COS 423: Theory of Algorithms

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Theory of Algorithms

Algorithm. (webster.com)

- A procedure for solving a mathematical problem (as of finding the greatest common divisor) in a finite number of steps that frequently involves repetition of an operation.
- Broadly: a step-by-step procedure for solving a problem or accomplishing some end especially by a computer.

"Great algorithms are the poetry of computation."

Etymology.

- "algos" = Greek word for pain.
- "algor" = Latin word for to be cold.
- Abu Ja'far al-Khwarizmi's = 9th century Arab scholar.
  - his book "Al-Jabr wa-al-Muqabilah" evolved into today's high school algebra text

Imagine: A World With No Algorithms

Fast arithmetic.

- Cryptography.

Quicksort.

- Databases.

FFT.

- Signal processing.

Huffman codes.

- Data compression.

Network flow.

- Routing Internet packets.

Linear programming.

- Planning, decision-making.

What is COS 423?

Introduction to design and analysis of computer algorithms.

- Algorithmic paradigms.
- Analyze running time of programs.
- Data structures.
- Understand fundamental algorithmic problems.
- Intrinsic computational limitations.
- Models of computation.
- Critical thinking.

Prerequisites.

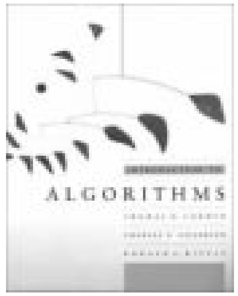
- COS 226 (array, linked list, search tree, graph, heap, quicksort).
- COS 341 (proof, induction, recurrence, probability).

Administrative Stuff

Lectures: (Kevin Wayne)
- Monday, Wednesday 10:00 - 10:50, COS 104.

TA's: (Edith Elkind, Sumeet Sobti)

Textbook: Introduction to Algorithms (CLR).



Grading:
- Weekly problem sets.
- Collaboration, no-collaboration.
- Class participation, staff discretion.
- Undergrad / grad.

Course web site: courseinfo.princeton.edu/courses/COS423_S2001/
- Fill out questionnaire.

Approximate Lecture Outline

Algorithmic paradigms.
- Divide-and-conquer.
- Greed.
- Dynamic programming.
- Reductions.

Analysis of algorithms.
- Amortized analysis.

Data structures.
- Union find.
- Search trees and extensions.

Graph algorithms.
- Shortest path, MST.
- Max flow, matching.

Approximate Lecture Outline

NP completeness.
- More reductions.
- Approximation algorithms.

Other models of computation.
- Parallel algorithms.
- Randomized algorithms.

Miscellaneous.
- Numerical algorithms.
- Linear programming.

College Admissions

Sample problem.
Algorithm.
Analysis.

References:

The Stable Marriage Problem by Dan Gusfield and Robert Irving, MIT Press, 1989.

Introduction to Algorithms by Jon Kleinberg and Éva Tardos.

College Admissions

Goal: Design a self-reinforcing college admissions process.

Given a set of preferences among colleges and applicants, can we assign applicants to colleges so that for every applicant X, and every college C that X is not attending, either:

- C prefers every one of its admitted students to X;
- X prefers her current situation to the situation in which she is attending college C.

If this holds, the outcome is STABLE.

- Individual self-interest prevents any applicant / college to undermine assignment by joint action.

Love, Marriage, and Lying

Standard disclaimer.

Stable Matching Problem

Problem: Given N men and N women, find a "suitable" matching between men and women.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.

Men's Preference List

Man	1st	2nd	3rd	4th	5th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

best ... worst

Stable Matching Problem

Problem: Given N men and N women, find a "suitable" matching between men and women.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference.

Women's Preference List

Woman	1st	2nd	3rd	4th	5th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

best ... worst

Stable Matching Problem

Problem: Given N men and N women, find a "suitable" matching between men and women.

- PERFECT MATCHING: everyone is matched monogamously.
  - each man gets exactly one woman
  - each woman gets exactly one man
- STABILITY: no incentive for some pair of participants to undermine assignment by joint action.
  - in matching M, an unmatched pair (m,w) is UNSTABLE if man m and woman w prefer each other to current partners
  - unstable pair could each improve by dumping spouses and eloping

STABLE MATCHING = perfect matching with no unstable pairs.
(Gale and Shapley, 1962)

Example

Men's Preference List

Man	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

Women's Preference List

Woman	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Lavender assignment is a perfect matching.
Are there any unstable pairs?

✐ Yes. Bertha and Xavier form an unstable pair.
They would prefer each other to current partners.

Example

Men's Preference List

Man	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

Women's Preference List

Woman	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Green assignment is a stable matching.

Example

Men's Preference List

Man	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

Women's Preference List

Woman	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Orange assignment is also a stable matching.

Stable Roommate Problem

Not obvious that any stable matching exists.

Consider related "stable roommate problem."

- 2N people.
- Each person ranks others from 1 to 2N-1.
- Assign roommate pairs so that no unstable pairs.

Preference List

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

All 3 perfect matchings have unstable pair.

E.g., A-C forms unstable pair in lavender matching.

No stable matching.

Propose-And-Reject Algorithm

Intuitive method that guarantees to find a stable matching.

Gale-Shapley Algorithm (men propose)

```
Initialize each person to be free.
while (some man m is free and hasn't proposed to every woman)
    w = first woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
```


Implementation and Running Time Analysis

Engagements.

- Maintain two arrays wife[m], and husband[w]; set equal to 0 if participant is free.
- Store list of free men on a stack (queue).

Preference lists.

- For each man, create a linked list of women, ordered from favorite to worst.
  - men propose to women at top of list, if rejected goto next
- For each woman, create a "ranking array" such that m^{th} entry in array is woman's ranking of man m.
  - allows for queries of the form: does woman w prefer m to m' ?

Resource consumption.

- Time = $\Theta(N^2)$.
- Space = $\Theta(N^2)$.
- Optimal.

$$\Phi = \left| \{ (m,w) \ : \ m \text{ has proposed to } w \} \right|$$

A Worst Case Instance

Number of proposals $\leq$ n(n-1) + 1.

- Algorithm terminates when last woman gets first proposal.

Number of proposals = n(n-1) + 1 for following family of instances.

Men's Preference List

Man	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

Women's Preference List

Man	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

Proof of Correctness

Observation 1. Men propose to their favorite women first.

Observation 2. Once a woman is matched, she never becomes unmatched. She only "trades up."

Fact 1. All men and women get matched (perfect).

- Suppose upon termination Zeus is not matched.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. (contradiction)

Proof of Correctness

Observation 1. Men propose to their favorite women first.

Observation 2. Once a woman is matched, she never becomes unmatched. She only "trades up."

Fact 2. No unstable pairs.

- Suppose (Amy, Zeus) is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

S*
Amy-Yancey
Bertha-Zeus

- Case 1. Zeus never proposed to Amy.
  - ⇒ Zeus must prefer Bertha to Amy (Observation 1)
  - ⇒ (Amy, Zeus) is stable. (contradiction)

- Case 2. Zeus proposed to Amy.
  - ⇒ Amy rejected Zeus (right away or later)
  - ⇒ Amy prefers Yancey to Zeus (women only trade up)
  - ⇒ (Amy, Zeus) is stable (contradiction)

Understanding the Solution

For a given problem instance, there may be several stable matchings.

- Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Fact 3. Yes. Gale-Shapley finds MAN-OPTIMAL stable matching!

- Man m is a valid partner of woman w if there exists some stable matching in which they are married.
- Man-optimal assignment: every man receives best valid partner.
  - simultaneously best for each and every man
  - there is no stable matching in which any single man individually does better

Proof of Fact 3

Proof.

- Suppose, for sake of contradiction, some man is paired with someone other than best partner.
  - since men propose in decreasing order of preference, some man is rejected by valid partner

S
Amy-Yancey
Bertha-Zeus

- Let Yancey be first such man, and let Amy be first valid partner that rejects him.
- When Yancey is rejected, Amy forms (or reaffirms) engagement with man, say Zeus, whom she prefers to Yancey.
- Let Bertha be Zeus' partner in S.
- Zeus not rejected by any valid partner at the point when Yancey is rejected by Amy (since Yancey is first to be rejected by valid partner). Thus, Zeus prefers Amy to Bertha.
- But Amy prefers Zeus to Yancey.
- Thus (Amy, Zeus) is unstable pair in S.

Understanding the Solution

Fact 4. **Gale-Shapley finds WOMAN-PESSIMAL matching.**

- Each woman married to worst valid partner.
  - simultaneously worst for each and every woman.
  - there is no stable matching in which any single woman individually does worse

Proof.

- Suppose (Amy, Zeus) matched in S*, but Zeus is not worst valid partner for Amy.
- There exists stable matching S in which Amy is paired with man, say Yancey, whom she likes less than Zeus.
- Let Bertha be Zeus' partner in S.
- Zeus prefers Amy to Bertha (man optimality).
- (Amy, Zeus) form unstable pair in S.

S
Amy-Yancey
Bertha-Zeus

Understanding the Solution

Fact 5. **The man-optimal stable matching is weakly Pareto optimal.**

- There is no other perfect matching (stable or unstable), where every man does strictly better.

Proof.

- Let Amy be last woman in some execution of Gale-Shapley (men propose) algorithm to receive a proposal.
- No man is rejected by Amy since algorithm terminates when last woman receives first proposal.
- No man matched to Amy will be strictly better off than in man-optimal stable matching.

Extensions: Unacceptable Partners

Yeah, but in real-world every woman is not willing to marry every man, and vice versa?

- Some participants declare others as "unacceptable."
  (prefer to be alone than with given partner)
- Algorithm extends to handle partial preference lists.

Matching S unstable if there exists man m and woman w such that:

- m is either unmatched in S, or strictly prefers w to his partner in S
- w is either unmatched in S, or strictly prefers m to her partner in S.

Fact 6. **Men and women are each partitioned into two sets:**

- those that have partners in all stable matchings;
- those that have partners in none.

Extensions: Sets of Unequal Size

Also, there may be an unequal number of men and women.

- E.g., |M| = 100 men, |W| = 90 women.
- Algorithm extends.
- WLOG, assume |W| < |M|.

Matching S unstable if there exists man m and woman w such that:

- m is either unmatched in S, or strictly prefers w to his partner in S;
- w is either unmatched in S, or strictly prefers m to her partner in S.

Fact 7. **All women are matched in every stable matching. Men are partitioned into two subsets:**

- men who are matched in every stable matching;
- men who are matched in none.

Extensions: Limited Polygamy

What about limited polygamy?

- E.g., Bill wants 3 women.
- Algorithm extends.

Matching S unstable if there exists man m and woman w such that:

- either w is unmatched, or w strictly prefers m to her partner;
- either m does not have all its "places" filled in the matching, or m strictly prefers w to at least one of its assigned residents.

Application: Matching Residents to Hospitals

Sets of unequal size, unacceptable partners, limited polygamy.

Matching S unstable if there exists hospital h and resident r such that:

- h and r are acceptable to each other;
- either r is unmatched, or r prefers h to her assigned hospital;
- either h does not have all its places filled in the matching, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

Matching medical school residents to hospitals. (NRMP)

- Hospitals ~ Men (limited polygamy allowed).
- Residents ~ Women.
- Original use just after WWII (predates computer usage).
- Ides of March, 13,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem:

Rural hospitals get exactly same residents in every stable matching!

Deceit: Machiavelli Meets Gale-Shapley

Is there any incentive for a participant to misrepresent his/her preferences?

- Assume you know men's propose-and-reject algorithm will be run.
- Assume that you know the preference lists of all other participants.

Fact 8. No, for any man yes, for some women!

Men's Preference List

Man	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

Women's True Preference List

Woman	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Amy Lies

Woman	1st	2nd	3rd
Amy	Y	Z	X
Bertha	X	Y	Z
Clare	X	Y	Z

Lessons Learned

Powerful ideas learned in COS 423.

- **Isolate underlying structure of problem.**
- **Create useful and efficient algorithms.**

- **Sometimes deep social ramifications.**
  - **Historically, men propose to women. Why not vice versa?**
  - **Men: propose early and often.**
  - **Men: be more honest.**
  - **Women: ask out the guys.**
  - **Theory can be socially enriching and fun!**
  - **CS majors get the best partners!!!**