Representations

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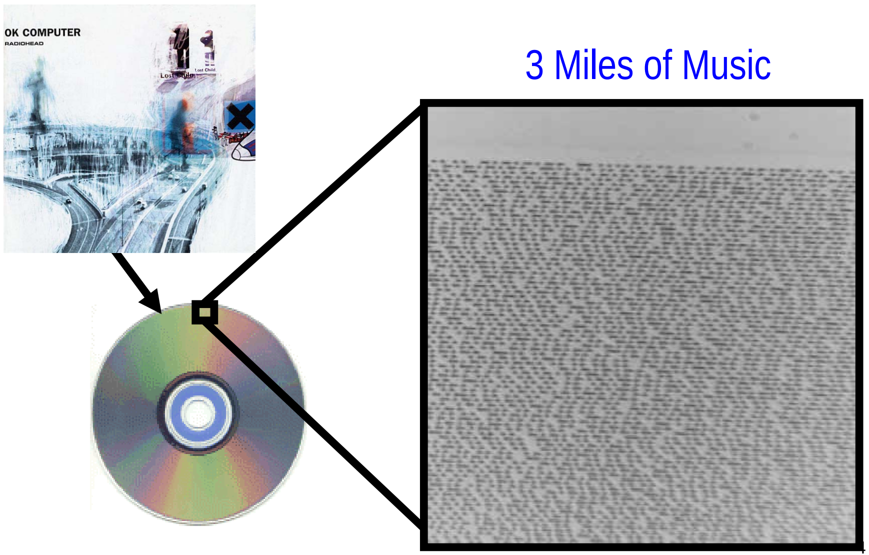
COS 217

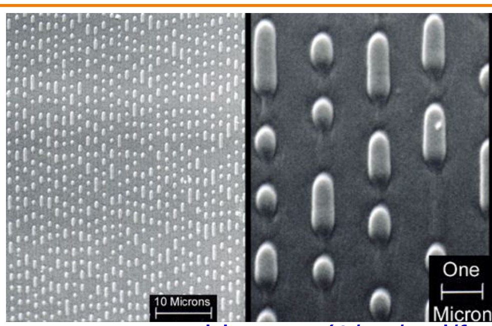
Goals of Today's Lecture

- Representations
  - Why binary?
  - Converting base 10 to base 2
  - Octal and hexadecimal
- Integers
  - Unsigned integers
  - Integer addition, subtraction
  - Signed integers
- C bit operators
  - And, or, not, and xor
  - Shift-left and shift-right
  - Function for counting the number of 1 bits
  - Function for XOR encryption of a message

Radiohead - OK Computer CD

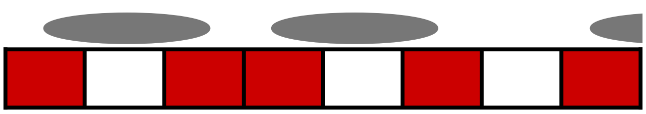


Pits and Lands



Transition represents a bit state (1/on/red/female/heads)

No change represents other state (0/off/white/male/tails)



Interpretation

0	1	1	1	0	1	0	1

As Music:

01110101_2 = 117/256 position of speaker

As Number:

$01110101_2 = 1 + 4 + 16 + 32 + 64 = 117_{10} = 75_{16}$

(Get comfortable with base 2, 8, 10, and 16.)

As Text:

01110101_2 = 117th character in the ASCII codes = “u”

Interpretation – ASCII

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	¶	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	^	126	~
031	(cursor down)	US	063	?	095	_	127	⌂

Computer Science Building West Wall

Interpretation: Code and Data (Hello World!)

- Programs consist of Code and Data
- Code and Data are Encoded in Bits

IA-64 Binary (objdump)

```
00000000: 7f45 4c46 0201 0100 0000 0000 0000 0000  .ELF............
...
00000260: 5002 0000 0000 0000 006c 6962 632e 736f  P........libc.so
00000270: 2e36 2e31 0070 7269 6e74 6600 5f5f 6c69  .6.1.printf.__li
00000280: 6263 5f73 7461 7274 5f6d 6169 6e00 474c  bc_start_main.GL
00000290: 4942 435f 322e 3200 0000 0200 0200 0000  IBC_2.2.........
...
00000860: 4865 6c6c 6f20 576f 726c 6421 0d00 0000  Hello World!....
...
4000000000000690 <main>:
4000000000000690:        00 10 15 08 80 05      [MII]       alloc r34=ar.pfs,5,4,0
4000000000000696:        30 02 30 00 42 20                  mov r35=r12
400000000000069c:        04 00 c4 00                        mov r33=b0
40000000000006a0:        0a 20 81 03 00 24      [MMI]       addl r36=96,r1;;
40000000000006a6:        40 02 90 30 20 00                  ld8 r36=[r36]
40000000000006ac:        04 08 00 84                        mov r32=r1
40000000000006b0:        1d 00 00 00 01 00      [MFB]       nop.m 0x0
40000000000006b6:        00 00 00 02 00 00                  nop.f 0x0
40000000000006bc:        b8 fd ff 58                        br.call.sptk.many b0=4000000000000460;;
40000000000006c0:        00 08 40 00 21         [MII]       mov r1=r32
40000000000006c6:        80 00 00 00 42 00                  mov r8=r0
40000000000006cc:        20 02 aa 00                        mov.i ar.pfs=r34
40000000000006d0:        00 00 00 00 01 00      [MII]       nop.m 0x0
40000000000006d6:        00 08 05 80 03 80                  mov b0=r33
40000000000006dc:        01 18 01 84                        mov r12=r35
40000000000006e0:        1d 00 00 00 01 00      [MFB]       nop.m 0x0
40000000000006e6:        00 00 00 02 00 80                  nop.f 0x0
40000000000006ec:        08 00 84 00                        br.ret.sptk.many b0;;
```

Interpretation: Numbers

- Base 10
  - Each digit represents a power of 10
  - $4173 = 4 \times 10^3 + 1 \times 10^2 + 7 \times 10^1 + 3 \times 10^0$
- Base 2
  - Each bit represents a power of 2
  - $10110 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 22$

Divide repeatedly by 2 and keep remainders

12/2 = 6	R = 0
6/2 = 3	R = 0
3/2 = 1	R = 1
1/2 = 0	R = 1

Result = 1100

Writing Bits is Tedious for People

- Octal (base 8)
  - Digits 0, 1, …, 7
  - In C: 00, 01, …, 07
- Hexadecimal (base 16)
  - Digits 0, 1, …, 9, A, B, C, D, E, F
  - In C: 0x0, 0x1, …, 0xf

0000 = 0	**1000 = 8**
0001 = 1	**1001 = 9**
0010 = 2	**1010 = A**
0011 = 3	**1011 = B**
0100 = 4	**1100 = C**
0101 = 5	**1101 = D**
0110 = 6	**1110 = E**
0111 = 7	**1111 = F**

Thus the 16-bit binary number

1011 0010 1010 1001

converted to hex is

B2A9

Interpretation: Colors

- Three primary colors
  - Red
  - Green
  - Blue
- Strength
  - 8-bit number for each color (e.g., two hex digits)
  - So, 24 bits to specify a color
- In HTML, on the course Web page
  - Red: <i>Symbol Table Assignment Due</i>
  - Blue: <i>Fall Recess</i>
- Same thing in digital cameras
  - Each pixel is a mixture of red, green, and blue

Binary Representation of Integers

- Fixed number of bits in memory
  - char: 8 bits
  - short: usually 16 bits
  - int: 16 or 32 bits
  - long: 32 bits
  - long long: 64 bits
- Unsigned integers
  - Always positive or 0
  - All arithmetic is modulo 2n
  - unsigned char
  - unsigned short
  - unsigned int
  - unsigned long
  - unsigned long long

Binary	Decimal
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
…	…
1{n}	2^n-1

Size and Overflow in Unsigned Integers

Bits	Integer Range
8	0 - 255
16	0 - 65,535
32	0 - 4,294,967,295
64	0 - 18,446,744,073,709,551,615

Binary	Decimal
0	0
1	1
10	2
…	…
1{n}	2^n-1

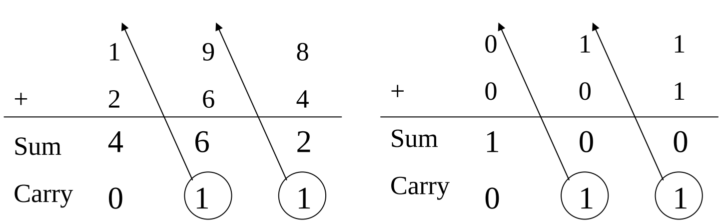
Number of bits determines unsigned integer range

Overflow:

- 8-bit integer → 11111111_2 (255_{10})
- Add 1
- What happens?

Adding Two Integers: Base 10

- From right to left, we add each pair of digits
- We write the sum, and add the carry to the next column



Binary Sums and Carries

a	b	Sum
0	0	0
0	1	1
1	0	1
1	1	0

XOR

a	b	Carry
0	0	0
0	1	0
1	0	0
1	1	1

AND

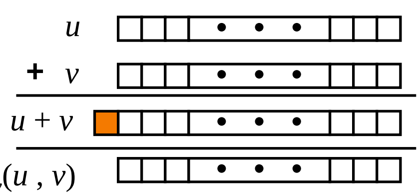
```
  0100 0101   <--  69
+ 0110 0111   <--  103
-----------
  1010 1100   <--  172
```


Overflow in Unsigned Addition

Operands: *w* bits

True Sum: *w* + 1 bits

Discard Carry: *w* bits

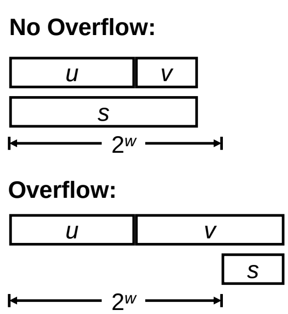


$$UAdd_w(u,v) = \begin{cases} u+v & u+v<2^w \\ u+v-2^w & u+v\geq 2^w \end{cases}$$

Modulo Arithmetic: $\text{UAdd}_w(u, v) = u + v \bmod 2^w$

Detecting Unsigned Overflow

- Task:
  - Given s = $\text{UAdd}_w(u, v)$
  - Determine if s = u + v
- Claim:
  - Overflow iff s < u
  - ovf = (s < u)
  - By symmetry iff s < v
- Proof:
  - $0 \leq v < 2^w$
  - No overflow $\Rightarrow$ $s = u + v \geq u + 0 = u$
  - Overflow $\Rightarrow$ $s = u + v - 2^w < u + 0 = u$



Modulo Arithmetic

- Consider only numbers in a range
  - E.g., five-digit car odometer: 0, 1, …, 99999
  - E.g., eight-bit numbers 0, 1, …, 255
- Roll-over when you run out of space
  - E.g., car odometer goes from 99999 to 0, 1, …
  - E.g., eight-bit number goes from 255 to 0, 1, …
- Adding 2^n doesn't change the answer
  - For eight-bit number, n=8 and 2^n=256
  - E.g., (37 + 256) mod 256 is simply 37
- This can help us do subtraction…
  - Suppose you want to compute a – b
  - Note that this equals a + (256 -1 - b) + 1

Modulo Arithmetic

Modulo Addition Forms an Abelian Group

- Closed under addition
  - $0 \leq UAdd_w(u, v) \leq 2^w - 1$
- Commutative
  - $UAdd_w(u, v) = UAdd_w(v, u)$
- Associative
  - $UAdd_w(t, UAdd_w(u, v)) = UAdd_w(UAdd_w(t, u), v)$
- 0 is additive identity
  - $UAdd_w(u, 0) = u$
- Every element has additive inverse
  - Let $UComp_w(u) = 2^w - u$
  - $UAdd_w(u, UComp_w(u)) = 0$

(negatives…)

What about Negative Numbers?

Bits	Patterns
8	256
16	65,536
32	4,294,967,296
64	18,446,744,073,709,551,616

Binary	Pattern
0	1
1	2
10	3
…	…
1{n}	2^n

- We have been looking at unsigned numbers
- What about negative or signed numbers?
- Need new interpretation of bits
- Some patterns interpreted as negative numbers

Key Standard Pattern Assignments

Bit Pattern	Sign Magnitude	One's Complement	Two's Complement
000	+0	+0	0
001	+1	+1	+1
010	+2	+2	+2
011	+3	+3	+3
100	-0	-3	-4
101	-1	-2	-3
110	-2	-1	-2
111	-3	-0	-1

- Which one is best?
  - Balance
  - Zeros
  - Ease of operations

Most Common: Two's Complement

Bit Pattern	Two's Complement
000	0
001	+1
010	+2
011	+3
100	-4
101	-3
110	-2
111	-1

- "Invert and Add 1" to negate
- Sign Bit
- Zeros, Range
- What about arithmetic?

Unsigned and Two's Complement

- Unsigned Values

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

- Two's Complement

Sign Bit

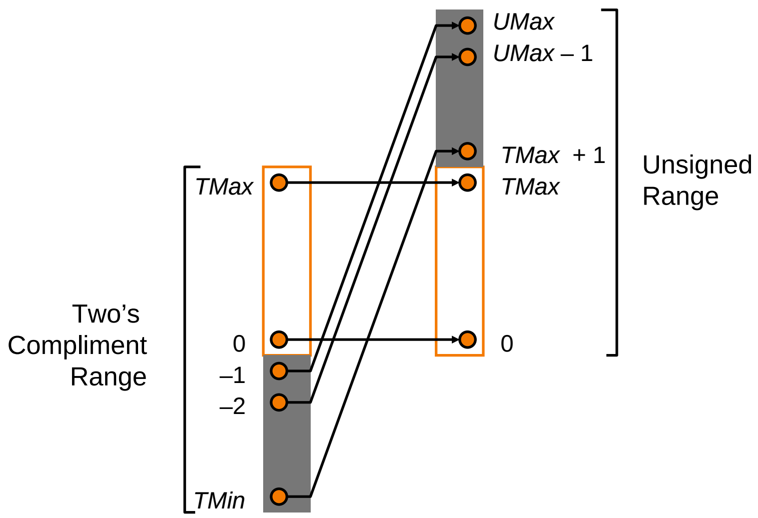
$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

- **UMin** = 0
- **UMax** = $2^w - 1$
- **TMin** = -2^{w-1}
- **TMax** = $2^{w-1} - 1$

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Representation Relationship



Sizes and C Data Types

C Data Type	MIPS, x86	Alpha
char	8 bits	8 bits
short	16 bits	16 bits
int	32 bits	32 bits
long int	32 bits	64 bits

`char, short, int, long int`

- Refer to number of bits of integer
- Most machines: signed two's complement

`unsigned <type>`

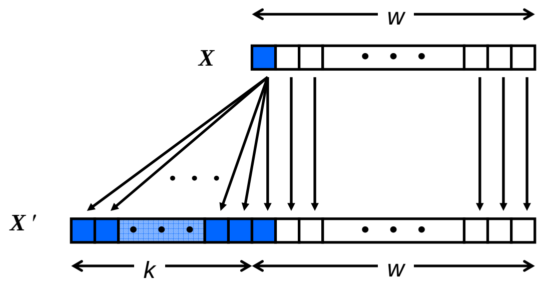
- Same number of bits as signed counterparts
- Unsigned integer

Sign Extension

```
char minusFour = -4;
short moreBits;
moreBits = (short) minusFour;
```

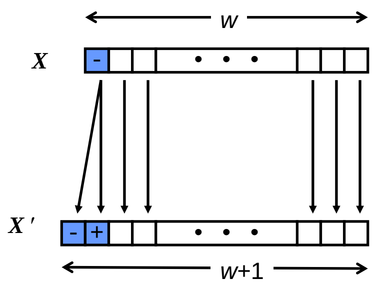
Given w bit signed integer, return equivalent w+k bit signed integer

Sign Extend:



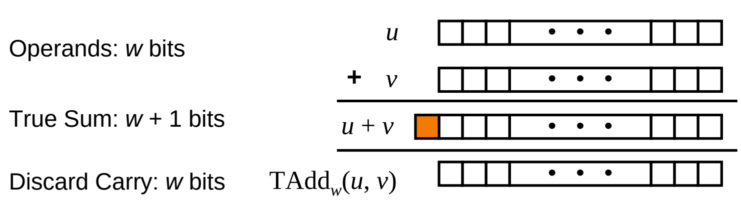
Sign Extension
Proof of Correctness Outline

- Prove Correctness by Induction on k
- Induction Step: extending by single bit maintains value



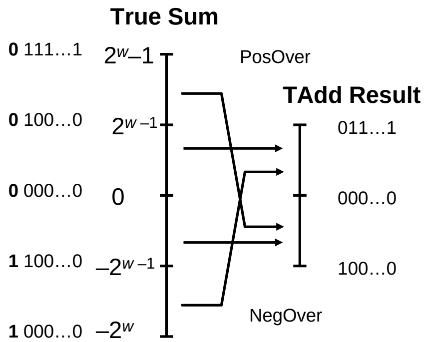
Two's Complement Addition

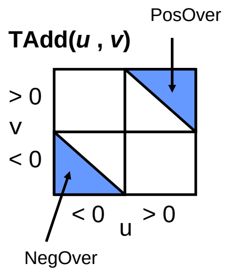
- TAdd and UAdd have identical Bit-Level Behavior!



Characterizing TAdd

- True sum requires w+1 bits
- Drop MSB

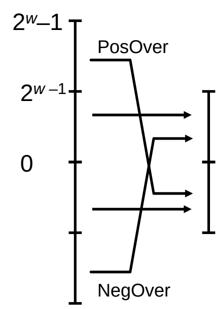




$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \text{ (NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \text{ (PosOver)} \end{cases}$$

Detecting Two's Complement Overflow

- Task:
  - Given s = $TAdd_w$(u, v)
  - Determine if s = Add_w(u, v)
- Claim:
  - Overflow iff either:
    - u, v < 0, s ≥ 0 (NegOver)
    - u, v ≥ 0, s < 0 (PosOver)
  - ovf = (u<0 == v<0) && (u<0 != s<0);



- Proof:
  - Obviously, if u ≥ 0 and v < 0, then $TMin_w$ ≤ u + v ≤ $TMax_w$
  - Symmetrically if u < 0 and v ≥ 0
  - Other cases from analysis of TAdd

Negation vs. Inversion

Inversion:

- A bit-wise operation
- Flip all 0's to 1's and vice versa: 0011 => 1100
- What does this do to the two's complement value?

Negation:

- Two's complement: invert all bits and add 1
- Example:

  3_{10} = 0011

  invert(0011) + 1 → 1100 + 1 → 1101

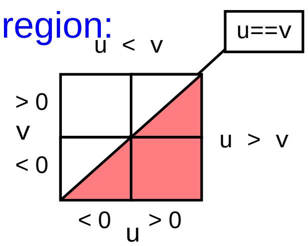
  1101 = -3_{10}

Two's Complement Negation

- Mostly like Integer Negation
  - TComp(u) = –u
- TMin is Special Case
  - TComp(TMin) = TMin
  - Note Also: TComp(0) = 0
- Negation in C (`x = -x;`) is Actually TComp

Comparing Two's Complements

- Given signed numbers u, v
- Determine whether or not u > v
- Return true for shaded region:



- Bad Approach:
  - Test (u – v) > 0
  - Problem: Thrown off by Overflow

Representation: A Collection of Bits

- Treat unsigned int as a collection 32 independent bits
- Good for tracking 32 individual binary conditions
  - True/False
  - Yes/No
  - Black/White
- Can also treat unsigned in as:
  - 16 2-bit values
  - 8 4-bit values
  - 4 8-bit values
  - 8 1-bit value, 4 2-bit values, 2 4-bit values, and 1 8-bit value

Bitwise Operators: AND and OR

- Bitwise AND (&)

&	0	1
0	0	0
1	0	1

  - Mod on the cheap!
    - E.g., h = 53 & 15;

53	0	0	1	1	0	1	0	1
& 15	0	0	0	0	1	1	1	1
5	0	0	0	0	0	1	0	1

- Bitwise OR (|)

|	0	1
0	0	1
1	1	1

Bitwise Operators: Not and XOR

- One's complement (~)
  - Turns 0 to 1, and 1 to 0
  - E.g., set last three bits to 0
    - x = x & ~7;
- XOR (^)
  - 0 if both bits are the same
  - 1 if the two bits are different

^	0	1
0	0	1
1	1	0

Bitwise Operators: Shift Left/Right

- Shift left (<<): Multiply by powers of 2
  - Shift some # of bits to the left, filling the blanks with 0

53	0	0	1	1	0	1	0	0
53<<2	1	1	0	1	0	0	0	0

- Shift right (>>): Divide by powers of 2
  - Shift some # of bits to the right
    - For unsigned integer, fill in blanks with 0
    - What about signed integers? Varies across machines…
      - Can vary from one machine to another!

53	0	0	1	1	0	1	0	0
53>>2	0	0	0	0	1	1	0	1

Count Number of 1s in an Integer

- Function bitcount(unsigned x)
  - Input: unsigned integer
  - Output: number of bits set to 1 in the binary representation of x
- Main idea
  - Isolate the last bit and see if it is equal to 1
  - Shift to the right by one bit, and repeat

```
int bitcount(unsigned int x) {
   int b;
   for (b = 0; x != 0; x >>= 1)
      if (x & 1)
         b++;
   return b;
}
```


XOR Encryption

- Program to encrypt text with a key
  - Input: original text in stdin
  - Output: encrypted text in stdout
- Use the same program to decrypt text with a key
  - Input: encrypted text in stdin
  - Output: original text in stdout
- Basic idea
  - Start with a key, some 8-bit number (e.g., 0110 0111)
  - Do an operation that can be inverted
    - E.g., XOR each character with the 8-bit number

```
  0100 0101          0010 0010
^ 0110 0111        ^ 0110 0111
-----------        -----------
  0010 0010          0100 0101
```


XOR Encryption, Continued

- But, we have a problem
  - Some characters are control characters
  - These characters don't print
- So, let's play it safe
  - If the encrypted character would be a control character
  - … just print the original, unencrypted character
  - Note: the same thing will happen when decrypting, so we're okay
- C function **`iscntrl()`**
  - Returns true if the character is a control character

XOR Encryption, C Code

```
#define KEY '&'
int main(void) {
   int orig_char, new_char;

   while ((orig_char = getchar()) != EOF) {
      new_char = orig_char ^ KEY;
      if (iscntrl(new_char))
         putchar(orig_char);
      else
         putchar(new_char);
   }
   return 0;
}
```


Stupid Programmer Tricks

- Where do I use bitwise & most?
  - Bit vectors
- What's a bit vector?
  - Lots of booleans packed into an int/long
  - Often used to indicate some condition(s)
  - Less storage space than lots of fields
  - More explicit storage than compiled-defined bit fields
- Your compiler can do this?

```
typedef struct Blah {
  int b_onoff:1;
  int b_temperature:7;
  char b_someChar;
}
```


Example From Real Code

- #define DONTCACHE_REQNOSTORE 0x000001
- #define DONTCACHE_AUTHORIZED 0x000002
- #define DONTCACHE_MISSINGVARIANTHDR 0x000004
- #define DONTCACHE_USERORPASS 0x000008
- #define DONTCACHE_BYPASSFILTER 0x000010
- #define DONTCACHE_NONCACHEMETHOD 0x000020
- #define DONTCACHE_CTLPRIVATE 0x000040
- #define DONTCACHE_CTLNOSTORE 0x000080
- #define DONTCACHE_ISQUERY 0x000100
- #define DONTCACHE_EARLYEXPIRE 0x000200
- #define DONTCACHE_NOLASTMOD 0x000400
- #define DONTCACHE_NONEGCACHING 0x000800
- #define DONTCACHE_INSTANTEXPIRE 0x001000
- #define DONTCACHE_FILETOOBIG 0x002000
- #define DONTCACHE_FILEGREWTOOBIG 0x004000
- #define DONTCACHE_ICPPROXYONLY 0x008000
- #define DONTCACHE_LARGEFILEBLAST 0x010000
- #define DONTCACHE_PERSISTLOGLOADING 0x020000
- #define DONTCACHE_NEWERCOPYEXISTS 0x040000
- #define DONTCACHE_BADVARYFIELDS 0x080000
- #define DONTCACHE_SETCOOKIE 0x100000
- #define DONTCACHE_HTTPSTATUSCODE 0x200000
- #define DONTCACHE_OBJECTINCOMPLETE 0x400000

Conclusions

- Computer represents everything in binary
  - Integers, floating-point numbers, characters, addresses, …
  - Pixels, sounds, colors, etc.
- Binary arithmetic through logic operations
  - Sum (XOR) and Carry (AND)
  - Two's complement for subtraction
- Binary operations in C
  - AND, OR, NOT, and XOR
  - Shift left and shift right
  - Useful for efficient and concise code, though sometimes cryptic