# COS 341: Discrete Mathematics

Homework #4 — Fall 2006
Counting — Due: Friday, October 20

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See instructions on the "assignments" web-page on how and when to turn in homework, and be sure to read the collaboration and late policy for this course. Approximate point values are given in parentheses. Be sure to show your work and justify your answers.

**1.** (10) Prove that at any party of $n$ people, there must be two people who have the same number of friends at the party. (Assume friendship is symmetric, meaning that if $X$ is a friend of $Y$, then $Y$ is a friend of $X$.)

**2.** (15) The purpose of this problem is to prove a generalization of the inclusion-exclusion principle. Let $A_1,\ldots,A_n$ be finite sets. Then for odd $r$:

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| \;\leq\; & \sum_{1 \leq i_1 \leq n} |A_{i_1}| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots \\ & + \sum_{1 \leq i_1 < \cdots < i_r \leq n} |A_{i_1} \cap \cdots \cap A_{i_r}|. \end{aligned} \tag{1}$$

And for even $r$:

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| \;\geq\; & \sum_{1 \leq i_1 \leq n} |A_{i_1}| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots \\ & - \sum_{1 \leq i_1 < \cdots < i_r \leq n} |A_{i_1} \cap \cdots \cap A_{i_r}|. \end{aligned} \tag{2}$$

a. Consider an element $a \in A_1 \cup \cdots \cup A_n$. Suppose that it belongs to exactly $m$ sets $A_i$. Prove that element $a$ is counted

$$\sum_{k=1}^{r} (-1)^{k+1} \binom{m}{k}$$

times on the right hand sides of the formulas above. (Note that, for any natural number $m$ and any integer $k$, $\binom{m}{k}$ is defined to be zero if $k < 0$ or $k > m$.)

b. As proved in the book and class, Pascal's identity states that, for positive integer $m$, and any integer $k$,

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}.$$

Use Pascal's identity to prove that

$$\sum_{k=0}^{r} (-1)^k \binom{m}{k} = (-1)^r \binom{m-1}{r}.$$

c. Prove formulas (1) and (2). Also prove the inclusion-exclusion principle itself (i.e., that either inequality becomes equality when $r = n$).

**3.** (32)

a. How many different solutions over the natural numbers are there to the following equation?

$$x_1 + x_2 + ... + x_8 = 100$$

A solution is a specification of the value of each variable $x_i$. Two solutions are different if different values are specified for some variable $x_i$.

b. In how many different ways can $2n$ students be paired up?

c. In how many different ways can one choose $n$ out of $2n$ objects, given that $n$ of the $2n$ objects are identical and the other $n$ are all unique?

d. The working days in the next year can be numbered $1, 2, \dots, 300$. Homer wants to avoid as many as possible.

- On even-numbered days, Homer will say he's sick.
- On remaining days that are a multiple of 3, he will say he's stuck in traffic.
- On remaining days that are a multiple of 5, he will refuse to come out from under the blankets.

In total, how many work days will he *avoid* in the coming year?

e. How many of the billion numbers in the range from 1 up to and including 1,000,000,000 contain the digit 1?

f. Consider the set of $n$-digit sequences of digits $0, 1, \dots, 9$. Two sequences are said to be of the same *type* if the digits of one are a permutation of the digits of the other. How many types of the $10^n$ $n$-digit sequences are there?

g. How many ways are there to order the 26 letters of the alphabet so that no two of the vowels $a$, $e$, $i$, $o$, $u$ appear consecutively and the last letter in the ordering is not a vowel?

h. How many ordered pairs $(A, B)$ of subsets of $\{1, 2, ..., n\}$ are there such that the intersection $A \cap B$ contains exactly one element?