deal with the links.

Dynamic-array implementation.

- Every operation takes constant amortized time.
- Less wasted space.



- stacks
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- **queues**
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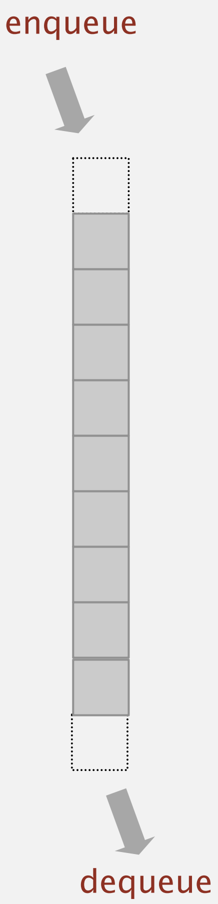
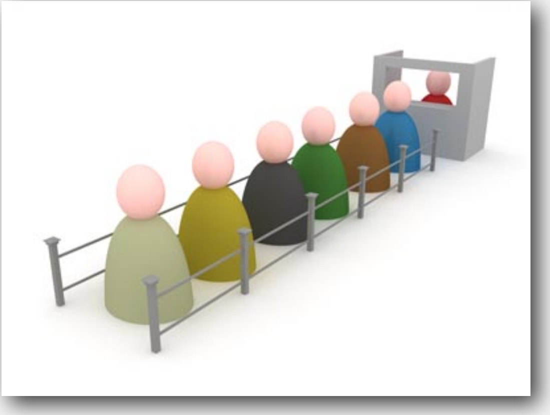
Queue API



`public class QueueOfStrings`

	`QueueOfStrings()`	*create an empty queue*
`void`	`enqueue(String s)`	*insert a new item onto queue*
`String`	`dequeue()`	*remove and return the item least recently added*
`boolean`	`isEmpty()`	*is the queue empty?*
`int`	`size()`	*number of items on the queue*



Queue test client

```
public static void main(String[] args)
{
   QueueOfStrings q = new QueueOfStrings();
   while (!StdIn.isEmpty())
   {
      String item = StdIn.readString();
      if (item.equals("-")) StdOut.print(q.dequeue());
      else                  q.enqueue(item);
   }
}
```

```
% more tobe.txt
to be or not to - be - - that - - - is

% java QueueOfStrings < tobe.txt
to be or not to be
```

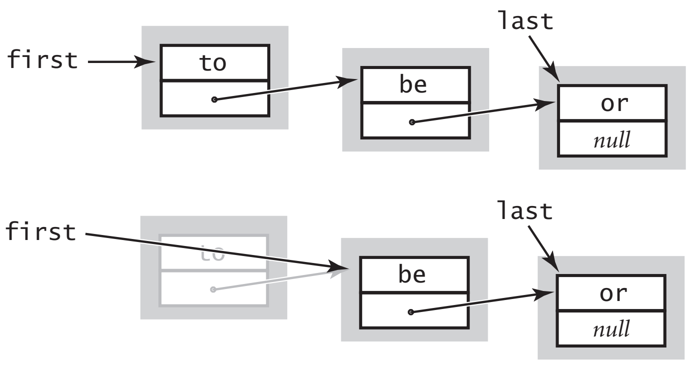
Queue dequeue: linked-list implementation

save item to return

```
String item = first.item;
```

save item to return

```
first = first.next;
```



return saved item

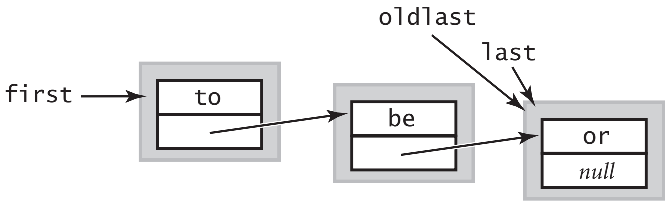
```
return item;
```

Remark. Identical code to linked-list stack `pop()`.

Queue enqueue: linked-list implementation

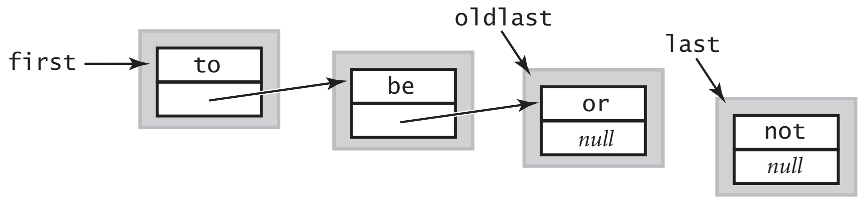
save a link to the last node

```
Node oldlast = last;
```



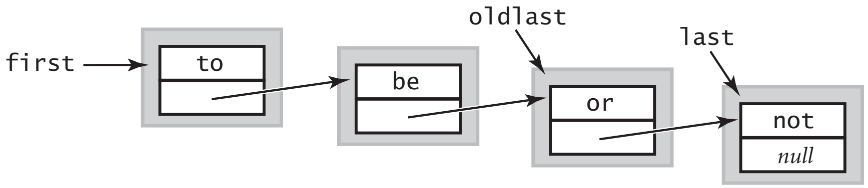
create a new node for the end

```
Node last = new Node();
last.item = "not";
last.next = null;
```



link the new node to the end of the list

```
oldlast.next = last;
```

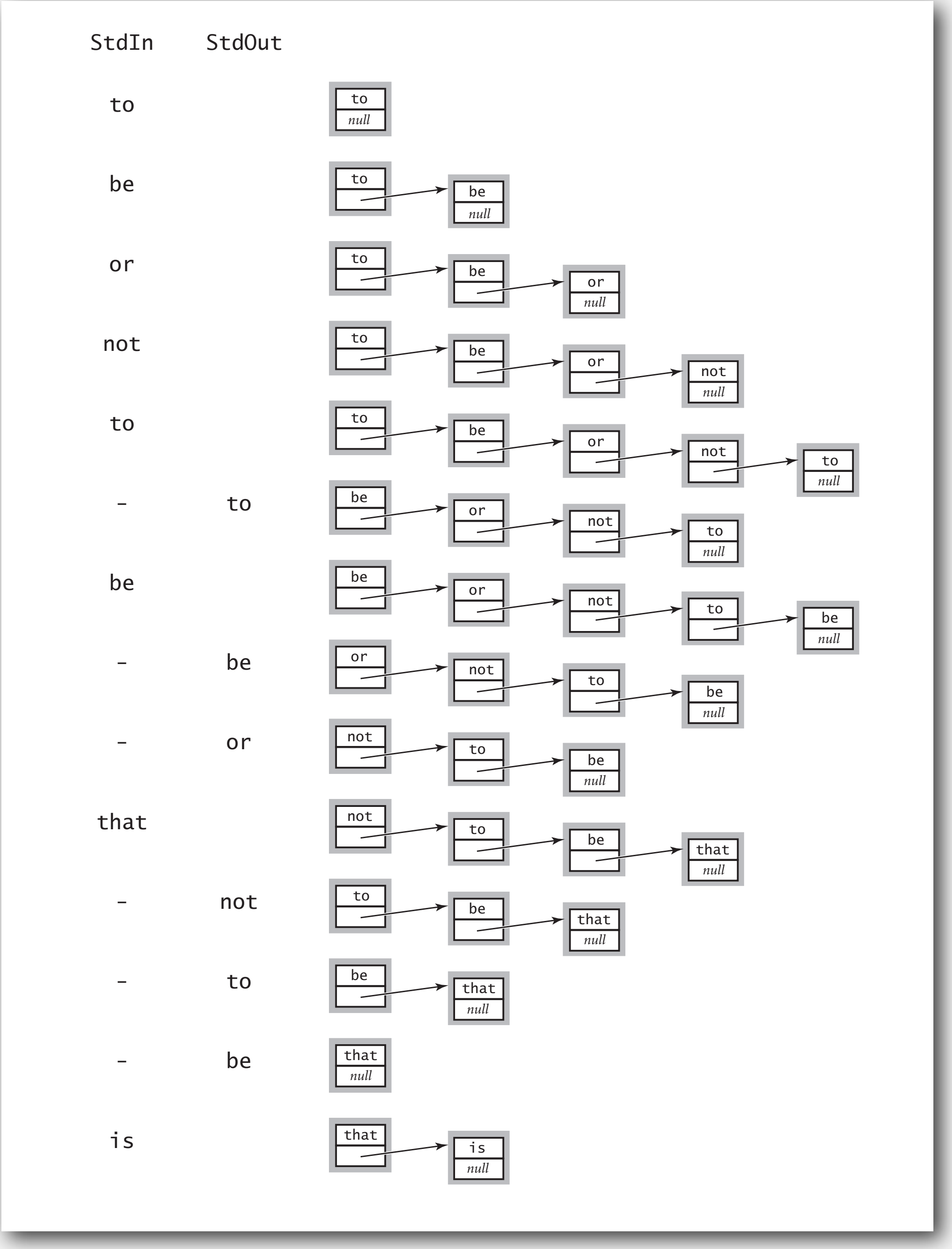


Queue: linked-list implementation in Java

```
public class QueueOfStrings
{
   private Node first, last;

   private class Node
   {  /* same as in StackOfStrings */   }

   public boolean isEmpty()
   {  return first == null;   }

   public void enqueue(String item)
   {
      Node oldlast = last;
      last = new Node();
      last.item = item;
      last.next = null;
      if (isEmpty()) first = last;
      else           oldlast.next = last;
   }

   public String dequeue()
   {
      String item = first.item;
      first       = first.next;
      if (isEmpty()) last = null;
      return item;
   }
}
```

special cases for
empty queue

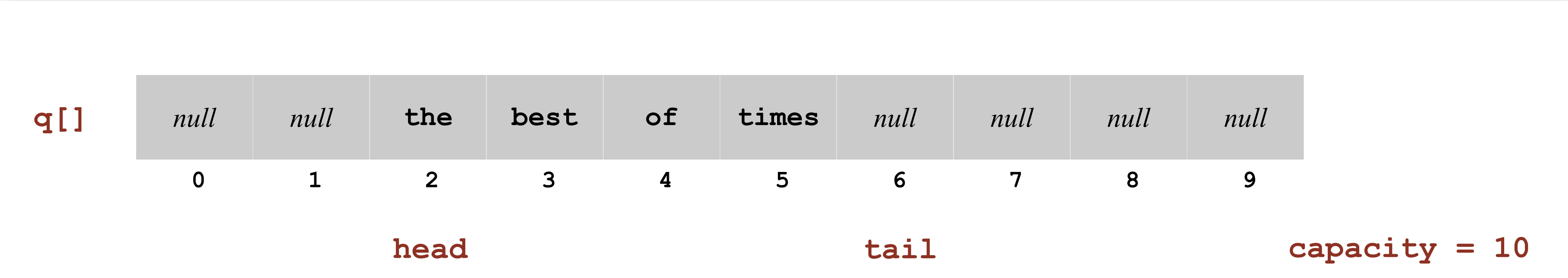
Queue: linked-list trace



Queue: dynamic array implementation

Array implementation of a queue.

- Use array `q[]` to store items in queue.
- `enqueue()`: add new item at `q[tail]`.
- `dequeue()`: remove item from `q[head]`.
- Update `head` and `tail` modulo the `capacity`.
- Add dynamic resizing.



`q[]`	*null*	*null*	`the`	`best`	`of`	`times`	*null*	*null*	*null*	*null*
	0	1	2	3	4	5	6	7	8	9
			`head`				`tail`			

`capacity = 10`

- stacks
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Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`, etc.?

Attempt 1. Implement a separate stack class for each type.

- Rewriting code is tedious and error-prone.
- Maintaining cut-and-pasted code is tedious and error-prone.

@#$*! most reasonable approach until Java 1.5.

Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`, etc.?

Attempt 2. Implement a stack with items of type `Object`.

- Casting is required in client.
- Casting is error-prone: run-time error if types mismatch.

```
StackOfObjects s = new StackOfObjects();
Apple  a = new Apple();
Orange b = new Orange();
s.push(a);
s.push(b);
a = (Apple) (s.pop());
```

run-time error

Parameterized stack

We implemented: `StackOfStrings`.

We also want: `StackOfURLs`, `StackOfInts`, `StackOfVans`, etc.?

Attempt 3. Java generics.

- Avoid casting in client.
- Discover type mismatch errors at compile-time instead of run-time.

```
Stack<Apple> s = new Stack<Apple>();
Apple  a = new Apple();
Orange b = new Orange();
s.push(a);
s.push(b);
a = s.pop();
```

type parameter

compile-time error

Guiding principles. Welcome compile-time errors; avoid run-time errors.

Generic stack: linked-list implementation

```
public class LinkedStackOfStrings
{
   private Node first = null;

   private class Node
   {
      String item;
      Node next;
   }

   public boolean isEmpty()
   {  return first == null;   }

   public void push(String item)
   {
      Node oldfirst = first;
      first = new Node();
      first.item = item;
      first.next = oldfirst;
   }

   public String pop()
   {
      String item = first.item;
      first = first.next;
      return item;
   }
}
```

```
public class Stack<Item>
{
   private Node first = null;

   private class Node
   {
      Item item;
      Node next;
   }

   public boolean isEmpty()
   {  return first == null;   }

   public void push(Item item)
   {
      Node oldfirst = first;
      first = new Node();
      first.item = item;
      first.next = oldfirst;
   }

   public Item pop()
   {
      Item item = first.item;
      first = first.next;
      return item;
   }
}
```

generic type name

Generic stack: array implementation

```
public class ArrayStackOfStrings
{
   private String[] s;
   private int N = 0;

   public StackOfStrings(int capacity)
   {  s = new String[capacity];   }

   public boolean isEmpty()
   {  return N == 0;   }

   public void push(String item)
   {  s[N++] = item;   }

   public String pop()
   {  return s[--N];   }
}
```

the way it should be

```
public class ArrayStack<Item>
{
   private Item[] s;
   private int N = 0;

   public Stack(int capacity)
   {  s = new Item[capacity];   }

   public boolean isEmpty()
   {  return N == 0;   }

   public void push(Item item)
   {  s[N++] = item;   }

   public Item pop()
   {  return s[--N];   }
}
```

@#$*! generic array creation not allowed in Java

Generic stack: array implementation

```
public class ArrayStackOfStrings
{
   private String[] s;
   private int N = 0;

   public StackOfStrings(int capacity)
   {  s = new String[capacity];  }

   public boolean isEmpty()
   {  return N == 0;  }

   public void push(String item)
   {  s[N++] = item;  }

   public String pop()
   {  return s[--N];  }
}
```

the way it is

```
public class ArrayStack<Item>
{
   private Item[] s;
   private int N = 0;

   public Stack(int capacity)
   {  s = (Item[]) new Object[capacity]; }

   public boolean isEmpty()
   {  return N == 0;  }

   public void push(Item item)
   {  s[N++] = item;  }

   public Item pop()
   {  return s[--N];  }
}
```

the ugly cast

Generic data types: autoboxing

Q. What to do about primitive types?

Wrapper type.

- Each primitive type has a wrapper object type.
- Ex: `Integer` is wrapper type for `int`.

Autoboxing. Automatic cast between a primitive type and its wrapper.

Syntactic sugar. Behind-the-scenes casting.

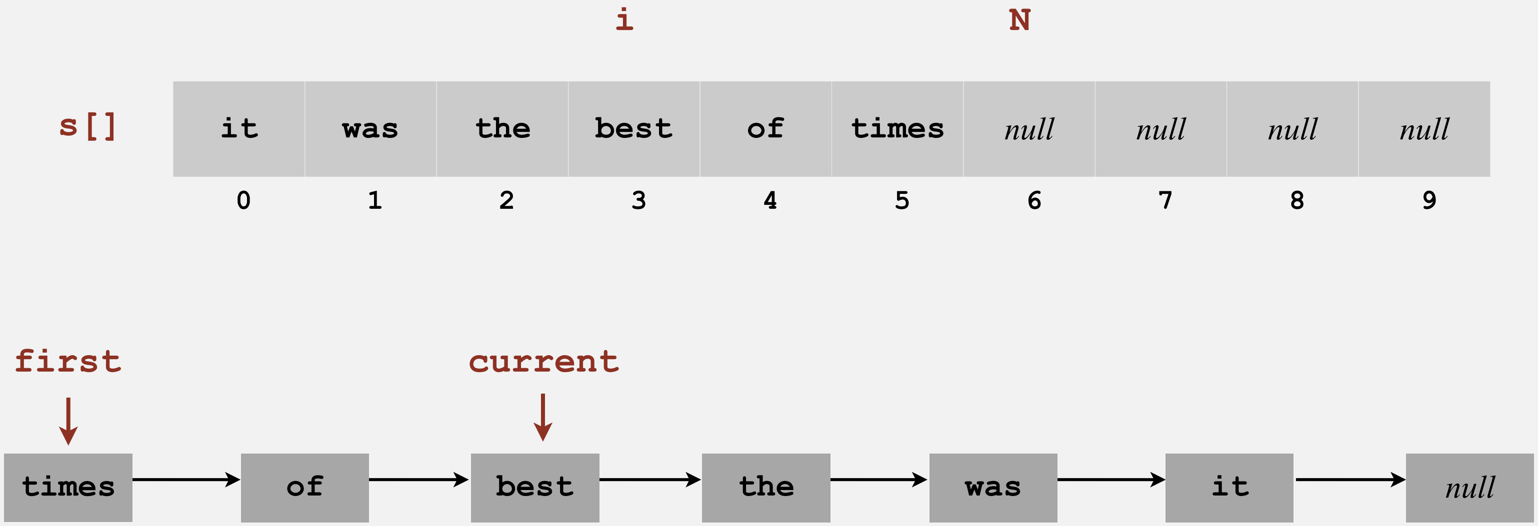
```
Stack<Integer> s = new Stack<Integer>();
s.push(17);          // s.push(new Integer(17));
int a = s.pop();     // int a = s.pop().intValue();
```

Bottom line. Client code can use generic stack for any type of data.

- stacks
- dynamic resizing
- queues
- generics
- **iterators**
- applications

Iteration

Design challenge. Support iteration over stack items by client, without revealing the internal representation of the stack.



Java solution. Make stack implement the `Iterable` interface.

Iterators

Q. What is an `Iterable` ?

A. Has a method that returns an `Iterator`.

Iterable interface

```
public interface Iterable<Item>
{
   Iterator<Item> iterator();
}
```

Q. What is an `Iterator` ?

A. Has methods `hasNext()` and `next()`.

Iterator interface

```
public interface Iterator<Item>
{
   boolean hasNext();
   Item next();
   void remove();
}
```

← optional; use at your own risk

Q. Why make data structures `Iterable` ?

A. Java supports elegant client code.

"foreach" statement

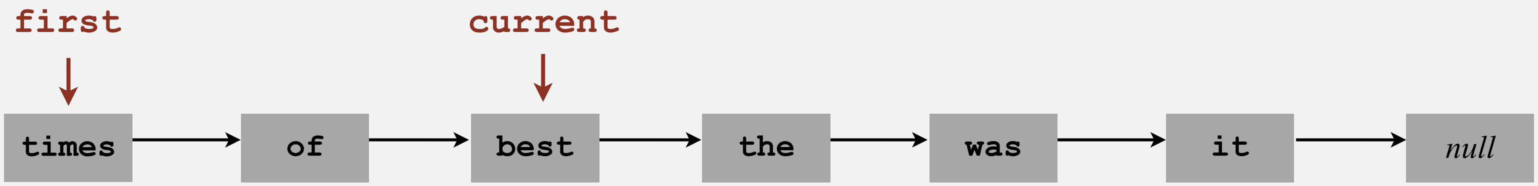
```
for (String s : stack)
   StdOut.println(s);
```

equivalent code

```
Iterator<String> i = stack.iterator();
while (i.hasNext())
{
   String s = i.next();
   StdOut.println(s);
}
```

Stack iterator: linked-list implementation

```
import java.util.Iterator;

public class Stack<Item> implements Iterable<Item>
{
    ...

    public Iterator<Item> iterator() { return new ListIterator();   }

    private class ListIterator implements Iterator<Item>
    {
        private Node current = first;

        public boolean hasNext() {  return current != null;  }
        public void remove()     {  /* not supported */      }
        public Item next()
        {
            Item item = current.item;
            current   = current.next;
            return item;
        }
    }
}
```



Stack iterator: array implementation

```
import java.util.Iterator;

public class Stack<Item> implements Iterable<Item>
{
    …

    public Iterator<Item> iterator() { return new ArrayIterator(); }

    private class ArrayIterator implements Iterator<Item>
    {
        private int i = N;

        public boolean hasNext() {  return i > 0;        }
        public void remove()     {  /* not supported */  }
        public Item next()       {  return s[--i];       }
    }
}
```

				i			**N**			
s[]	it	was	the	best	of	times	*null*	*null*	*null*	*null*
	0	1	2	3	4	5	6	7	8	9

Bag API

When order doesn't matter:

`public class Bag<Item> implements Iterator<Item>`

	`Bag()`	*create an empty bag*
`void`	`add(Item x)`	*insert a new item onto bag*
`int`	`size()`	*number of items in bag*
`Iterable<Item>`	`iterator()`	*iterator for all items in bag*

- stacks
- dynamic resizing
- queues
- generics
- iterators
- **applications**

Java collections library

List interface. `java.util.List` is API for ordered collection of items.

`public interface List<Item> implements Iterator<Item>`

	`List()`	*create an empty list*
`boolean`	`isEmpty()`	*is the list empty?*
`int`	`size()`	*number of items*
`void`	`add(Item item)`	*append item to the end*
`Item`	`get(int index)`	*return item at given index*
`Item`	`remove(int index)`	*return and delete item at given index*
`boolean`	`contains(Item item)`	*does the list contain the given item?*
`Iteartor<Item>`	`iterator()`	*iterator over all items in the list*
`...`		

Implementations. `java.util.ArrayList` uses dynamic array; `java.util.LinkedList` uses linked list.

Java collections library

`java.util.Stack`.

- Supports `push()`, `pop()`, `size()`, `isEmpty()`, and iteration.
- Also implements `java.util.List` interface from previous slide, including, `get()`, `remove()`, and `contains()`.
- Bloated and poorly-designed API ⇒ don't use.

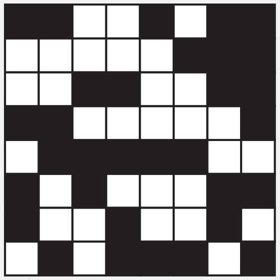
`java.util.Queue`. An interface, not an implementation of a queue.

Best practices. Use our implementations of `Stack`, `Queue`, and `Bag`.

War story (from COS 226)

Generate random open sites in an N-by-N percolation system.

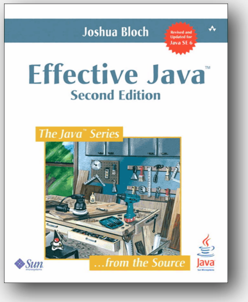
- Jenny: pick $(i,\ j)$ at random; if already open, repeat.
  Takes ~ $c_1 N^2$ seconds.
- Kenny: create a `java.util.LinkedList` of N^2 open sites.
  Pick an index at random and delete.
  Takes ~ $c_2 N^4$ seconds.



Kenny

Lesson. Don't use a library until you understand its API!

This course. Can't use a library until we've implemented it in class.

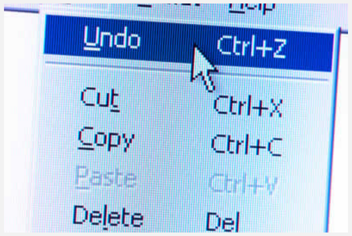


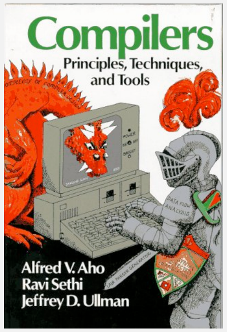
Stack applications

- Parsing in a compiler.
- Java virtual machine.
- Undo in a word processor.
- Back button in a Web browser.
- PostScript language for printers.
- Implementing function calls in a compiler.
- ...

Java

Adobe® PostScript® 3™




Compilers
Principles, Techniques, and Tools
Alfred V. Aho
Ravi Sethi
Jeffrey D. Ullman

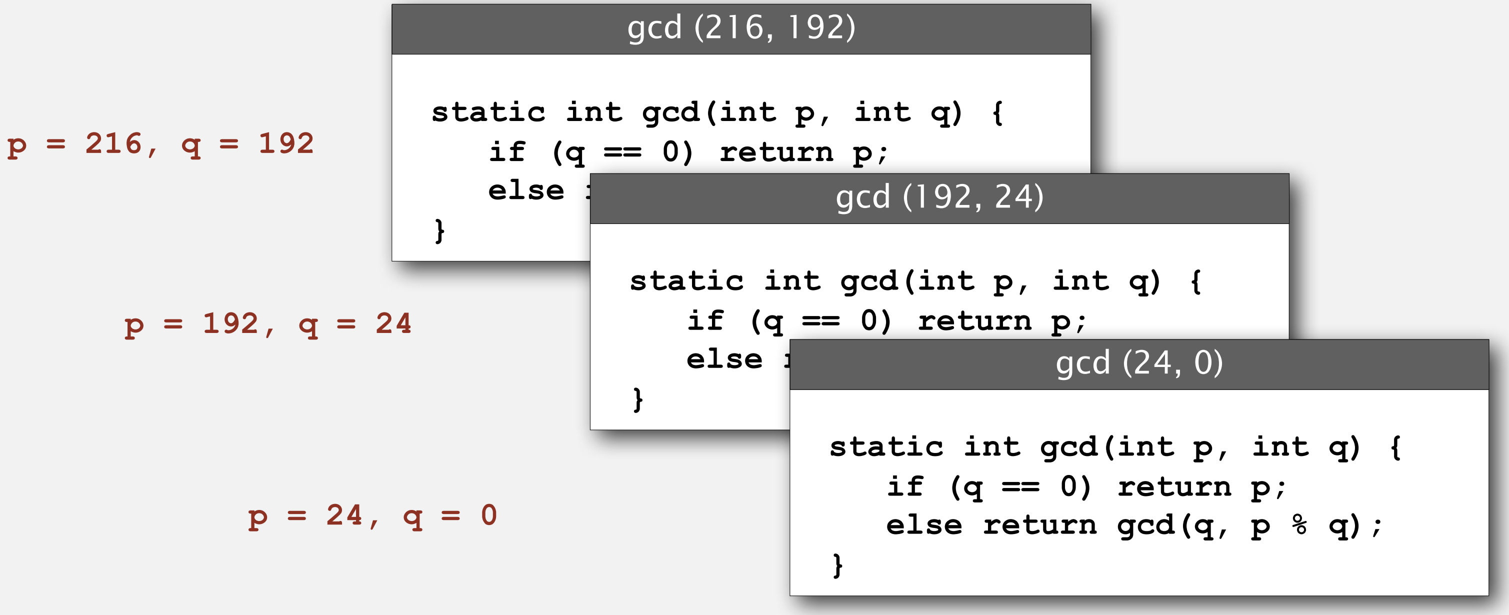
Function calls

How a compiler implements a function.

- Function call: push local environment and return address.
- Return: pop return address and local environment.

Recursive function. Function that calls itself.

Note. Can always use an explicit stack to remove recursion.



p = 216, q = 192

gcd (216, 192)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else
}
```

p = 192, q = 24

gcd (192, 24)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else
}
```

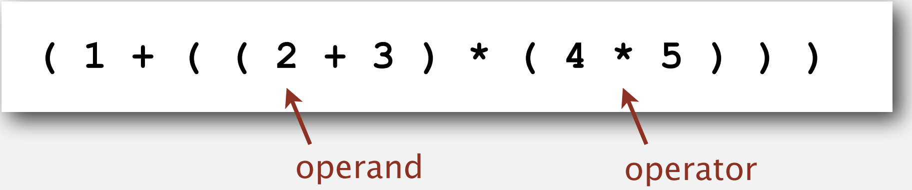
p = 24, q = 0

gcd (24, 0)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

Arithmetic expression evaluation

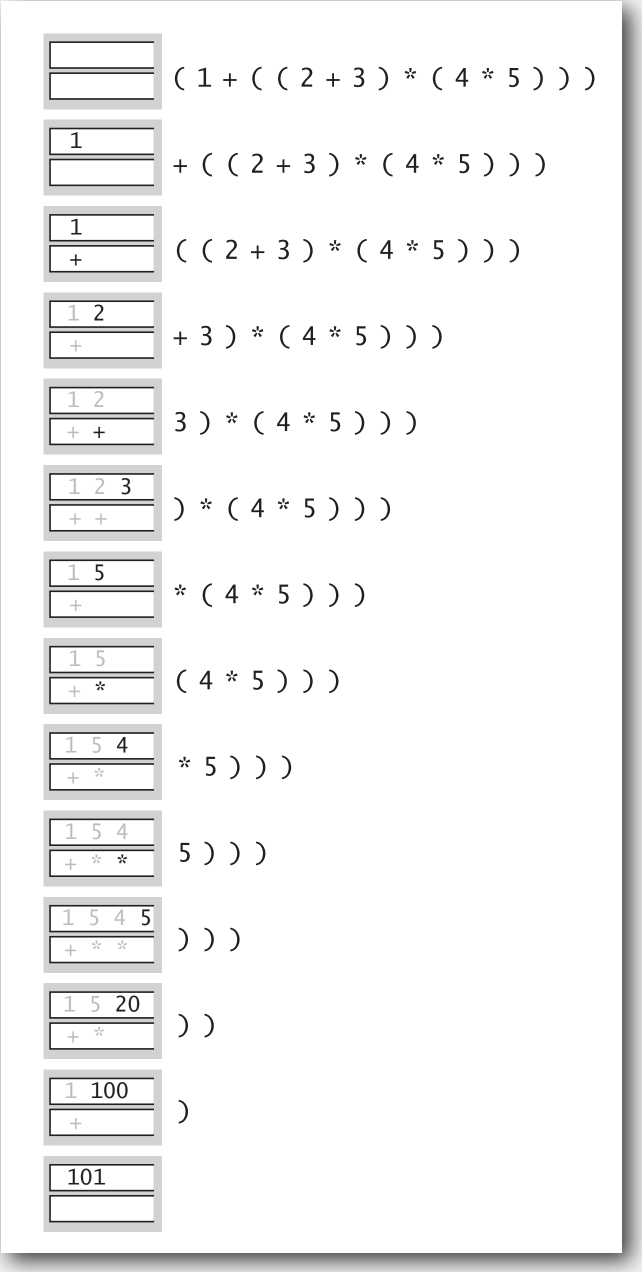
Goal. Evaluate infix expressions.


`( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )`

Two-stack algorithm. [E. W. Dijkstra]

- Value: push onto the value stack.
- Operator: push onto the operator stack.
- Left parens: ignore.
- Right parens: pop operator and two values; push the result of applying that operator to those values onto the operand stack.

Context. An interpreter!


value stack
operator stack

		(1 + ((2 + 3) * (4 * 5)))
1		+ ((2 + 3) * (4 * 5)))
1	+	((2 + 3) * (4 * 5)))
1 2	+	+ 3) * (4 * 5)))
1 2	+ +	3) * (4 * 5)))
1 2 3	+ +	) * (4 * 5)))
1 5	+	* (4 * 5)))
1 5	+ *	(4 * 5)))
1 5 4	+ *	* 5)))
1 5 4	+ * *	5)))
1 5 4 5	+ * *	)))
1 5 20	+ *	))
1 100	+	)
101		

Arithmetic expression evaluation

```
public class Evaluate
{
   public static void main(String[] args)
   {
      Stack<String> ops  = new Stack<String>();
      Stack<Double> vals = new Stack<Double>();
      while (!StdIn.isEmpty()) {
         String s = StdIn.readString();
         if      (s.equals("("))               ;
         else if (s.equals("+"))    ops.push(s);
         else if (s.equals("*"))    ops.push(s);
         else if (s.equals(")"))
         {
            String op = ops.pop();
            if      (op.equals("+")) vals.push(vals.pop() + vals.pop());
            else if (op.equals("*")) vals.push(vals.pop() * vals.pop());
         }
         else vals.push(Double.parseDouble(s));
      }
      StdOut.println(vals.pop());
   }
}
```

```
% java Evaluate
( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )
101.0
```

Correctness

Q. Why correct?

A. When algorithm encounters an operator surrounded by two values within parentheses, it leaves the result on the value stack.

```
( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )
```

as if the original input were:

```
( 1 + ( 5 * ( 4 * 5 ) ) )
```

Repeating the argument:

```
( 1 + ( 5 * 20 ) )
( 1 + 100 )
101
```

Extensions. More ops, precedence order, associativity.

Stack-based programming languages

Observation 1. The 2-stack algorithm computes the same value if the operator occurs after the two values.

```
( 1 ( ( 2 3 + ) ( 4 5 * ) * ) + )
```

Observation 2. All of the parentheses are redundant!

```
1 2 3 + 4 5 * * +
```

Jan Lukasiewicz

Bottom line. Postfix or "reverse Polish" notation.

Applications. Postscript, Forth, calculators, Java virtual machine, …

PostScript

PostScript. [Warnock-Geschke 1980s]

- Postfix program code.
- Turtle graphics commands.
- Variables, types, text, loops, conditionals, functions, ...

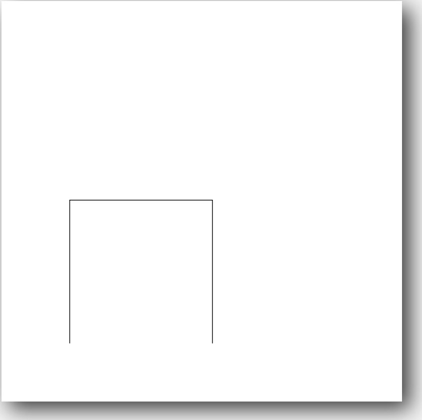
units are points (72 per inch)

```
%!
 100 100 moveto
 100 300 lineto
 300 300 lineto
 300 100 lineto
 stroke
```

define a path

draw the path

a PostScript program



its output

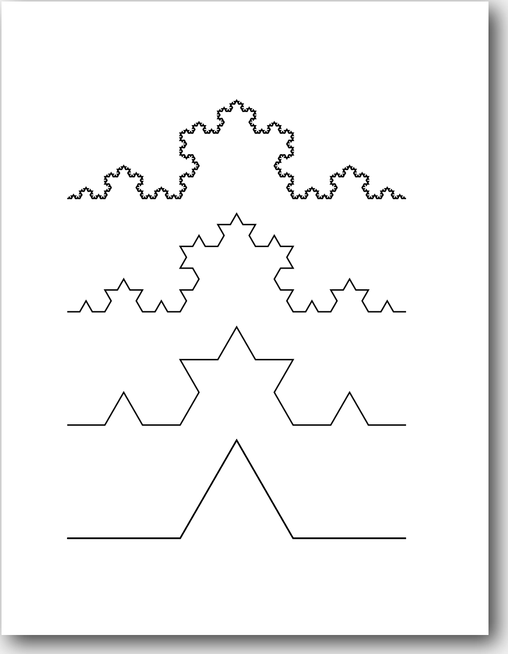
Simple virtual machine, but not a toy.

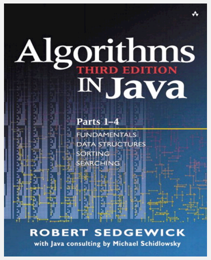
- Easy to specify published page.
- Easy to implement in printers.
- Revolutionized the publishing world.

PostScript applications

Algorithms, 3rd edition. Figures created directly in PostScript.

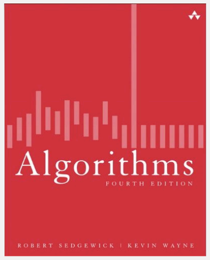
```
%!
72 72 translate

/kochR
  {
    2 copy ge { dup 0 rlineto }
      {
        3 div
        2 copy kochR 60 rotate
        2 copy kochR -120 rotate
        2 copy kochR 60 rotate
        2 copy kochR
      } ifelse
    pop pop
  } def

0   0 moveto   81 243 kochR
0  81 moveto   27 243 kochR
0 162 moveto    9 243 kochR
0 243 moveto    1 243 kochR
stroke
```





see page 218

Algorithms, 4th edition. Figures created using enhanced version of `stdDraw` that saves to PostScript for vector graphics.



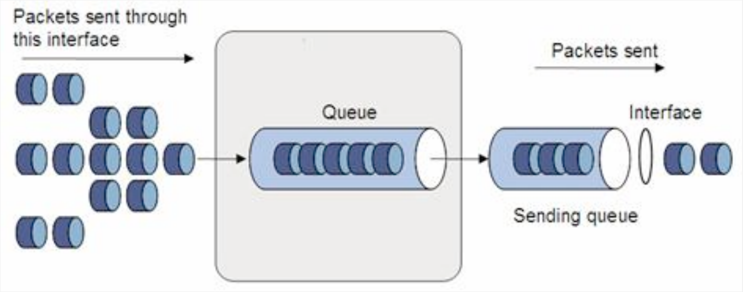
Queue applications

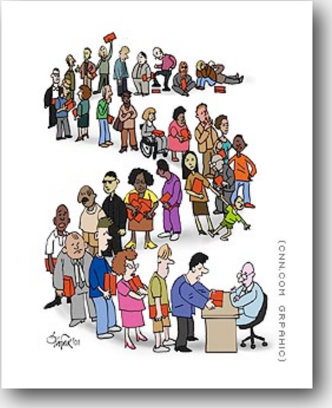
Familiar applications.

- iTunes playlist.
- Data buffers (iPod, TiVo).
- Asynchronous data transfer (file IO, pipes, sockets).
- Dispensing requests on a shared resource (printer, processor).

Simulations of the real world.

- Traffic analysis.
- Waiting times of customers at call center.
- Determining number of cashiers to have at a supermarket.





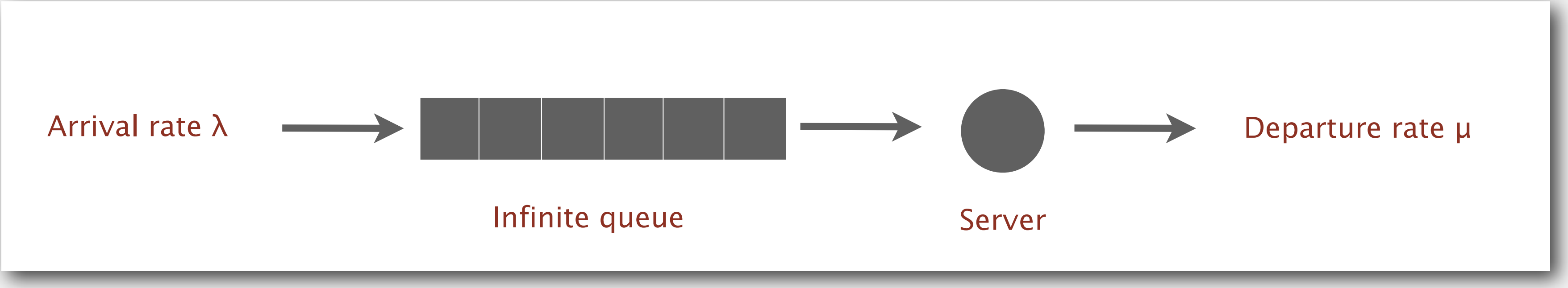
M/M/1 queuing model

M/M/1 queue.

- Customers arrive according to Poisson process at rate of λ per minute.
- Customers are serviced with rate of μ per minute.

interarrival time has exponential distribution $\Pr[X \leq x] = 1 - e^{-\lambda x}$

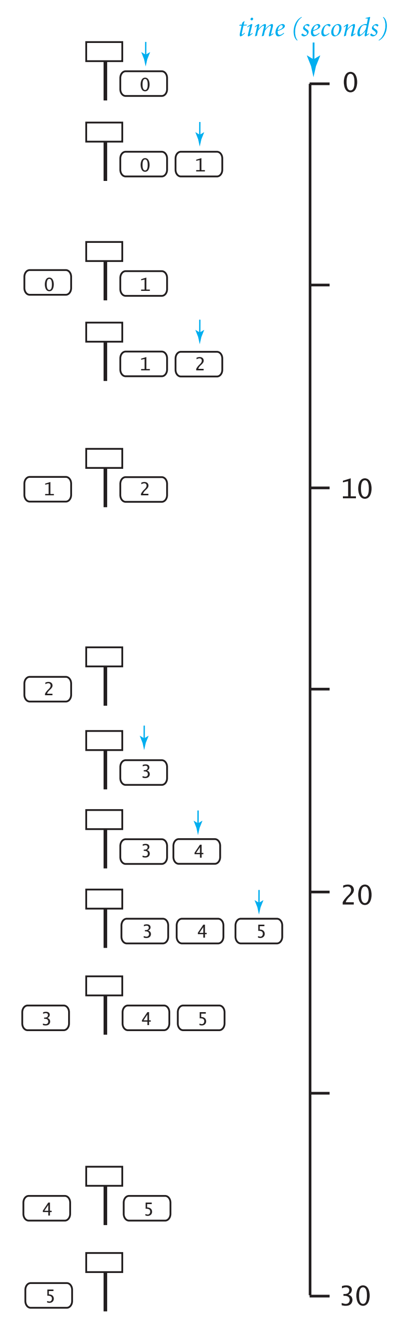
service time has exponential distribution $\Pr[X \leq x] = 1 - e^{-\mu x}$



Q. What is average wait time W of a customer in system?

Q. What is average number of customers L in system?

M/M/1 queuing model: example simulation



	arrival	departure	wait
0	0	5	5
1	2	10	8
2	7	15	8
3	17	23	6
4	19	28	9
5	21	30	9

M/M/1 queuing model: event-based simulation

```
public class MM1Queue
{
    public static void main(String[] args) {
        double lambda = Double.parseDouble(args[0]);   // arrival rate
        double mu     = Double.parseDouble(args[1]);   // service rate
        double nextArrival = StdRandom.exp(lambda);
        double nextService = nextArrival + StdRandom.exp(mu);

        Queue<Double> queue = new Queue<Double>();
        Histogram hist = new Histogram("M/M/1 Queue", 60);

        while (true)
        {
            while (nextArrival < nextService)
            {
                queue.enqueue(nextArrival);
                nextArrival += StdRandom.exp(lambda);
            }

            double arrival = queue.dequeue();
            double wait = nextService - arrival;
            hist.addDataPoint(Math.min(60,  (int) (Math.round(wait))));
            if (queue.isEmpty()) nextService = nextArrival + StdRandom.exp(mu);
            else                 nextService = nextService + StdRandom.exp(mu);
        }
    }
}
```

queue of arrival times

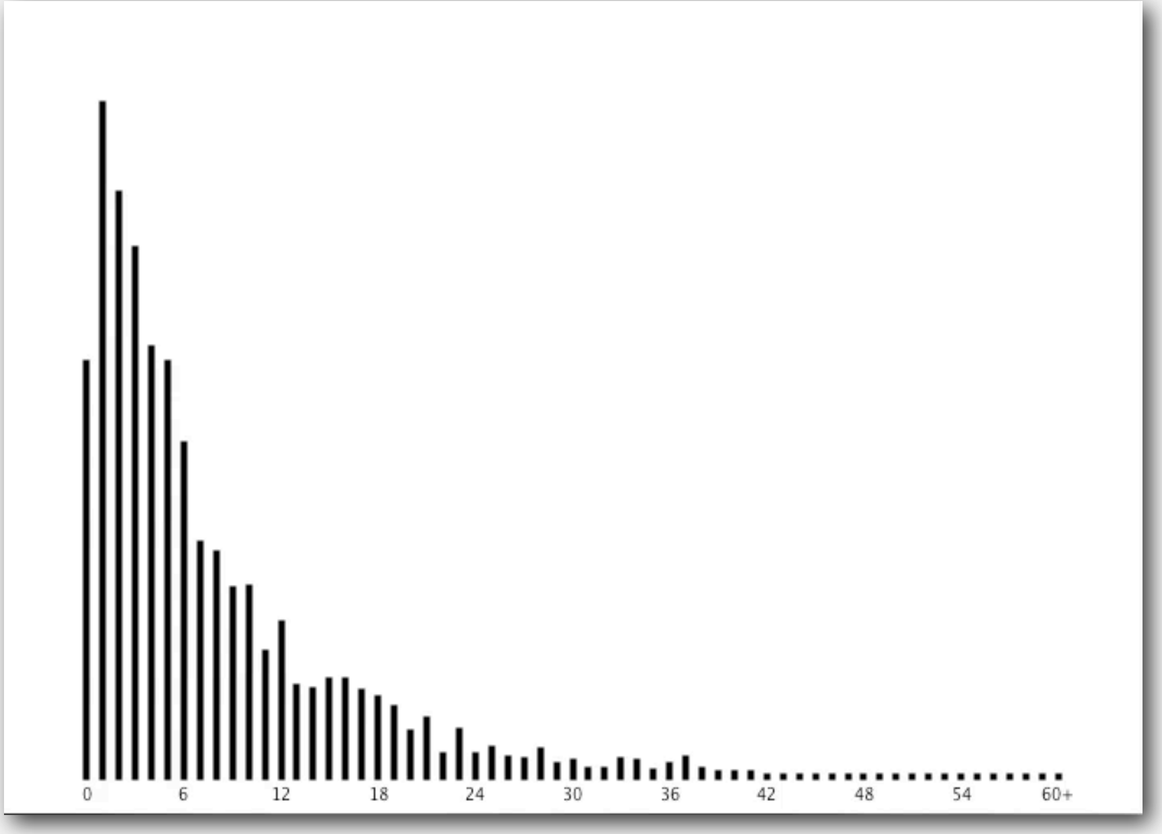
next event is an arrival

next event is a service completion

M/M/1 queuing model: experiments

Observation. If service rate μ is much larger than arrival rate λ, customers gets good service.

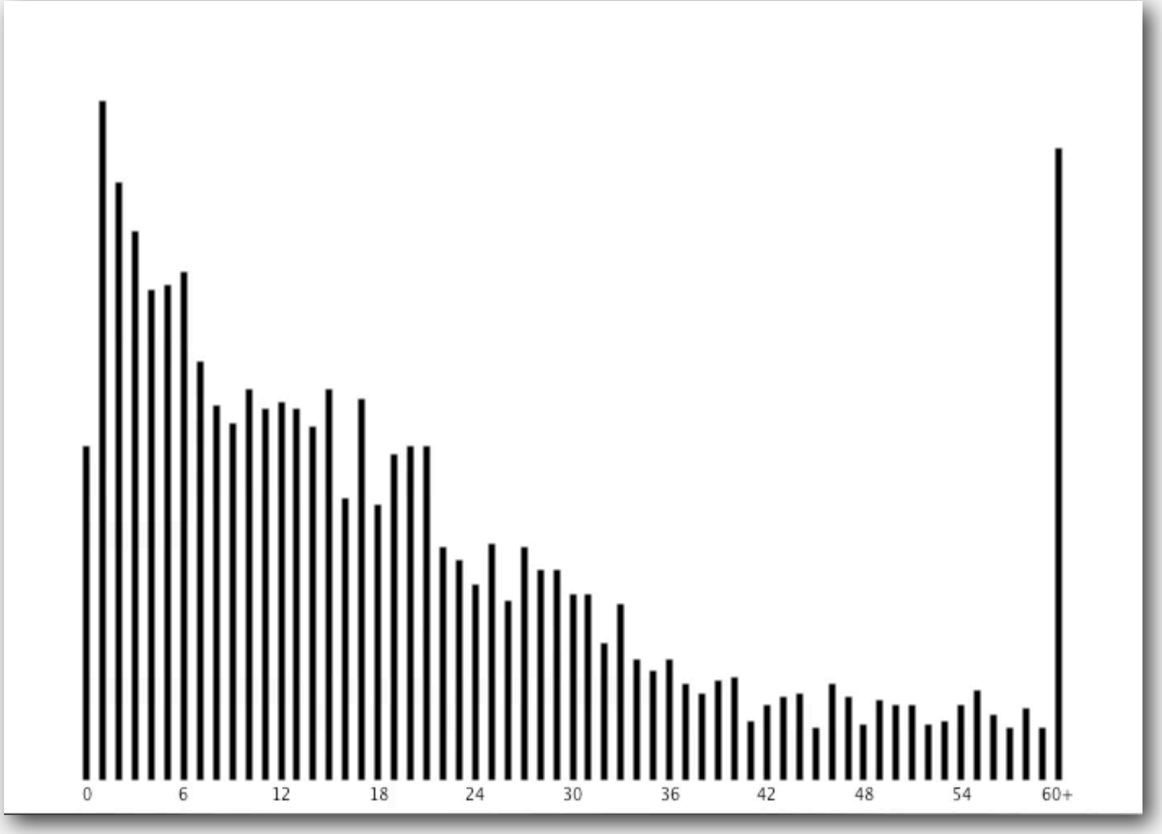
```
% java MM1Queue .2 .333
```



M/M/1 queuing model: experiments

Observation. As service rate μ approaches arrival rate λ, services goes to h***.

```
% java MM1Queue .2 .25
```



M/M/1 queuing model: experiments

Observation. As service rate μ approaches arrival rate λ, services goes to h***.

```
% java MM1Queue .2 .21
```

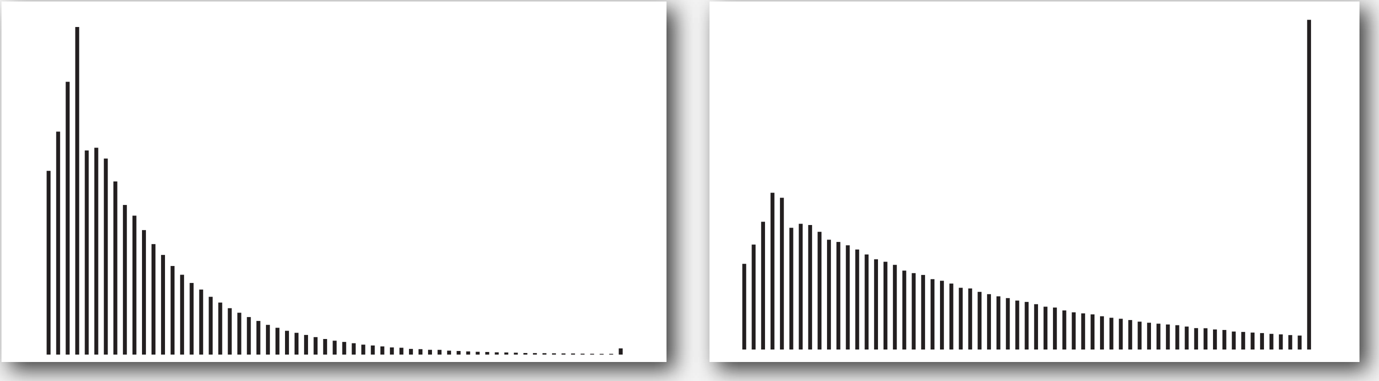
M/M/1 queuing model: analysis

M/M/1 queue. Exact formulas known.

wait time W and queue length L approach infinity
as service rate approaches arrival rate

Little's Law

$$W = \frac{1}{\mu - \lambda}, \quad L = \lambda W$$



More complicated queueing models. Event-based simulation essential!

Queueing theory. See ORF 309.