EXPLICIT UNIQUE-NEIGHBOR EXPANDERS

Extended Abstract

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Abstract

We present a simple, explicit construction of an infinite family $\mathcal{F}$ of bounded-degree 'unique-neighbor' expanders Γ; i.e., there are strictly positive constants α and ϵ, such that all $\Gamma = (X, E(\Gamma)) \in \mathcal{F}$ satisfy the following property. For each subset S of X with no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in $X \setminus S$ that are adjacent in Γ to exactly one vertex in S. The construction of $\mathcal{F}$ is simple to specify, and each $\Gamma \in \mathcal{F}$ is 6-regular. We then extend the technique and present easy to describe explicit infinite families of 4-regular and 3-regular unique-neighbor expanders, as well as explicit families of bipartite graphs with non equal color classes and similar properties. This has several applications and settles an open problem considered by various researchers.

1 Introduction

We call a graph $\Gamma = (X, E(\Gamma))$ an (α, ϵ)-*unique neighbor expander* if, for each subset S of X such that $|S| \leq \alpha|X|$, there are at least $\epsilon|S|$ vertices in $X \setminus S$ that are adjacent to exactly one vertex in S. The construction of an infinite family of (α, ϵ)-unique neighbor expanders for some positive α, ϵ has been an open question for a while (see [2], [11]), perhaps because the 'second eigenvalue' method, which has been the main tool used in guaranteeing the expansion properties of a graph (see [1], [12]), does not appear to be strong enough to prove that an infinite family of bounded-degree graphs are unique-neighbor expanders (see [7]). Here we answer this question by first presenting, for strictly positive constants α and ϵ, explicit infinite families $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ of 6-regular, 4-regular, and 3-regular unique-neighbor expanders. We further present a simple, explicit family $\mathcal{G}$ of bounded-degree bipartite graphs Γ with sides X and Y, where Y is a constant fraction smaller than X, and for every subset S of X with no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in Y adjacent to exactly one vertex in S. Although these graphs are not unique-neighbor expanders in the sense defined here (as for some subsets S of Y there are no vertices of X having a unique neighbor in S), we call them, with a slight abuse of notation, bipartite unique-neighbor expanders.

The unique-neighbor property of these graphs can be useful for the design of networks that support very simple distributed algorithms. Specifically, for any $\Gamma = (X, Y, E(\Gamma))$ in $\mathcal{G}$ and any subset S of X (where X is the larger side) that is no larger than $\alpha|X|$, there is a simple algorithm for finding a matching in Γ that saturates S in $O(\log|S|)$ time; at each step, each vertex in S not already matched pairs up with one of its neighbors that is unshared with any other vertex in S not already matched, if such a vertex exists. Thus, these graphs can be used to significantly simplify the routing networks and algorithms in [2] and [11].

The classes $\mathcal{F}, \mathcal{F}'$ and $\mathcal{F}''$ are also useful. For example, $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ support simple parallel algorithms for solving a load-balancing problem. More specifically, first, let Γ be an (α, ϵ) unique neighbor expander on n vertices. Next, let us suppose that there is a distribution of pebbles on the vertices of Γ, such that each vertex of Γ has somewhere between none and d pebbles, for some $d \in O(1)$. Further suppose that there are no more than αn pebbles total. Then for arbitrary such distributions, there exists an $O(d \log n/\epsilon)$-step parallel algorithm for redistributing the pebbles among the vertices of Γ such that each vertex of Γ has at most 1 pebble (at each step, each pebble may either move across an edge or stay, and at most one pebbel may

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move across an edge during a step). We defer the details until the full version.

Unique-neighbor expanders can be useful in additional scenarios, similar to those discussed in [3] and its references.

The construction of each Γ in both $\mathcal{F}$ and $\mathcal{F}'$ is a simple appropriately defined graph product of a 'small' graph C on 8 vertices with a 'large' graph $\Lambda \in \mathcal{H}$, where $\mathcal{H}$ is an explicit infinite family of 8-regular Ramanujan graphs constructed in [9] (see also [8]). The construction of each 3-regular $\Gamma \in \mathcal{F}''$ is a slightly different graph product of a cycle on 8 vertices with a 'large' graph $\Lambda \in \mathcal{H}''$, where $\mathcal{H}''$ is an explicit infinite family of 4-regular Ramanujan graphs, also constructed in [9] and [8]. Finally, the construction of each $\Gamma \in \mathcal{G}$ is a certain graph product of a small graph H with a large graph $\Lambda \in \mathcal{H}'$, where $\mathcal{H}'$ is an explicit infinite family of 44-regular Ramanujan graphs constructed in [9]. The graph H is a bipartite graph with 44 vertices on one side and 21 on the other, and satisfies certain properties.

To show that each $\Gamma \in \mathcal{F}$ is indeed a unique-neighbor expander, we first present properties of each $\Lambda \in \mathcal{H}$ that were established by Kahale [7], whose analysis uses the 'second eigenvalue' method, strengthening the basic approach of [1]. We later present and establish a property of an appropriate small graph C. Then we show that the combined properties of each $\Lambda \in \mathcal{H}$ and C guarantee that each $\Gamma \in \mathcal{F}$ is indeed a unique-neighbor expander. We use similar analysis to show that each $\Gamma \in \mathcal{G}$ is as claimed.

Our results are related to those in the recent paper [4], where the authors construct explicitly, for any absolute positive constant $c < 1$, infinite families of bounded degree bipartite graphs with classes of vertices A and B, with $|B| \leq (1-c)|A|$, in which for every set $S \subset A$ of size at most $\delta|A|$ there are many vertices in B with a unique neighbor in S. In fact, the construction in [4] provides such graphs in which every vertex of A has degree d, and every sufficiently small set $S \subset A$ has at least $(1-\epsilon)d|S|$ neighbors in B (and hence many of them necessarily have a unique neighbor in S). However, these constructions do not have the unique neighbor property, as there are small subsets of B so that no vertex in A has a unique neighbor in them, and it does not seem that the method can be extended to construct such graphs. Moreover, the constants involved in these constructions are large, and the description in [4] is based on some nontrivial properties of extractors. Our constructions here are much simpler, and have much lower degrees.

Our notation is the following. $\mathbf{Z}$ denotes the set of integers. For any positive integer ι, $\mathbf{N}_{\leq \iota}$ denotes the set of $\iota + 1$ nonnegative integers no larger than ι. For any subset V' of $V(G)$, $G[V']$ denotes the induced subgraph of G on V'. A $q+1$-regular graph Λ is *Ramanujan* if all but one of the eigenvalues of the adjacency matrix of Λ are no larger than $2\sqrt{q}$. For a group G and a subset Σ of G, where $\pi \in \Sigma$ if and only if $\pi^{-1} \in \Sigma$, define the *Cayley graph* Λ' on G *with respect to* Σ to be the $|\Sigma|$-regular graph whose vertex-set is G, and whose edge-set is $\{\{\nu, \pi\nu\} | \ \nu \in G \text{ and } \pi \in \Sigma\}$. (Call the elements $\pi \in \Sigma$ the *generators* of Λ', and Σ the set of generators of Λ'.) If Σ generates G, then Λ' is connected; otherwise, the components of Λ' are the right cosets of the subgroup of G generated by Σ.

Finally, for every ring R, $\mathrm{PGL}_2(R)$ denotes the quotient group $\mathrm{GL}_2(R)/Z$, where $\mathrm{GL}_2(R)$ is the group of 2×2 matrices σ with entries in R such that the determinant of σ is in $R^\times$, and Z is the subgroup of $\mathrm{GL}_2(R)$ of the form

$$Z = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} | \ \lambda \in R^\times \right\}, \tag{1}$$

where $R^\times$ is the group of elements of R with a multiplicative inverse.

The rest of this paper is organized as follows. In §2 we present and analyze $\mathcal{F}$ and $\mathcal{F}'$. In §3 we construct and analyze $\mathcal{F}''$. In §4 we construct and analyze the family of bipartite expanders $\mathcal{G}$. The final §5 contains some concluding remarks and open problems.

The proofs in Sections 2 and 3 are short, and are given here with essentially all details. Due to space limitations, we omit the somewhat lengthy proof of one of the lemmas in Section 4, and postpone it to the full version of the paper.

2 6 and 4-Regular Unique Neighbor Expanders

In this section we illustrate our basic techniques. For the rest of this section, let $\mathcal{H}$ be the infinite family of 8-regular Cayley Ramanujan graphs explicitly constructed in [9], (see also [8], [5]), and let C be the 3-regular, 8 vertex graph on $\mathbf{N}_{\leq 7}$, where ι and ι' are adjacent in C if and only if $|\iota - \iota'|$ equals either 1 or 7, or 4. We present the construction of $\mathcal{F}$ by constructing, for each $\Lambda = (V, E) \in \mathcal{H}$, a graph Γ on $|E|$ vertices, and then by proving Theorem 2.1 below.

Let $\Lambda = (V, E)$ be a graph in $\mathcal{H}$. For each vertex $\nu \in V$, let E_ν be the set $\{\gamma \in E | \ \gamma \text{ is incident to } \nu\}$, and let H_ν be any graph that has E_ν as its vertex-set, and is isomorphic to C. Finally, let Γ be the graph whose vertex-set is E, where γ and γ' are adjacent in Γ if and only if there exists a $\nu \in V$ where γ and γ' are both incident to ν, and γ and γ' are adjacent in H_ν.

Theorem 2.1 *Γ is an $(\alpha, 1/10)$-unique-neighbor 6-regular expander, where α is as in Corollary 2.3 below.*

The proof of Theorem 2.1 uses the theorem below, proved by Kahale [6].

Theorem 2.2 *Let $\bar{\Lambda} = (\bar{V}, \bar{E})$ be a k-regular Ramanujan graph. Then for all $\delta > 0$, and for any nonempty subset X of size at most $k^{-1/\delta}|\bar{V}|$, the average degree d of $\bar{\Lambda}[X]$ satisfies*

$$d \leq (1 + \sqrt{k-1})(1 + O(\delta)). \qquad (2)$$

□

This implies the following two corollaries, which we will use.

Corollary 2.3 *Let $\Lambda = (V, E)$ be a graph in $\mathcal{H}$. There exists a strictly positive α such that, for any subset S of E, where $|S| \leq \alpha|E|$, the average degree of $\Lambda[V_S]$ is at most 38/10, where V_S is the set of vertices $\nu \in V$ such that ν is incident to at least one edge in S.* □

Corollary 2.4 *Let $\Lambda = (V, E)$ be a graph in $\mathcal{H}$. Then there exists a positive α such that for any subset V' of V, where $|V'| \leq 2\alpha|V|$, the average degree of $\Lambda[V']$ is at most 38/10.* □

We also need the following observation about the graph C presented at the beginning of this section.

Lemma 2.5 *Let C be the 3-regular graph on $\mathbf{N}_{\leq 7}$ where ι and ι' are adjacent if and only if $|\iota' - \iota|$ is either 1,or 4, or 7. Then, for every nonempty subset J of $\mathbf{N}_{\leq 7}$, the quantity $|J| + |J'|$ is at least 4, where J' is the set of vertices ι in $\mathbf{N}_{\leq 7} \setminus J$ such that ι is adjacent in C to exactly one vertex in J.*

The assertion of this lemma is obvious for $|J| = 1$ and for $|J| \geq 4$. For sets J of cardinality 2 or 3 it can be easily checked. □

With Corollary 2.3 and Lemma 2.5 at hand we are now ready to prove Theorem 2.1.

Let S be a subset of E, $|S| \leq \alpha|E|$. Let V_S be the set of all endpoints of the members of S in the graph Λ. By Corollary 2.3 there are at least $|V_S|/5$ vertices $\nu \in V_S$ that have in Λ at most 3 neighbors in V_S. Let ν be such a vertex, and put $J = S \cap E_\nu$. Let J' be the set of all vertices of H_ν which are adjacent (in H_ν) to a unique vertex of J. Note that all elements in $J \cup J'$ are in fact edges of Λ, incident with ν. By Lemma 2.5 $|J|+|J'| \geq 4$, implying (since ν has at most 3 neighbors in Λ that are in V') that there is at least one member $\gamma \in J \cup J'$ whose other endpoint (besides ν) does not lie in V_S. Clearly $\gamma \notin J$ (since otherwise $\gamma \in S$ and then both its ends lie in V_S). It follows that γ is not in S, and has a unique neighbor in S. Thus, altogether, there are at least $|V_S|/5$ vertices of Γ with a unique neighbor in S. As the average degree of the induced subgraph of Λ on V_S is at most 4, $|V_S|/5 \geq |S|/10$, completing the proof of Theorem 2.1. □

We next present a family $\mathcal{F}'$ of 4-regular unique-neighbor expanders by constructing Γ, and by proving Theorem 2.6 below.

Let $\Lambda = (V, E)$ be a graph in $\mathcal{H}$ and let Σ be the set of 8 generators of Λ. Let H be any graph with vertex-set Σ that is isomorphic to C. Put $X = V \times \Sigma$. Finally, let Γ be the graph on X, where $x = (\nu, \pi)$ and $x' = (\nu', \pi')$ are adjacent in Γ if and only if either

(I) $\nu = \nu'$, and π and π' are adjacent in H, or

(II) $\pi\nu = \nu'$, and $\nu = \pi'\nu'$.

Theorem 2.6 *Γ is an $(\alpha/4, 1/40)$-unique-neighbor expander and is 4-regular, where α is as in Corollary 2.4.*

Proof: For each vertex $\nu \in \Lambda$, let X_ν denote the set $\{(\nu, \pi) \in X | \ \pi \in \Sigma\}$. Clearly:

(A′) each vertex $x = (\nu, \pi)$ in each X_ν is adjacent in Γ to exactly one vertex x' outside of X_ν, namely, $x' = (\pi\nu, \pi^{-1})$.

Let S be a subset of X where $|S| \leq \alpha|X|/4$. Let V_S be the set of all vertices $\nu \in V$ such that $X_\nu \cap S$ is nonempty. Then $|V_S|$ is no larger than $2\alpha|V|$. Thus, by Corollary 2.4 there are at least $|V_S|/5$ vertices $\nu \in V_S$ that have in Λ at most 3 neighbors in V_S. Let ν be such a vertex, let $J = X_\nu \cap S$, and let J' be the vertices in $X_\nu \setminus J$ that are adjacent in Γ to a unique vertex in J. Note that $\Gamma[X_\nu]$ is isomorphic to the graph C described in Lemma 2.5, so $|J| + |J'| \geq 4$. Thus, there is at least one vertex (ν, π) in $J \cup J'$ such that $\pi\nu \notin V_S$ (since ν has at most 3 neighbors in Λ that are in V_S) or equivalently, $X_{\pi\nu} \cap S$ is empty. Suppose then that (ν, π) is in J. Then, because $X_{\pi\nu} \cap S$ is empty, the vertex $(\pi\nu, \pi^{-1})$ is not in S, and (ν, π) is the unique neighbor of $(\pi\nu, \pi^{-1})$ in S, by **(A′)**. On the other hand, if (ν, π) is in J', then (ν, π) is not in S. Also, because $X_{\pi\nu} \cap S$ is empty, all neighbors of (ν, π) in Γ that are in S are in J. But there is only one such neighbor, by the definition of J'. Thus altogether there are at least $|V_S|/5$ vertices in $\Gamma \setminus S$ with a unique neighbor in S. However, $|V_S|/5 \geq |S|/40$, since there are at most 8 vertices in $X_\nu \cap S$ for each $\nu \in V_S$. This completes the proof. □

Actually, we can prove that there are at least $|S|/20$ vertices in $\Gamma \setminus S$ that have a unique neighbor in S. For each vertex $\nu \in V_S$, let J_ν be the set $X_\nu \cap S$, and let J'_ν be the set of vertices in $X_\nu \setminus J_\nu$ that are adjacent in Γ to exactly one vertex in J_ν. Then set $f_S(\nu) = |J_\nu \cup J'_\nu|$. Next, let $\delta_S(\nu)$ be the number of vertices in V_S that are adjacent in Λ to ν. By using the reasoning used in the proof of Theorem 2.6, we can show that the number M of vertices in $\Gamma \setminus S$ that have a unique neighbor in S satisfies

$$M \geq \sum_{\nu \in V_S} (f_S(\nu) - \delta_S(\nu)). \tag{3}$$

However, $f_S(\nu) \geq 4$ for each $\nu \in V_S$, but

$$\sum_{\nu \in V_S} \delta_S(\nu) \leq 38|V_S|/10 \leq \frac{3.8}{4} \sum_{\nu \in V_S} f_S(\nu).$$

Since $\sum_{\nu \in V_S} f_S(\nu) \geq |S|$ we conclude that $M \geq \frac{0.2}{4} \sum_{\nu \in V_S} f_S(\nu) \geq \frac{|S|}{20}$, as needed.

In fact, as each $\Lambda \in \mathcal{H}$ has a simple algebraic construction [5, 8], we can give an explicit, self-contained description of $\mathcal{F}'$ as follows. Let $l \geq 3$ be any integer. Set $V = \mathrm{PGL}_2(\mathbf{Z}/17^l\mathbf{Z})$. Then, fix an arbitrary square root ε of -1 in the ring $\mathbf{Z}/17^l\mathbf{Z}$, and define a 8-element subset Σ of V where Σ is the set

$$\left\{ \begin{pmatrix} 2+\varepsilon a_1 & a_3+\varepsilon a_3 \\ -a_2+\varepsilon a_3 & 2-\varepsilon a_1 \end{pmatrix} | a_1, a_2, a_3 \in \{1, -1\} \right\}. \tag{4}$$

Next, let $X = V \times \Sigma$, and let τ be an arbitrary bijection from Σ to $\mathbf{N}_{\leq 7}$. Finally, let Γ be the graph on X, where (ν, π) and (ν', π') are adjacent in Γ if and only if either
(I) $\nu = \nu'$, and $|\tau\pi - \tau\pi'|$ is either 1,7, or 4, or
(II) $\pi\nu = \nu'$, and $\pi'\nu = \nu$.

By the discussion above, Γ is an $(\alpha/4, 1/20)$-unique neighbor expander.

3 3-Regular Unique-Neighbor Expanders

In this section we construct an infinite family $\mathcal{F}''$ of 3-regular unique-neighbor expanders. To do so, we use a slightly more exotic graph product of a 4-regular Ramanujan graph with a cycle on 8 vertices.

For the rest of this section, let C'' be the cycle on $\mathbf{N}_{\leq 7}$ where ι and ι' are adjacent in C'' if $|\iota' - \iota|$ is either 1 or 7. Let $\mathcal{H}''$ be the infinite family of 4-regular Cayley Ramanujan graphs explicitly constructed in [8], [9]. Although this is not necessary, it will be convenient to assume that each element π in the generating set Σ of each of these graphs is of order 2, that is $\pi^{-1} = \pi$. We will see later that indeed we can choose such generating sets. We construct $\mathcal{F}''$ by constructing, for each $\Lambda = (V, E) \in \mathcal{H}''$, a graph Γ on $4|E| = 8|V|$ vertices, and then by proving Theorem 3.1 below.

Let $\Lambda = (V, E)$ be a graph in $\mathcal{H}''$, and let us write the set Σ of 4 generators of Λ as $\Sigma = \{\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4\}$. For each $\iota \in \mathbf{N}_{\leq 7}$, let π_ι be a specific element in Σ; specifically, set

$$\pi_1 = \pi_5 = \hat{\pi}_1; \quad \pi_2 = \pi_7 = \hat{\pi}_2;$$

and

$$\pi_3 = \pi_6 = \hat{\pi}_3; \quad \pi_0 = \pi_4 = \hat{\pi}_4.$$

Put $X = V \times \mathbf{N}_{\leq 7}$. Let Γ be the graph on X where (ν, ι) is adjacent to (ν', ι') in Γ if and only if
(I) $\nu = \nu'$, and $|\iota' - \iota|$ is either 1 or 7, or equivalently, ι' and ι are adjacent in C'', or
(II) $\iota = \iota'$, and $\pi_\iota\nu = \nu'$, or equivalently, $\nu = \pi_\iota\nu'$.

Theorem 3.1 Γ *is a 3-regular* $(\alpha''/8, 1/40)$*-unique-neighbor expander, where* α'' *is as in Corollary 3.2, stated below.*

The proof of Theorem 3.1 uses Corollary 3.2, which follows from Theorem 2.2.

Corollary 3.2 *Let* $\Lambda = (V, E)$ *be a graph in* $\mathcal{H}''$*. There exists a strictly positive* α'' *such that, for any subset* V' *of* V*, where* $|V'| \leq \alpha''|V|$*, the average degree of* $\Lambda[V']$ *is at most 14/5.* $\square$

We need another lemma, which is similar in spirit to Lemma 2.5. We first specify 4 subsets J_1, J_2, J_3, and J_4 of $\mathbf{N}_{\leq 7}$; set $J_1 = \{1, 5\}$; $J_2 = \{2, 7\}$; $J_3 = \{3, 6\}$, and $J_4 = \{0, 4\}$. Thus $\pi_\iota = \hat{\pi}_i$ if and only if ι is in J_i. For any subset J of $\mathbf{N}_{\leq 7}$, define the *rank* of J, written as $r(J)$, to be the number of J_i that J intersects. The following lemma can be proved by a simple case analysis.

Lemma 3.3 *Let* C'' *be the cycle on* $\mathbf{N}_{\leq 7}$ *where* ι *and* ι' *are adjacent in* C'' *if and only if* $|\iota' - \iota|$ *is either 1 or 7. Set* $J_1 = \{1, 5\}$*;* $J_2 = \{2, 7\}$*;* $J_3 = \{3, 6\}$*, and* $J_4 = \{0, 4\}$*, and for any subset* J *of* $\mathbf{N}_{\leq 7}$*, define* $r(J)$ *to be the number of* J_i *specified above that* J *intersects. Then for every nonempty subset* J *of* $\mathbf{N}_{\leq 7}$*,* $r(J \cup J')$ *is at least 3, where* J' *is the set of vertices* ι *in* $\mathbf{N}_{\leq 7} \setminus J$ *such that* ι *is adjacent in* C'' *to exactly one vertex in* J*.* $\square$

We now combine Corollary 3.2 and Lemma 3.3 to complete the proof of Theorem 3.1. For each $\nu \in V$, let X_ν denote the set $\{(\nu, \iota) | \ \iota \in \mathbf{N}_{\leq 7}\}$. Clearly:

(A$''$), For each vertex $\nu \in V$, each $x = (\nu, \iota) \in X_\nu$ is adjacent to exactly one vertex x' outside of X_ν; namely, $x' = (\pi_\iota \nu, \iota)$.

Let S be a subset of X; $|S| \leq \alpha''|V| = \alpha''|X|/8$, and let V_S denote the set $\{\nu \in V | X_\nu \cap S \neq \emptyset\}$. Obviously $|V_S| \leq \alpha''|V|$. By Corollary 3.2, there are at least $|V_S|/5$ vertices $\nu \in V_S$ such that ν is adjacent in Λ to at most 2 other vertices in V_S. Let ν be one such vertex. Let J be the set $\{\iota \in \mathbf{N}_{\leq 7} | \ (\nu, \iota) \in X_\nu \cap S\}$, and let J' be the set of ι in $N_{\leq 7}$ that have a unique neighbor in J in the graph C''. By Lemma 3.3, $r(J \cup J')$ is at least 3. Thus, there exists at least one $i \in \{1, 2, 3, 4\}$, such that $J \cup J'$ intersects J_i, where the J_i's are as defined in Lemma 3.3, but $\hat{\pi}_i \nu$ is not in V_S (because ν is adjacent in Λ to only two other vertices in V_S), or equivalently, $X_{\hat{\pi}_i \nu} \cap S$ is empty. Let i be such an integer, and let ι be an element in $(J \cup J') \cap J_i$. If ι is in J, then $(\pi_\iota \nu, \iota)$ $= (\hat{\pi}_i \nu, \iota)$ is a vertex not in S (because $X_{\hat{\pi}_i \nu} \cap S$ is empty) that has, by **(A$''$)**, (ν, ι) as its unique neighbor in S. On the other hand, if ι is in J', then (ν, ι) is a vertex not in S that has a unique neighbor in S. Thus, there must be at least $|V_S|/5$ vertices in $\Gamma \setminus S$ that have a unique neighbor in S. Since $X_\nu \cap S$ has at most 8 vertices of S, $|V_S|/5$ is at least $|S|/40$, and Theorem 3.1 follows. $\Box$

In fact, using the algebraic construction of $\mathcal{H}''$ presented in [5, 8], we can give the following explicit, self contained construction of $\mathcal{F}''$ as follows. Let l be any integer that is at least 3. Set $V = \mathrm{PGL}_2(\mathbf{Z}/17^l\mathbf{Z})$. Let ε be a square root of -1 in the ring $\mathbf{Z}/17^l\mathbf{Z}$, and set $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$, and $\hat{\pi}_4$ to be the following 4 matrices of V.

$$\hat{\pi}_1 = \begin{pmatrix} \varepsilon & 1+\varepsilon \\ -1+\varepsilon & -\varepsilon \end{pmatrix}; \tag{5}$$

$$\hat{\pi}_2 = \begin{pmatrix} \varepsilon & 1+\varepsilon \\ 1+\varepsilon & -\varepsilon \end{pmatrix}; \tag{6}$$

$$\hat{\pi}_3 = \begin{pmatrix} \varepsilon & -1-\varepsilon \\ -1-\varepsilon & -\varepsilon \end{pmatrix}; \tag{7}$$

and

$$\hat{\pi}_4 = \begin{pmatrix} \varepsilon & -1-\varepsilon \\ 1-\varepsilon & -\varepsilon \end{pmatrix}. \tag{8}$$

Note that $\pi = \pi^{-1}$ for each $\pi \in \{\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4\}$.

For each $\iota \in \mathbf{N}_{\leq 7}$, set π_ι to be one of $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$, and $\hat{\pi}_4$ as follows:

$$\pi_1 = \pi_5 = \hat{\pi}_1; \quad \pi_2 = \pi_7 = \hat{\pi}_2;$$

and

$$\pi_3 = \pi_6 = \hat{\pi}_3; \quad \pi_0 = \pi_4 = \hat{\pi}_4.$$

Put $X = V \times \mathbf{N}_{\leq 7}$. Finally, let Γ be the graph on X where (ν, ι) is adjacent to (ν', ι') in Γ if and only if

(I) $\nu = \nu'$, and $|\iota' - \iota|$ is either 1 or 7, or

(II) $\iota = \iota'$, and $\pi_\iota \nu = \nu'$.

Then Γ is a 3-regular $(\alpha''/8, 1/40)$-unique neighbor expander.

4 Bipartite Unique-Neighbor Expanders

In this section we construct, for some strictly positive α and ϵ, an infinite family $\mathcal{G}$ of bipartite graphs $\Gamma = (X, Y, E(\Gamma))$ where $|X| = 22|Y|/21$, that satisfies the following property: for every subset S of X with no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S. To do so, we use an appropriate graph product between a small graph H, and an infinite family $\mathcal{H}'$ of 44-regular Cayley Ramanujan graphs explicitly constructed in [9].

First, let H be the bipartite graph with a set W $= \{w_0, ..., w_{43}\}$ of 43 vertices on one side, and $T =$ $\{t_0, ..., t_{20}\}$ of 21 vertices on the other side, where $w_{\iota'}$ and t_ι are adjacent in H if and only if ι' and ι satisfy either (I) or (II) stated below.

(I) $\iota' \leq 24$, and either $\iota = 20$, or $\lfloor \frac{\iota'}{5} \rfloor \lfloor \frac{\iota}{5} \rfloor \equiv \iota - \iota'$ (mod 5).

(II) $\iota' \geq 25$, and $\iota' - 25 = \iota$.

We construct $\mathcal{G}$ by constructing for each Λ $=$ $(V, X) \in \mathcal{H}'$, a graph Γ with $|X|$ vertices on one side and $21|X|/22$ vertices on the other. Then we prove Theorem 4.1 stated below. First, let us write the 44 generators of Λ as $\pi_0, ..., \pi_{43}$. Next, let Γ be the bipartite graph where one side is the edge-set X of Λ, and the other side is $V \times N_{\leq 20}$, which we write as Y, and where $\gamma \in X$ and $\langle \nu, \iota \rangle \in Y$ are adjacent in Γ if and only if γ is incident to ν, and there exists an $\iota' \in N_{\leq 43}$ where $w_{\iota'}$ is adjacent to t_ι in H, and γ is incident to $\pi_{\iota'} \nu$.

Theorem 4.1 *The graph $\Gamma = (X, Y, E(\Gamma))$ satisfies the following properties.*

(i) Let α and ϵ are as in Corollary 4.2. Then, for each subset S of X of no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S.

(ii) $|X| = 22|Y|/21$.

(iii) Γ has maximum degree 25, and no more than $7|X|$ edges.

The proof of Theorem 4.1 uses Corollary 4.2 and Lemma 4.3. Corollary 4.2 follows from Theorem 2.2.

Corollary 4.2 *Let $\Lambda = (V, X)$ be a graph in $\mathcal{H}'$. Then there exists strictly positive constants α and ϵ such that, for any subset S of E such that $|S| \leq \alpha|X|$, there are at least $\epsilon|S|$ vertices in Λ that are incident to at least 1, but no more than 7, edges of S.* □

Lemma 4.3 *The graph H satisfies the following property*
(**)*: if U is any nonempty subset of W that has no more than 7 vertices, there is at least one vertex in T that is adjacent in H to exactly one vertex in U.*

The proof of Lemma 4.3 requires some detailed arguments and will be given in the full version of the paper.

We now use Corollary 4.2 and Lemma 4.3 to prove Theorem 4.1. Let ν be an arbitrary vertex in V. Let X_ν denote the set of 44 vertices $\gamma \in X$ such that γ is incident to ν. Let Y_ν denote the set of 21 vertices $\langle \nu, \iota \rangle \in Y$, where $\iota \in \{0, 1, ..., 20\}$. We note that
(A) for each vertex ν in Λ every vertex in Y_ν is adjacent only to vertices in X_ν.
Furthermore,
(B) $\Gamma[X_\nu \cup Y_\nu]$ is isomorphic to H, where X_ν maps to W, and Y_ν maps to T.

Let S be any subset of X, $|S| \leq \alpha|X|$. By Corollary 4.2, there is a set Q of at least $\epsilon|S|$ vertices $\nu \in \Lambda$ such that $1 \leq |S \cap X_\nu| \leq 7$. By **(B)** and Lemma 4.3, for each $\nu \in Q$, there is at least one vertex $v \in Y_\nu$ that is adjacent in Γ to exactly one vertex in $S \cap X_\nu$. Thus, from **(A)**, v is adjacent in Γ to exactly one vertex in S. Therefore, there must be at least $|Q| \geq \epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S because the sets Y_ν are pairwise disjoint. This completes the proof of Theorem 4.1, part (i). The assertions of parts (ii) and (iii) follow easily from the construction.

5 Concluding Remarks

We have described several explicit families of low degree unique-neighbor expanders. In fact, by taking the double covers of the graphs in §3 and by omitting an appropriate sparse set of vertices in one of the sides, we can get explicitly infinite families of bipartite graphs with maximum degree 3, that have a property similar to the graphs constructed in §4 : there are some absolute positive constants $\delta_1, \delta_2, \delta_3$ such that for every graph in the family with classes of vertices A and B, $|B| \leq (1 - \delta_1)|A|$ and every set $S \subset A$ of size at most $\delta_2|A|$ has at least $\delta_3|S|$ vertices in B with a unique neighbor in S.

A related problem to the one considered here and in [4], is that of constructing, for every $\epsilon > 0$ an explicit infinite family of d-regular graphs $G = (V, E)$, where each set $S \subset V$ of size at most $\delta|V|$ has at least $(1 - \epsilon)d|S|$ neighbors, where here $d = d(\epsilon)$ and $\delta = \delta(\epsilon)$. The techniques here and in [4] do not supply such families and the problem of finding such an explicit construction remains open.

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