

1.2 Analysis of Algorithms



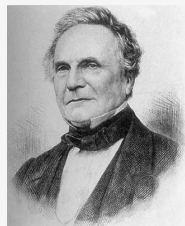
- ▶ estimating running time
- ▶ mathematical analysis
- ▶ order-of-growth hypotheses
- ▶ input models
- ▶ measuring space

Reference: Intro to Programming in Java, Section 4.1

Algorithms in Java, 4th Edition · Robert Sedgewick and Kevin Wayne · Copyright © 2009 · September 21, 2009 4:04:10 PM

Running time

“As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time?” — Charles Babbage



Charles Babbage (1864)



Analytic Engine

how many times do you have to turn the crank?

Cast of characters



Programmer needs to develop a working solution.



Student might play any or all of these roles someday.



Client wants problem solved efficiently.



Theoretician wants to understand.



Basic blocking and tackling is sometimes necessary. [this lecture]

Reasons to analyze algorithms

Predict performance.

Compare algorithms.

Provide guarantees.

Understand theoretical basis.

this course (COS 226)

theory of algorithms (COS 423)

Primary practical reason: avoid performance bugs.



client gets poor performance because programmer did not understand performance characteristics



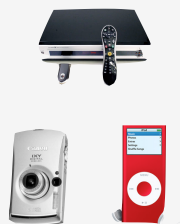
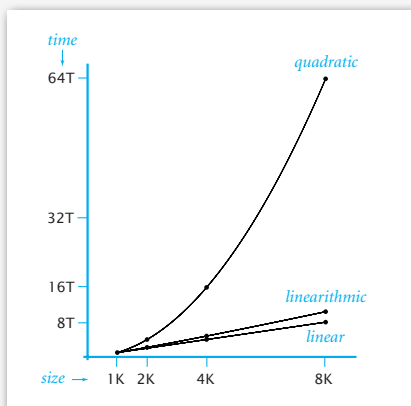
Some algorithmic successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics,
- Brute force: N^2 steps.
- FFT algorithm: $N \log N$ steps, **enables new technology.**



Friedrich Gauss
1805



5

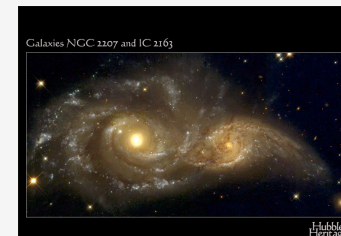
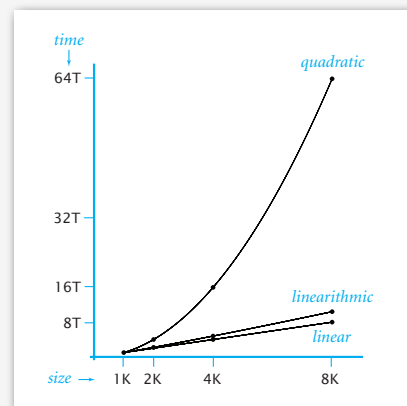
Some algorithmic successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: $N \log N$ steps, **enables new research.**



Andrew Appel
PU '81



6

- ▶ estimating running time
- ▶ mathematical analysis
- ▶ order-of-growth hypotheses
- ▶ input models
- ▶ measuring space

7

Scientific analysis of algorithms

A framework for predicting performance and comparing algorithms.

Scientific method.

- **Observe** some feature of the universe.
- **Hypothesize** a model that is consistent with observation.
- **Predict** events using the hypothesis.
- **Verify** the predictions by making further observations.
- **Validate** by repeating until the hypothesis and observations agree.

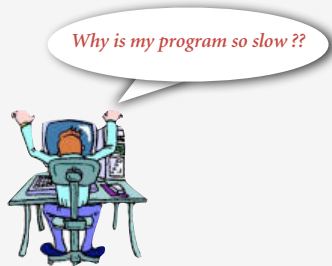
Principles.

- Experiments must be **reproducible**.
- Hypotheses must be **falsifiable**.

Universe = computer itself.

8

Every time you run a program you are doing an experiment!



First step. Debug your program!

Second step. Choose input model for experiments.

Third step. Run and time the program for problems of increasing size.

9

3-sum. Given N integers, find all triples that sum to exactly zero.

```
% more input8.txt
8
30 -30 -20 -10 40 0 10 5

% java ThreeSum < input8.txt
4
30 -30 0
30 -20 -10
-30 -10 40
-10 0 10
```

Context. Deeply related to problems in computational geometry.

10

3-sum: brute-force algorithm

```
public class ThreeSum
{
    public static int count(long[] a)
    {
        int N = a.length;
        int cnt = 0;
        for (int i = 0; i < N; i++)
            for (int j = i+1; j < N; j++)
                for (int k = j+1; k < N; k++)
                    if (a[i] + a[j] + a[k] == 0)
                        cnt++;
        return cnt;
    }

    public static void main(String[] args)
    {
        long[] a = StdArrayIO.readLong1D();
        StdOut.println(count(a));
    }
}
```

check each triple

11

Empirical analysis

Run the program for various input sizes and measure running time.

ThreeSum.java

N	time (seconds) †
1024	0.26
2048	2.16
4096	17.18
8192	137.76

† Running Linux on Sun-Fire-X4100

12

Measuring the running time

Q. How to time a program?

A. Manual.



```
% java ThreeSum < 1Kints.txt
0
tick tick tick

% java ThreeSum < 2Kints.txt
2
tick tick tick tick tick tick
tick tick tick tick tick tick
tick tick tick tick tick tick
tick tick tick tick tick tick
391930676 -763182495 371251819
-326747290 802431422 -475684132
```

13

Measuring the running time

Q. How to time a program?

A. Automatic.

```
Stopwatch stopwatch = new Stopwatch();

ThreeSum.count(a);

double time = stopwatch.elapsedTime();
StdOut.println("Running time: " + time + " seconds");
```

client code

```
public class Stopwatch
{
    private final long start = System.currentTimeMillis();

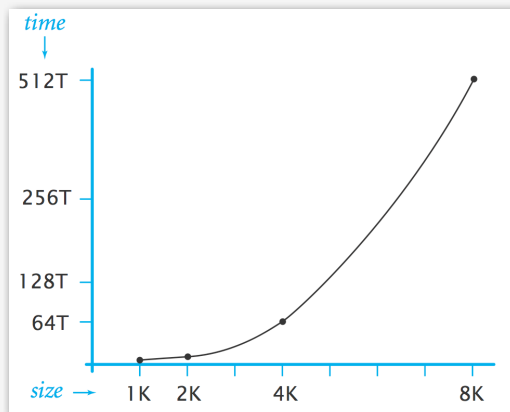
    public double elapsedTime()
    {
        long now = System.currentTimeMillis();
        return (now - start) / 1000.0;
    }
}
```

implementation (part of stdlib.jar, see <http://www.cs.princeton.edu/introcs/stdlib>)

14

Data analysis

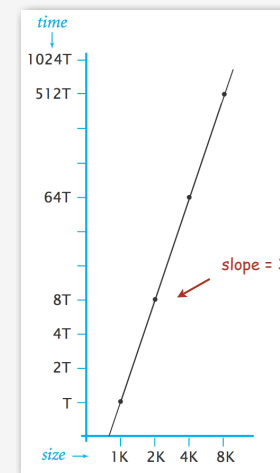
Plot running time as a function of input size N .



15

Data analysis

Log-log plot. Plot running time vs. input size N on log-log scale.



Regression. Fit straight line through data points: $a N^b$.

Hypothesis. Running time grows with the **cube** of the input size: $a N^3$.

power law

slope

16

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power law hypothesis.

Run program, **doubling** the size of the input.

N	time (seconds) †	ratio	lg ratio
512	0.03	-	
1024	0.26	7.88	2.98
2048	2.16	8.43	3.08
4096	17.18	7.96	2.99
8192	137.76	7.96	2.99

↑
seems to converge to a constant $b \approx 3$

Hypothesis. Running time is about $a N^b$ with $b = \lg \text{ratio}$.

Caveat. Can't identify logarithmic factors with doubling hypothesis.

17

Prediction and verification

Hypothesis. Running time is about $a N^3$ for input of size N .

Q. How to estimate a ?

A. Run the program!

N	time (seconds)
4096	17.18
4096	17.15
4096	17.17

$$17.17 = a \times 4096^3 \\ \Rightarrow a = 2.5 \times 10^{-10}$$

Refined hypothesis. Running time is about $2.5 \times 10^{-10} \times N^3$ seconds.

Prediction. 1,100 seconds for $N = 16,384$.

Observation.

N	time (seconds)
16384	1118.86

validates hypothesis!

18

Experimental algorithmics

Many obvious factors affect running time:

- Machine.
- Compiler.
- Algorithm.
- Input data.

More factors (not so obvious):

- Caching.
- Garbage collection.
- Just-in-time compilation.
- CPU use by other applications.

Bad news. It is often difficult to get precise measurements.

Good news. Easier than other sciences.

↖
e.g., can run huge number of experiments

19

War story (from COS 126)

Q. How long does this program take as a function of N ?

```
public class EditDistance
{
    String s = StdIn.readString();
    int N = s.length();
    ...
    for (int i = 0; i < N; i++)
        for (int j = 0; j < N; j++)
            distance[i][j] = ...
    ...
}
```

N	time	N	time
1024	0.11	256	0.5
2048	0.35	512	1.1
4096	1.6	1024	1.9
9182	6.5	2048	3.9

Jenny

Kenny

Jenny. $\sim c_1 N^2$ seconds.

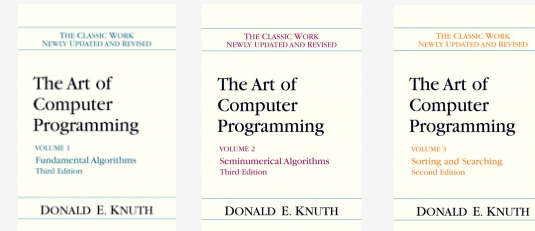
Kenny. $\sim c_2 N$ seconds.

20

Mathematical models for running time

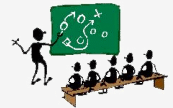
Total running time: sum of cost \times frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.



- › estimating running time
- › **mathematical analysis**
- › order-of-growth hypotheses
- › input models
- › measuring space

21

22

Cost of basic operations

operation	example	nanoseconds [†]
integer add	<code>a + b</code>	2.1
integer multiply	<code>a * b</code>	2.4
integer divide	<code>a / b</code>	5.4
floating point add	<code>a + b</code>	4.6
floating point multiply	<code>a * b</code>	4.2
floating point divide	<code>a / b</code>	13.5
sine	<code>Math.sin(theta)</code>	91.3
arctangent	<code>Math.atan2(y, x)</code>	129.0
...

[†] Running OS X on Macbook Pro 2.2GHz with 2GB RAM

23

Cost of basic operations

operation	example	nanoseconds [†]
variable declaration	<code>int a</code>	c_1
assignment statement	<code>a = b</code>	c_2
integer compare	<code>a < b</code>	c_3
array element access	<code>a[i]</code>	c_4
array length	<code>a.length</code>	c_5
1D array allocation	<code>new int[N]</code>	$c_6 N$
2D array allocation	<code>new int[N][N]</code>	$c_7 N^2$
string length	<code>s.length()</code>	c_8
substring extraction	<code>s.substring(N/2, N)</code>	c_9
string concatenation	<code>s + t</code>	$c_{10} N$

Novice mistake. Abusive string concatenation.

24

Example: 1-sum

Q. How many instructions as a function of N?

```
int count = 0;
for (int i = 0; i < N; i++)
    if (a[i] == 0) count++;
```

operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
increment	$\leq 2N$

between N (no zeros)
and 2N (all zeros)

25

Example: 2-sum

Q. How many instructions as a function of N?

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0) count++;
```

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$1/2 (N + 1) (N + 2)$
equal to compare	$1/2 N (N - 1)$
array access	$N (N - 1)$
increment	$\leq N^2$

$$0 + 1 + 2 + \dots + (N - 1) = \frac{1}{2} N (N - 1) = \binom{N}{2}$$

tedious to count exactly

26

Tilde notation

- Estimate running time (or memory) as a function of input size N .
- Ignore lower order terms.
 - when N is large, terms are negligible
 - when N is small, we don't care

Ex 1. $6N^3 + 20N + 16 \sim 6N^3$

Ex 2. $6N^3 + 100N^{4/3} + 56 \sim 6N^3$

Ex 3. $6N^3 + 17N^2 \lg N + 7N \sim 6N^3$

discard lower-order terms
(e.g., $N = 1000$ 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \rightarrow \infty} \frac{f(N)}{g(N)} = 1$

27

Example: 2-sum

Q. How long will it take as a function of N?

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0) count++;
```

"inner loop"

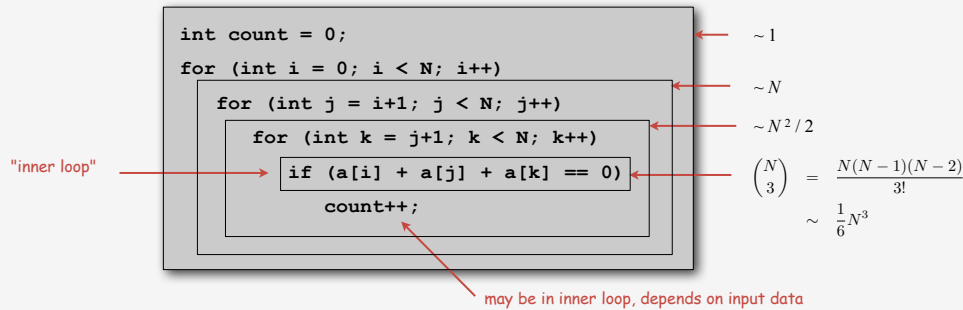
operation	frequency	time per op	total time
variable declaration	$\sim N$	c_1	$\sim c_1 N$
assignment statement	$\sim N$	c_2	$\sim c_2 N$
less than comparison	$\sim 1/2 N^2$	c_3	$\sim c_3 N^2$
equal to comparison	$\sim 1/2 N^2$		
array access	$\sim N^2$	c_4	$\sim c_4 N^2$
increment	$\leq N^2$	c_5	$\leq c_5 N^2$
total			$\sim c N^2$

depends on input data

28

Example: 3-sum

Q. How many instructions as a function of N?



Remark. Focus on instructions in **inner loop**; ignore everything else!

29

Bounding the sum by an integral trick

Q. How to estimate a discrete sum?

A. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \dots + N.$ $\sum_{i=1}^N i \sim \int_{x=1}^N x dx \sim \frac{1}{2}N^2$

Ex 2. $1 + 1/2 + 1/3 + \dots + 1/N.$ $\sum_{i=1}^N \frac{1}{i} \sim \int_{x=1}^N \frac{1}{x} dx = \ln N$

Ex 3. 3-sum triple loop. $\sum_{i=1}^N \sum_{j=i}^N \sum_{k=j}^N 1 \sim \int_{x=1}^N \int_{y=x}^N \int_{z=y}^N dz dy dx \sim \frac{1}{6}N^3$

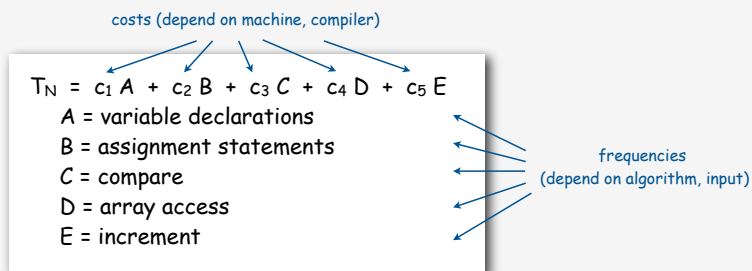
30

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.



Bottom line. We use **approximate** models in this course: $T_N \sim c N^3$.

31

- ▶ estimating running time
- ▶ mathematical analysis
- ▶ **order-of-growth hypotheses**
- ▶ input models
- ▶ measuring space

32

Common order-of-growth hypotheses

To determine order-of-growth:

- Assume a power law $T_N \sim a N^b$.
- Estimate exponent b with doubling hypothesis.
- Validate with mathematical analysis.

Ex. `ThreeSumDeluxe.java`

Food for precept. How is it implemented?

`ThreeSum.java`

N	time (seconds)
1024	0.26
2048	2.16
4096	17.18
8192	137.76

observations

`ThreeSumDeluxe.java`

N	time (seconds)
1,000	0.43
2,000	0.53
4,000	1.01
8,000	2.87
16,000	11.00
32,000	44.64
64,000	177.48

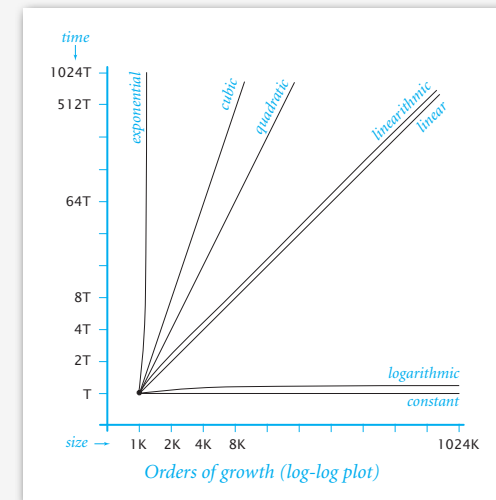
33

Common order-of-growth hypotheses

Good news. the small set of functions

1, $\log N$, N , $N \log N$, N^2 , N^3 , and 2^N

suffices to describe order-of-growth of typical algorithms.



34

Common order-of-growth hypotheses

growth rate	name	typical code framework	description	example	$T(2N) / T(N)$
1	constant	<code>a = b + c;</code>	statement	add two numbers	1
$\log N$	logarithmic	<pre>while (N > 1) { N = N / 2; ... }</pre>	divide in half	binary search	~ 1
N	linear	<pre>for (int i = 0; i < N; i++) { ... }</pre>	loop	find the maximum	2
$N \log N$	linearithmic	[see lecture 5]	divide and conquer	mergesort	~ 2
N^2	quadratic	<pre>for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) { ... }</pre>	double loop	check all pairs	4
N^3	cubic	<pre>for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) for (int k = 0; k < N; k++) { ... }</pre>	triple loop	check all triples	8
2^N	exponential	[see lecture 24]	exhaustive search	check all possibilities	$T(N)$

35

Practical implications of order-of-growth

growth rate	name	description	effect on a program that runs for a few seconds	
			time for 100x more data	size for 100x faster computer
1	constant	independent of input size	-	-
$\log N$	logarithmic	nearly independent of input size	-	-
N	linear	optimal for N inputs	a few minutes	100x
$N \log N$	linearithmic	nearly optimal for N inputs	a few minutes	100x
N^2	quadratic	not practical for large problems	several hours	10x
N^3	cubic	not practical for medium problems	several weeks	4-5x
2^N	exponential	useful only for tiny problems	forever	1x

36

- ▶ estimating running time
- ▶ mathematical analysis
- ▶ order-of-growth hypotheses
- ▶ **input models**
- ▶ measuring space

37

Types of analyses

Best case. Lower bound on cost.

- Determined by "easiest" input.
- Provides a goal for all inputs.

Worst case. Upper bound on cost.

- Determined by "most difficult" input.
- Provides guarantee for all inputs.

Average case. "Expected" cost.

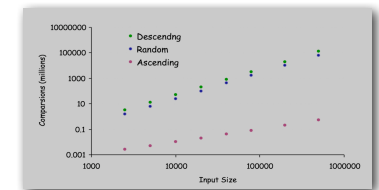
- Need a model for "random" input.
- Provides a way to predict performance.

Ex 1. Array accesses for brute-force 3-sum.

- Best: $\sim \frac{1}{2}N^3$
- Average: $\sim \frac{1}{3}N^3$
- Worst: $\sim \frac{1}{2}N^3$

Ex 2. Compares for insertion sort.

- Best (ascending order): $\sim N$.
- Average (random order): $\sim \frac{1}{4}N^2$
- Worst (descending order): $\sim \frac{1}{2}N^2$ (details in Lecture 4)



38

Commonly-used notations

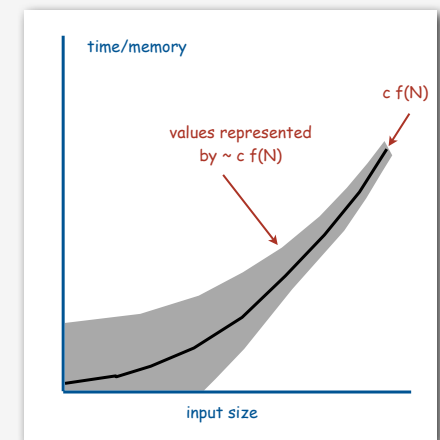
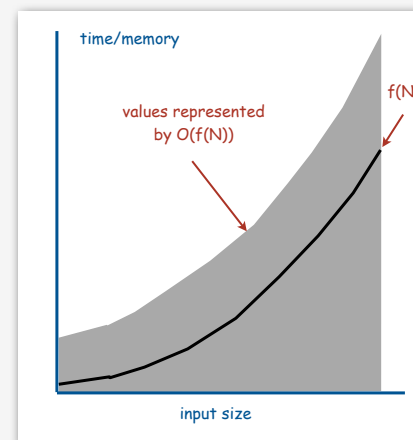
notation	provides	example	shorthand for	used to
Tilde	leading term	$\sim 10 N^2$	$10 N^2$ $10 N^2 + 22 N \log N$ $10 N^2 + 2 N + 37$	provide approximate model
Big Theta	asymptotic growth rate	$\Theta(N^2)$	N^2 $9000 N^2$ $5 N^2 + 22 N \log N + 3 N$	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	N^2 $100 N$ $22 N \log N + 3 N$	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$9000 N^2$ N^5 $N^3 + 22 N \log N + 3 N$	develop lower bounds

Common mistake. Interpreting big-Oh as an approximate model.

Tilde notation vs. big-Oh notation

We use tilde notation whenever possible.

- Big-Oh notation suppresses leading constant.
- Big-Oh notation only provides upper bound (not lower bound).



40

- › estimating running time
- › mathematical analysis
- › order-of-growth hypotheses
- › input models
- › **measuring space**

Typical memory requirements for primitive types in Java

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million bytes.

Gigabyte (GB). 1 billion bytes.

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

Typical memory requirements for arrays in Java

Array overhead. 16 bytes.

type	bytes
char[]	$2N + 16$
int[]	$4N + 16$
double[]	$8N + 16$

one-dimensional arrays

type	bytes
char[][]	$2N^2 + 20N + 16$
int[][]	$4N^2 + 20N + 16$
double[][]	$8N^2 + 20N + 16$

two-dimensional arrays

Typical memory requirements for objects in Java

Object overhead. 8 bytes.

Reference. 4 bytes.

Ex 1. A `Complex` object consumes 24 bytes of memory.

```
public class Complex
{
    private double re;
    private double im;
    ...
}
```

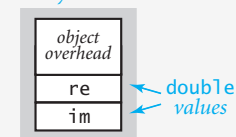
8 bytes overhead for object

8 bytes

8 bytes

24 bytes

24 bytes

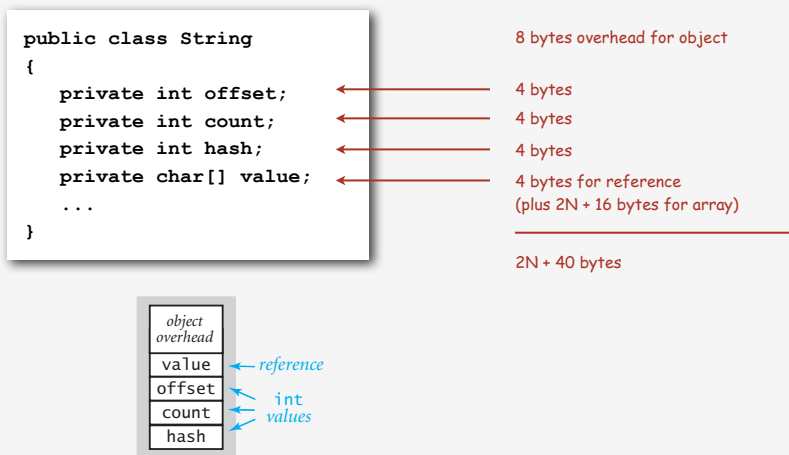


Typical memory requirements for objects in Java

Object overhead. 8 bytes.

Reference. 4 bytes.

Ex 2. A virgin string of length N consumes $2N + 40$ bytes.



45

Example 1

Q. How much memory does this data type use as a function of N ?

A.

```
public class QuickUWPC
{
    private int[] id;
    private int[] sz;

    public QuickUWPC(int N)
    {
        id = new int[N];
        sz = new int[N];
        for (int i = 0; i < N; i++) id[i] = i;
        for (int i = 0; i < N; i++) sz[i] = 1;
    }

    public boolean find(int p, int q)
    { ... }

    public void unite(int p, int q)
    { ... }
}
```

46

Example 2

Q. How much memory does this code fragment use as a function of N ?

A.

```
...
int N = Integer.parseInt(args[0]);
for (int i = 0; i < N; i++) {
    int[] a = new int[N];
    ...
}
```

Remark. Java automatically reclaims memory when it is no longer in use.

not always easy for Java to know

47

Turning the crank: summary

In principle, accurate mathematical models are available.

In practice, approximate mathematical models are easily achieved.

Timing may be flawed?

- Limits on experiments insignificant compared to other sciences.
- Mathematics might be difficult?
- Only a few functions seem to turn up.
- Doubling hypothesis cancels complicated constants.



Actual data might not match input model?

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

48