4.4 Hash Tables

- hash functions
- separate chaining
- linear probing
- applications

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Optimize judiciously

> *“ More computing sins are committed in the name of efficiency (without necessarily achieving it) than for any other single reason— including blind stupidity. ”* — William A. Wulf

> *“ We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil. ”* — Donald E. Knuth

> *“ We follow two rules in the matter of optimization:*
> *Rule 1: Don't do it.*
> *Rule 2 (for experts only). Don't do it yet - that is, not until you have a perfectly clear and unoptimized solution. ”* — M. A. Jackson

Reference: Effective Java by Joshua Bloch

ST implementations: summary

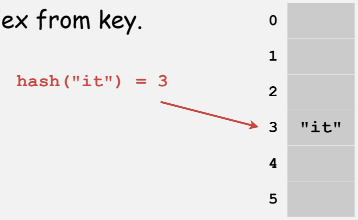
implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`

Q. Can we do better?

A. Yes, but with different access to the data.

Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).



Hash function. Method for computing array index from key.

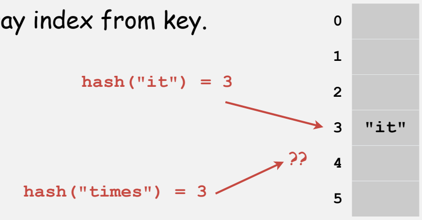
Issues.

- Computing the hash function.
- Equality test: Method for checking whether two keys are equal.

Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.



Issues.

- Computing the hash function.
- Equality test: Method for checking whether two keys are equal.
- Collision resolution: Algorithm and data structure to handle two keys that hash to the same array index.

Classic space-time tradeoff.

- No space limitation: trivial hash function with key as index.
- No time limitation: trivial collision resolution with sequential search.
- Limitations on both time and space: hashing (the real world).

- hash functions
- separate chaining
- linear probing
- applications

Equality test

Needed because hash methods do not use `compareTo()`.

All Java classes inherit a method `equals()`.

Java requirements. For any references `x`, `y` and `z`:

- Reflexive: `x.equals(x)` is `true`.
- Symmetric: `x.equals(y)` iff `y.equals(x)`.
- Transitive: if `x.equals(y)` and `y.equals(z)`, then `x.equals(z)`.
- Non-null: `x.equals(null)` is `false`.

equivalence relation

do `x` and `y` refer to the same object?

Default implementation. `(x == y)`

Customized implementations. `Integer`, `Double`, `String`, `File`, `URL`, `Date`, …

User-defined implementations. Some care needed.

Implementing equals for user-defined types

Seems easy

```
public       class Record
{
   private final String name;
   private final long val;
   ...

   public boolean equals(Record y)
   {





      Record that =          y;
      return (this.val == that.val) &&
             (this.name.equals(that.name));
   }
}
```

check that all significant fields are the same

Implementing equals for user-defined types

Seems easy, but requires some care.

```
public final class Record
{
   private final String name;
   private final long val;
   ...

   public boolean equals(Object y)
   {
      if (y == this) return true;

      if (y == null) return false;

      if (y.getClass() != this.getClass())
         return false;

      Record that = (Record) y;
      return (this.val == that.val) &&
             (this.name.equals(that.name));
   }
}
```

no safe way to use `equals()` with inheritance

must be `Object`.
Why? Experts still debate.

optimize for true object equality

check for `null`

objects must be in the same class

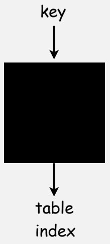
check that all significant
fields are the same

Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

- Efficiently computable.
- Each table index equally likely for each key.

thoroughly researched problem,
still problematic in practical applications



Ex 1. Phone numbers.

- Bad: first three digits.
- Better: last three digits.

Ex 2. Social Security numbers.

573 = California, 574 = Alaska
(assigned in chronological order within geographic region)

- Bad: first three digits.
- Better: last three digits.

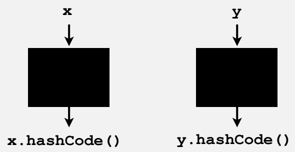
Practical challenge. Need different approach for each key type.

Java's hash code conventions

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

Requirement. If `x.equals(y)`, then `(x.hashCode() == y.hashCode())`.

Highly desirable. If `!x.equals(y)`, then `(x.hashCode() != y.hashCode())`.



Default implementation. Memory address of `x`.

Customized implementations. `Integer`, `Double`, `String`, `File`, `URL`, `Date`, …

User-defined types. Users are on their own.

Implementing hash code: integers and doubles

```
public final class Integer
{
   private final int value;
   ...

   public int hashCode()
   {  return value;  }
}
```

```
public final class Double
{
   private final double value;
   ...

   public int hashCode()
   {
      long bits = doubleToLongBits(value);
      return (int) (bits ^ (bits >>> 32));
   }
}
```

convert to IEEE 64-bit representation;
xor most significant 32-bits
with least significant 32-bits

Implementing hash code: strings

```
public final class String
{
   private final char[] s;
   ...

   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

ith character of s

char	Unicode
…	…
'a'	97
'b'	98
'c'	99
…	…

- Horner's method to hash string of length L: L multiplies/adds.
- Equivalent to $h = 31^{L-1} \cdot s^{0} + \ldots + 31^{2} \cdot s^{L-3} + 31^{1} \cdot s^{L-2} + 31^{0} \cdot s^{L-1}$.

Ex.

```
String s = "call";
int code = s.hashCode();
```

$3045982 = 99 \cdot 31^{3} + 97 \cdot 31^{2} + 108 \cdot 31^{1} + 108 \cdot 31^{0}$

$= 108 + 31 \cdot (108 + 31 \cdot (97 + 31 \cdot (99)))$

A poor hash code

Ex. Strings (in Java 1.1).

- For long strings: only examine 8-9 evenly spaced characters.
- Benefit: saves time in performing arithmetic.

```
public int hashCode()
{
   int hash = 0;
   int skip = Math.max(1, length() / 8);
   for (int i = 0; i < length(); i += skip)
      hash = s[i] + (37 * hash);
   return hash;
}
```

- Downside: great potential for bad collision patterns.

```
http://www.cs.princeton.edu/introcs/13loop/Hello.java
http://www.cs.princeton.edu/introcs/13loop/Hello.class
http://www.cs.princeton.edu/introcs/13loop/Hello.html
http://www.cs.princeton.edu/introcs/13loop/index.html
http://www.cs.princeton.edu/introcs/12type/index.html
```

Implementing hash code: user-defined types

```
public final class Record
{
   private String name;
   private int id;
   private double value;

   public Record(String name, int id, double value)
   {  /* as before */  }

   ...

   public boolean equals(Object y)
   {  /* as before */  }

   public int hashCode()
   {
      int hash = 17;
      hash = 31*hash + name.hashCode();
      hash = 31*hash + id;
      hash = 31*hash + Double.valueOf(value).hashCode();
      return hash;
   }
}
```

nonzero constant

typically a small prime

Hash code design

"Standard" recipe for user-defined types.

- Combine each significant field using the 31x + y rule.
- If field is a primitive type, use built-in hash code.
- If field is an array, apply to each element.
- If field is an object, apply rule recursively.

In practice. Recipe works reasonably well; used in Java libraries.
In theory. Need a theorem for each type to ensure reliability.

Basic rule. Need to use the whole key to compute hash code; consult an expert for state-of-the-art hash codes.

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31}-1$.

Hash function. An `int` between `0` and `M-1` (for use as array index).

typically a prime or power of 2

Bug.

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

1-in-a billion bug.

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;  }
```

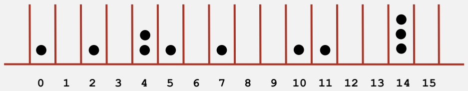
Correct.

```
private int hash(Key key)
{  return (key.hashCode() & 0x7fffffff) % M;  }
```

Uniform hashing assumption

Assumption J (uniform hashing hashing assumption).
Each key is equally likely to hash to an integer between 0 and M-1.

Bins and balls. Throw balls uniformly at random into M bins.



Birthday problem. Expect two balls in the same bin after $\sim \sqrt{\pi M / 2}$ tosses.

Coupon collector. Expect every bin has ≥ 1 ball after ~ M ln M tosses.

Load balancing. After M tosses, expect most loaded bin has $\Theta(\log M / \log \log M)$ balls.

Uniform hashing assumption

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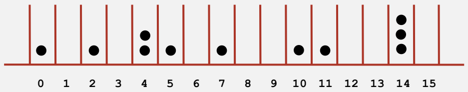
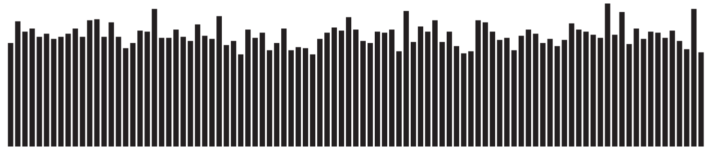




Hash value frequencies for words in Tale of Two Cities (M = 97)

Java's `String` data uniformly distribute the keys of Tale of Two Cities

---

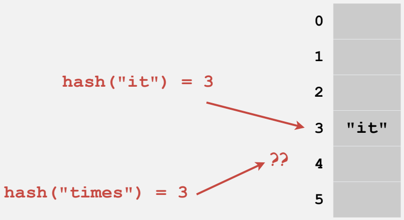
- hash functions
- **separate chaining**
- linear probing
- applications

Collisions

Collision. Two distinct keys hashing to same index.

- Birthday problem ⇒ can't avoid collisions unless you have a ridiculous amount (quadratic) of memory.
- Coupon collector + load balancing ⇒ collisions will be evenly distributed.

Challenge. Deal with collisions efficiently.

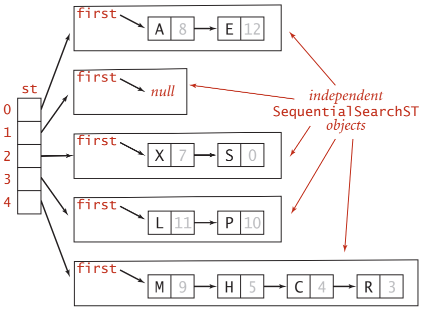


Separate chaining ST

Use an array of M < N linked lists. [H. P. Luhn, IBM 1953]

- Hash: map key to integer i between 0 and M-1.
- Insert: put at front of i^{th} chain (if not already there).
- Search: only need to search i^{th} chain.

key	hash	value
S	2	0
E	0	1
A	0	2
R	4	3
C	4	4
H	4	5
E	0	6
X	2	7
A	0	8
M	4	9
P	3	10
L	3	11
E	0	12



Hashing with separate chaining for standard indexing client

Separate chaining ST: Java implementation

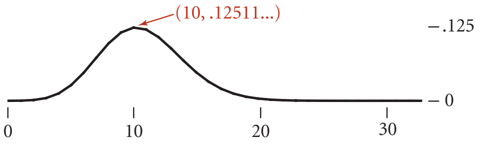
```
public class SeparateChainingHashST<Key, Value>
{
   private int N;      // number of key-value pairs
   private int M;      // hash table size
   private SequentialSearchST<Key, Value> [] st; // array of STs

   public SeparateChainingHashST()
   {  this(997);  }

   public SeparateChainingHashST(int M)
   {
      this.M = M;
      st = (SequentialSearchST<Key, Value>[]) new SequentialSearchST[M];
      for (int i = 0; i < M; i++)
         st[i] = new SequentialSearchST<Key, Value>();
   }
   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % M;   }

   public Value get(Key key)
   {  return st[hash(key)].get(key);  }

   public void put(Key key, Value val)
   {  st[hash(key)].put(key, val);  }

   public Iterable<Key> keys()
   {  return st[i].keys();  }
}
```

array doubling code omitted

Analysis of separate chaining

Proposition K. Under uniform hashing assumption, probability that the number of keys in a list is within a constant factor of N/M is extremely close to 1.

Pf sketch. Distribution of list size obeys a binomial distribution.



Binomial distribution $(N = 10^4, M = 10^3, \alpha = 10)$

`equals()` and `hashCode()`

Consequence. Number of probes for search/insert is proportional to N/M.

- M too large ⇒ too many empty chains.
- M too small ⇒ chains too long.
- Typical choice: M ~ N/5 ⇒ constant-time ops.

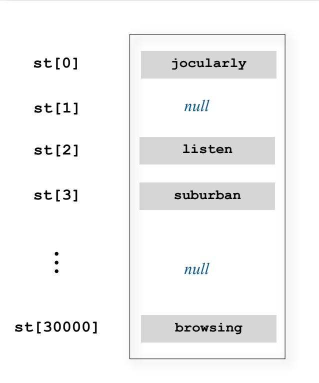
M times faster than sequential search

- hash functions
- separate chaining
- **linear probing**
- applications

Collision resolution: open addressing

Open addressing. [Amdahl-Boehme-Rocherster-Samuel, IBM 1953]

When a new key collides, find next empty slot, and put it there.



linear probing (M = 30001, N = 15000)

Linear probing

Use an array of size M > N.

- Hash: map key to integer i between 0 and M-1.
- Insert: put at table index i if free; if not try i+1, i+2, etc.
- Search: search table index i; if occupied but no match, try i+1, i+2, etc.

-	-	-	S	H	-	-	A	C	E	R	-	-
0	1	2	3	4	5	6	7	8	9	10	11	12

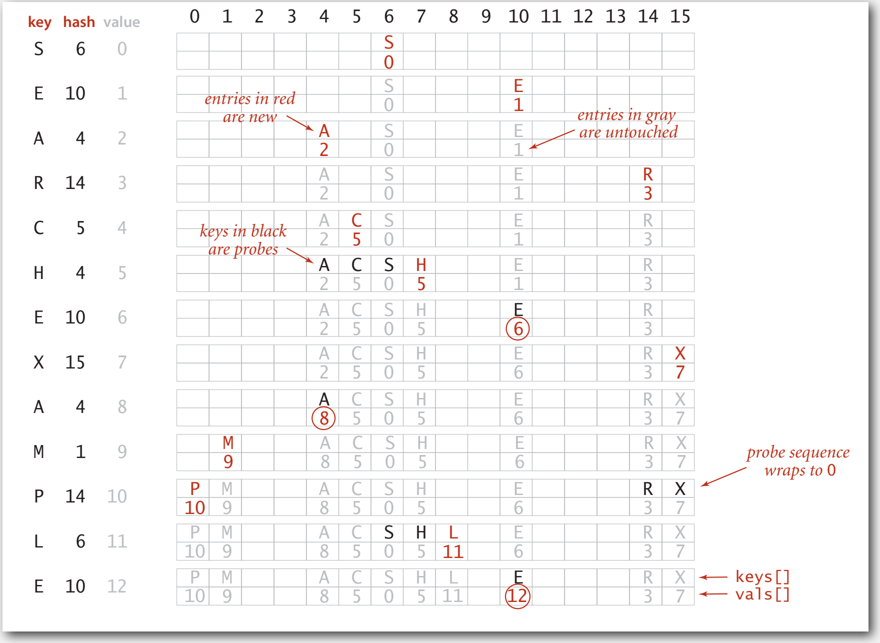
-	-	-	S	H	-	-	A	C	E	R	I	-
0	1	2	3	4	5	6	7	8	9	10	11	12

insert I
hash(I) = 11

-	-	-	S	H	-	-	A	C	E	R	I	N
0	1	2	3	4	5	6	7	8	9	10	11	12

insert N
hash(N) = 8

Linear probing: trace of standard indexing client



key	hash	value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S	6	0							S 0									
E	10	1							S 0				E 1					
A	4	2					A 2		S 0				E 1					
R	14	3					A 2		S 0				E 1				R 3	
C	5	4					A 2	C 5	S 0				E 1				R 3	
H	4	5					A 2	C 5	S 0	H 5			E 1				R 3	
E	10	6					A 2	C 5	S 0	H 5			E 6				R 3	
X	15	7					A 2	C 5	S 0	H 5			E 6				R 3	X 7
A	4	8					A 8	C 5	S 0	H 5			E 6				R 3	X 7
M	1	9		M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
P	14	10	P 10	M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
L	6	11	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 6				R 3	X 7
E	10	12	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 12				R 3	X 7

Linear probing ST implementation

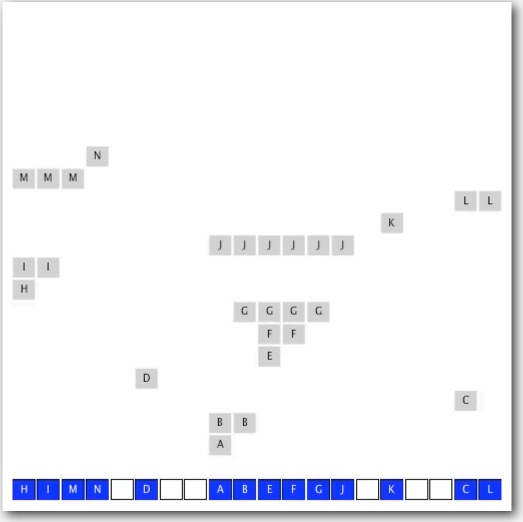
```
public class LinearProbingHashST<Key, Value>
{
   private int M = 30001;
   private Value[] vals = (Value[]) new Object[M];
   private Key[]   keys = (Key[])   new Object[M];

   private int hash(Key key) {  /* as before */  }

   public void put(Key key, Value val)
   {
      int i;
      for (i = hash(key); keys[i] != null; i = (i+1) % M)
         if (keys[i].equals(key))
             break;
      keys[i] = key;
      vals[i] = val;
   }

   public Value get(Key key)
   {
      for (int i = hash(key); keys[i] != null; i = (i+1) % M)
         if (key.equals(keys[i]))
             return vals[i];
      return null;
   }
}
```

array doubling code omitted

Clustering

Cluster. A contiguous block of items.

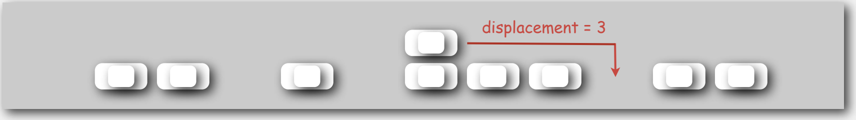
Observation. New keys likely to hash into middle of big clusters.



Knuth's parking problem

Model. Cars arrive at one-way street with M parking spaces. Each desires a random space i: if space i is taken, try i+1, i+2, …

Q. What is mean displacement of a car?



Empty. With M/2 cars, mean displacement is ~ 3/2.

Full. With M cars, mean displacement is ~ $\sqrt{\pi M / 8}$

Analysis of linear probing

Proposition M. Under uniform hashing assumption, the average number of probes in a hash table of size M that contains N = α M keys is:

$$\sim \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \qquad \sim \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

search hit — search miss / insert

Pf. [Knuth 1962] A landmark in analysis of algorithms.

Parameters.

- M too large ⇒ too many empty array entries.
- M too small ⇒ search time blows up.
- Typical choice: α = N/M ~ $\frac{1}{2}$.

probes for search hit is about 3/2
probes for search miss is about 5/2

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`
hashing	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`

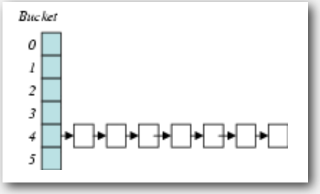
* under uniform hashing assumption

Algorithmic complexity attacks

Q. Is the uniform hashing assumption important in practice?

A. Obvious situations: aircraft control, nuclear reactor, pacemaker.

A. Surprising situations: denial-of-service attacks.



malicious adversary learns your hash function (e.g., by reading Java API) and causes a big pile-up in single slot that grinds performance to a halt

Real-world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem.
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Algorithmic complexity attack on Java

Goal. Find family of strings with the same hash code.

Solution. The base-31 hash code is part of Java's string API.

key	hashCode()
"Aa"	2112
"BB"	2112

key	hashCode()
"AaAaAaAa"	-540425984
"AaAaAaBB"	-540425984
"AaAaBBAa"	-540425984
"AaAaBBBB"	-540425984
"AaBBAaAa"	-540425984
"AaBBAaBB"	-540425984
"AaBBBBAa"	-540425984
"AaBBBBBB"	-540425984

key	hashCode()
"BBAaAaAa"	-540425984
"BBAaAaBB"	-540425984
"BBAaBBAa"	-540425984
"BBAaBBBB"	-540425984
"BBBBAaAa"	-540425984
"BBBBAaBB"	-540425984
"BBBBBBAa"	-540425984
"BBBBBBBB"	-540425984

2^N strings of length 2N that hash to same value!

Diversion: one-way hash functions

One-way hash function. Hard to find a key that will hash to a desired value, or to find two keys that hash to same value.

Ex. MD4, MD5, SHA-0, SHA-1, SHA-2, WHIRLPOOL, RIPEMD-160.

known to be insecure

```
String password = args[0];
MessageDigest sha1 = MessageDigest.getInstance("SHA1");
byte[] bytes = sha1.digest(password);

/* prints bytes as hex string */
```

Applications. Digital fingerprint, message digest, storing passwords.

Caveat. Too expensive for use in ST implementations.

Separate chaining vs. linear probing

Separate chaining.

- Easier to implement delete.
- Performance degrades gracefully.
- Clustering less sensitive to poorly-designed hash function.

Linear probing.

- Less wasted space.
- Better cache performance.

Hashing: variations on the theme

Many improved versions have been studied.

Two-probe hashing. (separate chaining variant)

- Hash to two positions, put key in shorter of the two chains.
- Reduces average length of the longest chain to log log N.

Double hashing. (linear probing variant)

- Use linear probing, but skip a variable amount, not just 1 each time.
- Effectively eliminates clustering.
- Can allow table to become nearly full.

Hashing vs. balanced trees

Hashing.

- Simpler to code.
- No effective alternative for unordered keys.
- Faster for simple keys (a few arithmetic ops versus log N compares).
- Better system support in Java for strings (e.g., cached hash code).

Balanced trees.

- Stronger performance guarantee.
- Support for ordered ST operations.
- Easier to implement `compareTo()` correctly than `equals()` and `hashCode()`.

Java system includes both.

- Red-black trees: `java.util.TreeMap`, `java.util.TreeSet`.
- Hashing: `java.util.HashMap`, `java.util.IdentityHashMap`.