Recall that the diameter of our body is poly($n$) since we started with a body whose diameter was $n^2$ and then we placed a grid of size $1/n^2$ or so. Combining all the above information, we get that

$$\phi \geq \frac{1}{\text{poly}(n)}.$$

Therefore, the mixing time is $\mathcal{O}(\frac{1}{\phi^2}\log N) = \mathcal{O}(\text{poly}(n))$.

# 4 Dimension Reduction

Now we describe a central result of high-dimensional geometry (at least when distances are measured in the $\ell_2$ norm). Problem: Given $n$ points $z^1, z^2, ..., z^n$ in $\Re^n$, we would like to find $n$ points $u^1, u^2, ..., u^n$ in $\Re^m$ where $m$ is of low dimension (compared to $n$) and the metric restricted to the points is almost preserved, namely:

$$\|z^i - z^j\|_2 \leq \|u^i - u^j\|_2 \leq (1+\epsilon)\|z^j - z^j\|_2 \ \forall i, j. \tag{2}$$

The following main result is by Johnson & Lindenstrauss :

Theorem 4
*In order to ensure (2), $m = \mathcal{O}(\frac{\log n}{\epsilon^2})$ suffices.*

Note: In class, we used the notation of $k$ vectors $z^1...z^k$ in $\Re^n$, but we can always embed the $k$ vectors in a $k$-dimensional space, so here I assume that $n = k$ and use only $n$.
Proof:
Choose $m$ vectors $x^1, ..., x^m \in \Re^n$ at random by choosing each coordinate randomly from $\{\sqrt{\frac{1+\epsilon}{m}}, -\sqrt{\frac{1+\epsilon}{m}}\}$. Then consider the mapping from $\Re^n$ to $\Re^m$ given by

$$z \longrightarrow (z \cdot x^1, z \cdot x^2, \ldots, z \cdot x^m).$$

In other words $u^i = (z^i \cdot x^1, z^i \cdot x^2, ..., z^i \cdot x^m)$ for $i = 1, \ldots, k$. We want to show that with positive probability, $u^1, ..., u^k$ has the desired properties. This would mean that there exists at least one choice of $u^1, ..., u^k$ satisfying inequality 2. To show this, first we write the expression $\|u^i - u^j\|$ explicitly:

$$\|u^i - u^j\|^2 = \sum_{k=1}^{m}\left(\sum_{l=1}^{n}(z_l^i - z_l^j)x_l^k\right)^2.$$

Denote by $z$ the vector $z^i - z^j$, and by $u$ the vector $u^i - u^j$. So we get:

$$\|u\|^2 = \|u^i - u^j\|^2 = \sum_{k=1}^{m}\left(\sum_{l=1}^{n} z_l x_l^k\right)^2.$$

Let $X_k$ be the random variable $(\sum_{l=1}^n z_l x_l^k)^2$. Its expectation is $\mu = \frac{1+\epsilon}{m}\|z\|^2$ (can be seen similarly to the proof of lemma 1). Therefore, the expectation of $\|u\|^2$ is $(1+\epsilon)\|z\|^2$. If we show that $\|u\|^2$ is concentrated enough around its mean, then it would prove the theorem. More formally, we state the following Chernoff bound lemma:

LEMMA 5
*There exist constants $c_1 > 0$ and $c_2 > 0$ such that:*

1. $Pr[\|u\|^2 > (1+\beta)\mu] < e^{-c_1\beta^2 m}$
2. $Pr[\|u\|^2 < (1-\beta)\mu] < e^{-c_2\beta^2 m}$

Therefore there is a constant $c$ such that the probability of a "bad" case is bounded by:

$$Pr[(\|u\|^2 > (1+\beta)\mu) \vee (\|u\|^2 < (1-\beta)\mu)] < e^{-c\beta^2 m}$$

Now, we have $\binom{n}{2}$ random variables of the type $\|u_i - u_j\|^2$. Choose $\beta = \frac{\epsilon}{2}$. Using the union bound, we get that the probability that any of these random variables is not within $(1 \pm \frac{\epsilon}{2})$ of their expected value is bounded by

$$\binom{n}{2} e^{-c\frac{\epsilon^2}{4}m}.$$

So if we choose $m > \frac{8(\log n + \log c)}{\epsilon^2}$, we get that with positive probability, all the variables are close to their expectation within factor $(1 \pm \frac{\epsilon}{2})$. This means that for all $i$,$j$:

$$(1-\frac{\epsilon}{2})(1+\epsilon)\|z^i - z^j\|^2 \le \|u^i - u^j\|^2 \le (1+\frac{\epsilon}{2})(1+\epsilon)\|z^i - z^j\|^2$$

Therefore,

$$\|z_i - z_j\|^2 \le \|u^i - u^j\|^2 \le (1+\epsilon)^2\|z^i - z^j\|^2,$$

and taking square root:

$$\|z^i - z^j\| \le \|u^i - u^j\| \le (1+\epsilon)\|z^i - z^j\|,$$

as required.

It remains to prove lemma 4. We prove the first part. Let $\alpha^2 = \frac{1+\epsilon}{m}$, so $\mu = \alpha^2 m\|z\|^2$ and we get the following equation:

$$\begin{aligned} P := Pr[\|u\|^2 > (1+\beta)\mu] &= Pr[\|u\|^2 > (1+\beta)\alpha^2 m\|z\|^2] \\ &= Pr[\|u\|^2 - (1+\beta)\alpha^2 m\|z\|^2 > 0] \\ &= Pr[t(\|u\|^2 - (1+\beta)\alpha^2 m\|z\|^2) > 0] \quad \forall t > 0 \\ &= Pr[\exp{(t(\|u\|^2 - (1+\beta)\alpha^2 m\|z\|^2))} > 1] \\ &\le E[\exp{(t(\|u\|^2 - (1+\beta)\alpha^2 m\|z\|^2))}] \quad \text{(Markov)} \end{aligned} \tag{3}$$

We calculate the last expectation:

$$\begin{aligned} P &\le E[\exp(t(\|u\|^2))]\exp(-t(1+\beta)\alpha^2 m\|z\|^2) \quad \text{(constant goes out)} \\ &= E[\exp(t(\sum_{k=1}^{m}(\sum_{l=1}^{n} z_l x_l^k)^2))]\exp(-t(1+\beta)\alpha^2 m\|z\|^2) \\ &= E[\exp(t(\sum_k(\sum_l z_l^2(x_l^j)^2)) + t(\sum_k(\sum_{l\neq h} z_l z_h x_l^k x_h^k)))]\exp(-t(1+\beta)\alpha^2 m\|z\|^2) \\ &= E[\exp(t\alpha^2 m\|z\|^2 + t(\sum_k(\sum_{l\neq h} z_l z_h x_l^k x_h^k)))]\exp(-t(1+\beta)\alpha^2 m\|z\|^2) \end{aligned} \tag{4}$$

The last step used the fact that $(x_l^k)^2 = \alpha^2$ and $\sum z_l^2 = \|z\|^2$. So continuing, we get:

$$P \leq E[\exp(t(\sum_k(\sum_{l\neq h} z_l z_h x_l^k x_h^k)))] \exp(-t\beta\alpha^2 m\|z\|^2) \tag{5}$$

The set of variables $\{x_l^k x_h^k\}_{l\neq h}$ are pairwise independent. Therefore the above expectation can be rewritten as a product of expectations:

$$P \leq \left(\prod_k \prod_{l\neq h} E[\exp(tz_l z_h x_l^k x_h^k)]\right) \exp(-t\beta\alpha^2 m\|z\|^2) \tag{6}$$

we notice that

$$E[\exp(tz_l z_h x_l^k x_h^k)] = \frac{1}{2}\exp(tz_l z_h \alpha^2) + \frac{1}{2}\exp(-tz_l z_h \alpha^2) < \exp(t^2 z_l^2 z_h^2 \alpha^4)$$

(the last inequality is easily obtained by Taylor expanding the exponent function). Plugging that in (6), we get:

$$\begin{aligned} P &< \left(\prod_k \prod_{l\neq h} \exp(t^2 z_l^2 z_h^2 \alpha^4)\right) \exp(-t\beta\alpha^2 m\|z\|^2) \\ &= \left(\prod_{l\neq h} \exp(t^2 z_l^2 z_h^2 \alpha^4)\right)^m \exp(-t\beta\alpha^2 m\|z\|^2) \\ &= \exp(mt^2 \sum_{l\neq h} z_l^2 z_h^2 \alpha^4 - t\beta\alpha^2 m\|z\|^2) \end{aligned} \tag{7}$$

Using simple analysis of quadratic function we see that the last expression obtains its minimum when

$$t = \frac{\beta\|z\|^2}{2\alpha^2 \sum_{l\neq h} z_l^2 z_h^2}.$$

Substituting for $t$, we get:

$$P < \exp(-\beta^2 m \frac{\|z\|^4}{4\sum_{l\neq h} z_l^2 z_h^2}) \tag{8}$$

Finally, the expression

$$\delta(z) = \left(\frac{\|z\|^4}{4\sum_{l\neq h} z_l^2 z_h^2}\right)$$

is bounded below by a constant $c_1$. To prove this, first note that $\delta(z) = \delta(\gamma z)$ for any $\gamma \neq 0$. So it is enough to consider the case $\|z\| = 1$. Then, using Lagrange multipliers technique, for example, we get that $\delta(z)$ obtains its minimum when $z_l = \frac{1}{\sqrt{n}}$ for each $l = 1..n$. Plugging this in the expression for $\delta(z)$ we see that it is bounded above by a constant $c_1$ that does not depend on $n$. This completes the proof. □