UNIQUENESS OF EQUILIBRIUM IN SEALED HIGH-BID AUCTIONS**

by

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INTRODUCTION

Although much has been written on the theory of auctions, most of this work focuses exclusively on the symmetric equilibrium of an auction in which bidders are *ex ante* the same in the sense that the joint distribution of buyers' types is symmetric. In previous work (Maskin and Riley (2000a and 2000b), we have begun exploring the theory in the absence of symmetry. [1] Specifically we have examined (i) the existence of equilibrium in a sealed high-bid auction and (ii) the differences between the equilibrium in high-bid and second-price auctions when buyers are asymmetric *ex ante*.

Here we turn to the question of uniqueness. With a symmetric distribution of types, it is well known that there is only one *symmetric* equilibrium (Milgrom and Weber, 1982, Maskin and Riley, 1984). However, it is not implausible to conjecture that, even in an *ex ante* symmetric setting, a particular buyer might establish a reputation as an aggressive bidder if it is in his interest to do so. Riley (1980) provides an example of the "war of attrition" in which this is indeed the case. In fact, there is a continuum of asymmetric equilibria in which one buyer bids "aggressively" and the other "passively." Furthermore, the greater the degree of aggression, the larger is the equilibrium expected gain of the aggressive buyer.

A second example of a continuum of equilibria occurs in a pure common-values setting, if the item is sold by open ascending bid. As first noted by Milgrom (1981) there is always a continuum of equilibria in the two-buyer case. Bikchandani and Riley (1991) also present an example in which, with n bidders, there is a continuum of equilibria.

[1] There is also a literature on efficient auctions (see Maskin, 2003, for a survey) that eschews the symmetry assumption.

For the symmetric high-bid auction with private values, however, we show that there can be no asymmetric equilibrium under the assumption that reservation prices are drawn independently from a distribution with finite support[2] and positive mass at the lower endpoint.[3] That is, equilibrium is unique.

When we drop the symmetry assumption, uniqueness continues to obtain under same assumptions if there are only 2 buyers. For more than two buyers, we need the additional fairly mild assumptions that buyers with the same reservation price have the same preferences, that absolute risk aversion is nonincreasing, and that the supports of the different buyers' distributions of reservation prices have the same upper endpoint.

The argument that equilibrium is unique is basically an application of the fundamental theorem of ordinary differential equations (FTODE). As we will see, the major problems with applying this theorem are (i) ensuring that buyers' (inverse) bid functions are differentiable, so that they satisfy a system of differential equations; and (ii) establishing that there exists a unique "boundary condition" for that system.

We describe the model in section 1. In section 2 we present characterization results. We use these in section 3 to derive our main theorems. Concluding remarks are in section 4.

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[2] If the support of the distribution is unbounded, we conjecture that there will be a continuum of asymmetric equilibria.

[3] This latter assumption is weak because it is satisfied automatically if the seller sets a reserve price that is even marginally above the lowest possible buyer reservation price.

1. THE MODEL

Throughout we shall make the following assumptions about the auction and the buyers participating in it. A single item is to be sold to the buyer who makes the highest non-negative sealed bid. If two or more bids tie, the winner is selected at random from among the high bidders. There are n potential buyers. Buyer *i* of type s_i obtains utility 0 if he loses and utility $U_i(b,s_i)$ if he wins with a bid of b, where U_i is twice continuously differentiable. We assume that

$$\frac{\partial U_i}{\partial b} < 0 \text{ and } \frac{\partial U_i}{\partial s_i} > 0 \text{ for all } i.$$

Without loss of generality, we can interpret s_i as buyer *i*'s reservation price. Hence $U_i\left(s_i, s_i\right) = 0$. Buyer *i*'s reservation price is drawn independently from a distribution with support $[\underline{s}_i, \overline{s}_i]$, where $\overline{s}_i > 0$, and c.d.f. $F_i(\cdot)$. We assume that F_i is twice continuously differentiable, that its derivative is strictly positive on $[\underline{s}_i, \overline{s}_i]$, and that $F_i(\underline{s}_i) > 0$ (see footnote 3).

Clearly it is a dominated strategy for a buyer to bid more than his reservation price. Hence, we will rule this out by assumption.

Assumption 1: Bidder *i* never bids more than his reservation price s_i in equilibrium.

If a buyer *i* has a negative reservation price, then it is a dominated strategy for him to bid at all, and so without loss of generality we can assume that $\underline{s}_i \geq 0$.

Let Π_i be the probability that bidder *i* wins. Then his expected utility is

$$E_i = \Pi_i U_i(b, s_i).$$

We shall assume throughout that the higher is a bidder's reservation price, the "flatter" are his indifference curves in bid-probability space. That is, the single-crossing property holds[4]. Given our assumptions, bidder *i*'s indifference curve, are as depicted in Figure 1.1. Specifically, at an indifference curve,

$$\left.\frac{db}{d\Pi}\right|_{E_i = const} = -\frac{\dfrac{\partial E_i}{\partial \Pi}}{\dfrac{\partial E_i}{\partial b}} = -\frac{1}{\Pi_i}\frac{U_i}{\dfrac{\partial U_i}{\partial b}}.$$

Thus, for single-crossing, we require the following assumption.

Assumption 2: **Single-crossing**

$$-\frac{\partial U_i}{\partial b}\Big/ U_i(b, s_i) \text{ is a decreasing function of } s_i.$$

Note that if U_i takes the form $U_i\left(b, s_i\right) = V_i\left(s_i - b\right)$, then Assumption 2 is satisfied provided that bidder *i* is risk-neutral or risk-averse, i.e., $V_i'' \leq 0$.

[4] In technical terms, this is the assumption that utility is log supermodular.

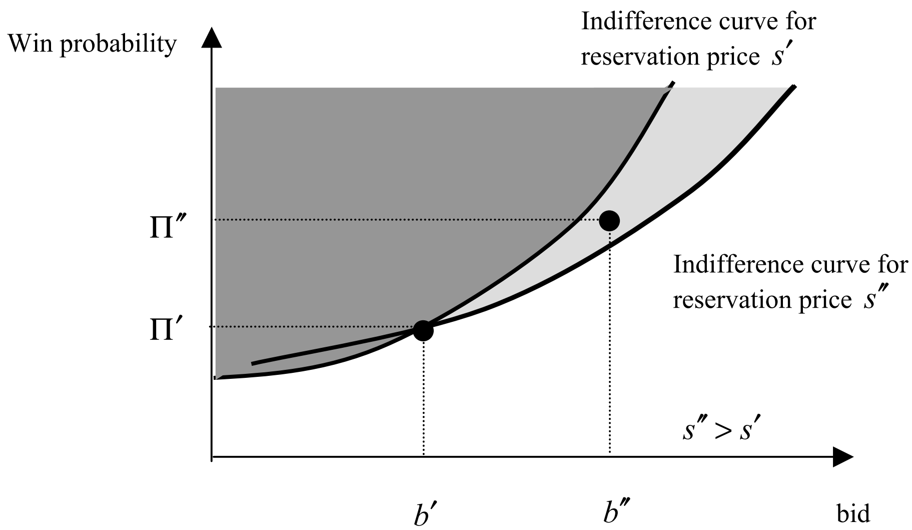

Figure 1.1: Single-crossing property

As we shall see in section 3, it will be helpful to define the "log cost" of having to bid to win the item, rather than getting it *gratis*:

$$c_i(b, s_i) \equiv \log U_i(0, s_i) - \log U_i(b, s_i)\,. \tag{1.1}$$

Then,

$$\frac{\partial c_i}{\partial b} = -\frac{\partial U_i}{\partial b} \Big/ U_i(b, s_i)\,, \tag{1.2}$$

and so Assumption 2 is equivalent to the assumption that the marginal log cost is lower for higher reservation prices. Given this assumption, buyer i's bidding behavior will be monotonic in s_i (see Lemma 2 below).

Since it will be useful below, we note that

$$\frac{\partial^2 c_i}{\partial b^2} = (A_i(b,s_i) + \frac{\partial c_i}{\partial b})\frac{\partial c_i}{\partial b} \ , \qquad (1.3)$$

where $A_i(b,s_i) = \frac{\partial^2 U_i}{\partial b^2} / \frac{\partial U_i}{\partial b}$ is buyer *i*'s coefficient of absolute risk aversion. Note that as long as a buyer is risk neutral or risk averse (and hence $A_i(b,s_i) \geq 0$), $c_i(b,s_i)$ is strictly convex for all $b \in [0, s_i)$.

2. CHARACTERIZING THE EQUILIBRIUM BID FUNCTIONS

From Maskin and Riley (2000a and b) we have the following two results:

Lemma 1: If Assumptions 1 and 2 hold, the distribution of winning bids in equilibrium has a support consisting of an interval $[b_*, b^*]$ and a c.d.f. $G_w(b)$ which is continuous on $(b_*, b^*]$ (see Proposition 3 of Maskin and Riley (2000b)).

Lemma 2: **Monotonicity**

If Assumptions 1 and 2 hold, then if $b_i(s_i)$ is a best response by buyer *i* with reservation price s_i to the other buyers' bidding strategies, it is non-decreasing in s_i (see Proposition 1 of Maskin and Riley (2000a)).

To understand Lemma 2 geometrically, consider Figure 1.1. If (b', Π') is optimal for a buyer with reservation price s', there can be no feasible alternatives in the heavily shaded region. Thus, any alternatives preferred to (b', Π') by the higher reservation price s'' must lie in the lightly shaded region, i.e., they must entail higher bids.

As our first preliminary result, we characterize b_*, the lower endpoint of the support of the distribution of winning bids.

Lemma 3: **Characterization of the minimum winning bid**

Without loss of generality, suppose that $\underline{s}_n \leq ... \leq \underline{s}_1$. If Assumption 1 and 2 hold, then the minimum bid satisfies

$$\underline{s}_2 \leq b_* \leq \underline{s}_1. \tag{2.1}$$

Furthermore, if $\underline{s}_2 < \underline{s}_1$, then

$$b_* = \max \arg \max_{b} \underset{i \neq 1}{\times} F_i(b) U_1(b, \underline{s}_1) \tag{2.2}$$

<u>Proof</u>: Suppose first that $b_* > \underline{s}_1$. Consider a buyer with reservation price $\hat{s} \in (\tfrac{1}{2}\underline{s}_1 + \tfrac{1}{2}b_*, b_*)$. Because $\hat{s} < b_*$, the lowest winning bid, the buyer has an equilibrium expected payoff of zero. But there is a positive probability that all other buyers have reservation prices less than $\tfrac{1}{2}\underline{s}_1 + \tfrac{1}{2}b_*$. Thus, from Assumption 1, our buyer has a strictly positive payoff if he bids $\tfrac{1}{2}\underline{s}_1 + \tfrac{1}{2}b_*$, a contradiction. We conclude that $b_* \leq s_1$.

Suppose next that $b_* < \underline{s}_2$. From Lemma 1, there are no mass points on $\left(b_*, b^*\right]$. Thus, buyers 1 and 2, regardless of their reservation prices, have strictly positive expected payoffs from bidding just above b_*. This means that if $I = \left\{ i \,\middle|\, \text{buyer } i \text{ bids } b_* \text{ or more with probability } 1 \right\}$, then $1,2 \in I$. For all $i \in I$, let p_i be the probability that buyer i bids b_*. If, for all $i \in I, p_i > 0$, then bidding b_* results in a tie with positive probability. Thus, buyer 1 is strictly better off bidding slightly above b_*, since this increases his probability of winning discontinuously. Hence, for some $i \in I, p_i = 0$. If $i \neq 1$, then buyer 1's probability of winning, and hence his expected utility, is approximately zero for bids near b_*. But we have already argued that buyer 1's equilibrium expected utility is strictly positive, a contradiction. Hence, $p_1 = 0$. But now the same contradiction pertains to buyer 2. We conclude that (2.1) holds.

Suppose that $\underline{s}_2 < \underline{s}_1$. From Assumption 1, if buyer 1 with reservation price $\underline{s}_1$ bids $b \neq b_*$, his expected payoff is at least $\underset{i \neq 1}{\times} F_i(b) U_1(b, \underline{s}_1)$. It follows that for b_* to be an equilibrium bid for him,

$$\underset{i \neq 1}{\times} F_i(b) U_1(b, \underline{s}_1) \leq \underset{i \neq 1}{\times} F_i(b_*) U_1(b_*, \underline{s}_1) \text{ for all } b.$$

Hence,

$$b_* \in \text{ arg max} \underset{i \neq 1}{\times} F_1(b) U_1(b, \underline{s}_1).$$

Finally, suppose that both b' and b'' solve this maximization problem and that $b' < b''$. Buyer 1 with reservation price $\underline{s}_1$ weakly prefers b'' to any lower bid. Given Assumption 2, all other buyer 1 types strictly prefer b'' to any lower bid. Thus the minimum bid for all reservation prices $s_1 > \underline{s}_1$ is at least b''. But then b' is not the lower endpoint of the support of the equilibrium distribution of winning bids. We conclude that (2.2) holds.

Q.E.D.

Lemma 4: **Strict monotonicity of the probability of winning**:

Suppose that $b' < b''$ and that b' and b'' are in the support of the distribution of winning bids in equilibrium. Then at least two buyers bid in the interval (b', b'') with positive probability.

<u>Proof</u>: From Lemma 1, the support of $G_w(b)$ is connected, and so all the bids in the interval (b', b'') are also in the support. This implies that at least one buyer bids in (b', b'') with positive probability. Suppose, contradicting the Lemma, that buyer *i* is the only one to do so. Specifically, assume that for reservation price s_i buyer *i* bids $\hat{b} \in (b', b'')$ in equilibrium. But buyer *i* can reduce his bid to $\hat{b} - \varepsilon \in (b', b'')$ without diminishing his probability of winning, a contradiction.

Q.E.D.

Let $(\tilde{b}_1(s_1), ..., \tilde{b}_n(s_n))$ be equilibrium bidding strategies (possibly mixed strategies). Because $G_w(b)$ is continuous, any deterministic selection $b_i(s_i)$ from $\tilde{b}_i(s_i)$ is strictly increasing at all s_i for which $b_i(s_i) > b_*$. It follows that

$$y_i(\cdot) = \tilde{b}_i^{-1}(\cdot)$$

is a nondecreasing function that is well defined at all $b > b_*$ for which there exists s_i with $b \in \operatorname{supp} \tilde{b}_i(s_i)$. Thus, for all $b > b_*$ we can define

$$\phi_i(b) = \sup\{y_i(\hat{b}) \mid \hat{b} \le b, y_i(\hat{b}) \text{ defined}\}. \tag{2.3}$$

Because $y_i(\cdot)$ is nondecreasing, $\phi_i(\cdot)$ is nondecreasing and continuous for all $b > b_*$. Note, furthermore, that buyer i's probability of winning can be written as

$$G_i(b) \equiv \underset{j \neq i}{\times} F_j(\phi_j(b)). \tag{2.4}$$

Because $\phi_j(b)$ is continuous for all j, so is $G_i(b)$. Any realization of $\tilde{b}_i(s_i)$ solves

$$\max_b E_i(b, s_i) = \max_b \underset{j \neq i}{\times} F_j(\phi_j(b)) U_i(b, s_i).$$

Equivalently, it solves:

$$\max_b \underset{j \neq i}{\times} F_j(\phi_j(b)) \frac{U_i(b, s_i)}{U_i(0, s_i)}.$$

That is, the bidder maximizes the ratio of his expected utility to his utility if he is simply given the item for free.

Define

$$p_i(b) \equiv \log F_i(\phi_i(b)). \tag{2.5}$$

Then, any realization of $\tilde{b}_i(s_i)$ solves

$$\max_b e_i(b, s_i),$$

where

$$e_i(b,s_i) = \log\,[\underset{j\neq i}{\times} F_j(\phi_j(b))\frac{U_i(b,s_i)}{U_i(0,s_i)}] = \sum_{j\neq i} p_j(b) - c_i(b,s_i),$$

and $c_i(b,s_i)$ is given by (1.1).

As a preliminary to establishing uniqueness, we now derive properties of $\phi_i(\cdot)$ and $\sum_{j\neq i} p_j(b)$. Proofs of Lemmas 5-8 can be found in the Appendix.

Lemma 5: **Strict monotonicity property of bid distributions.**

For any $b > b_*$ and any *i*, $\sum_{j\neq i} p_j(b)$ is strictly increasing at *b*.

Lemma 6: If $\phi_i(b)$ is strictly increasing to the right or left at $b = \hat{b} \geq b_*$, then $\hat{b}$ is a best response for buyer *i* with reservation price $\hat{s}_i = \phi_i(\hat{b})$.

Lemma 7: If $\phi_i(b)$ is strictly increasing to the right or to the left at $b = \hat{b} \geq b_*$, then $\sum_{j\neq i} p_j(b)$ is correspondingly right or left continuously differentiable at $\hat{b}$. Moreover, the right or left derivative satisfies

$$\sum_{j\neq i} p_j'(\hat{b}) = \frac{\partial c_i}{\partial b}(\hat{b}, \phi_i(\hat{b})). \tag{2.6}$$

Lemma 8: $\phi_i(b)$ is right or left continuously differentiable at all $b \geq b_*$.

Define the inverse function

$$h_i(\cdot) \equiv \left(\log F_i\right)^{-1}(\cdot) . \tag{2.7}$$

Then we can rewrite equation (2.6) as

$$\sum_{j \neq i} p_j{}'(b) = \frac{\partial}{\partial b} c_i\left(b, h_i\left(p_i(b)\right)\right). \tag{2.8}$$

We shall make important use of the following:

Lemma 9: Suppose that $\left(\overline{p}_1, ..., \overline{p}_n\right)$ and $\left(\hat{p}_1, ..., \hat{p}_n\right)$ are two solutions to the differential equation system

$$\sum_{j \neq i} p_j'(b) = \frac{\partial}{\partial b} c_i\left(b, h_i\left(p_i(b)\right)\right), \quad i = 1, ..., n \tag{2.9}$$

on the interval $\left(b^1, b^2\right]$. If for some $b_\circ \in \left(b^1, b^2\right], \overline{p}_i\left(b_\circ\right) < \hat{p}_i\left(b_\circ\right)$ for all i, then, for all $b \in \left(b^1, b_\circ\right)$,

$$\overline{p}_i(b) < \hat{p}_i(b) \text{ for all } i, \tag{2.10}$$

and

$$\sum_{j=1}^{n} \overline{p}_j'(b) > \sum_{j=1}^{n} \hat{p}_j'(b). \tag{2.11}$$

Proof: Dividing both sides of (2.11) by n-1 and then summing over i, we obtain

$$\sum_{j=1}^{n} p_j'(b) = \frac{1}{n-1} \sum_{j=1}^{n} c_j\left(b, h_j\left(p_j(b)\right)\right). \tag{2.12}$$

Subtracting (2.9) from (2.12), we have for all i,

$$p_i'(b) = \frac{1}{n-1}\left(\sum_{j \neq i} \frac{\partial}{\partial b} c_j\left(b, h_j\left(p_j(b)\right)\right) - (n-2)\frac{\partial}{\partial b} c_i\left(b, h_i\left(p_i(b)\right)\right)\right). \tag{2.13}$$

Suppose, contrary to (2.10), there exist i and $b \in \left(b^1, b_\circ\right)$ such that $\overline{p}_i(b) = \hat{p}_i(b)$. Let $\hat{b}$ be the biggest such b. Then

$$\overline{p}_i\left(\hat{b}\right) = \hat{p}_i\left(\hat{b}\right) \tag{2.14}$$

and

$$\overline{p}_j(b) < \hat{p}_j(b) \tag{2.15}$$

for all $b \in \left(\hat{b}, b_\circ\right)$ and $j = 1, \ldots, n$.

Now, from the fundamental theorem for ordinary differential equations (FTODE), there exists a unique solution $\left(p_1, \ldots, p_n\right)$ to (2.9) with the point condition $p_j\left(\hat{b}\right) = \overline{p}_j\left(\hat{b}\right)$ for all j.

Hence, from (2.14) and (2.15), there exists $k \neq i$ such that

$$\overline{p}_k\left(\hat{b}\right) < \hat{p}_k\left(\hat{b}\right). \tag{2.16}$$

From (2.13) and (2.14)

$$\overline{p}_i'\left(\hat{b}\right) - \hat{p}_i'\left(\hat{b}\right) = \frac{1}{n-1} \sum_{j \neq i} \frac{\partial}{\partial b}\left(c_j\left(\hat{b}, h_j\left(\overline{p}_j\left(\hat{b}\right)\right)\right) - c_j\left(\hat{b}, h_j\left(\hat{p}_j\left(\hat{b}\right)\right)\right)\right). \tag{2.17}$$

But from (2.15) and (2.16) and Assumption 2, the right-hand side of (2.17) is positive and hence $\overline{p}_i(b) > \hat{p}_i(b)$ for *b* in a right neighborhood of $\hat{b}$, contradicting (2.15). We conclude that (2.10) holds as claimed. Then (2.11) follows from (2.10), (2.12), and Assumption 2.

Q.E.D.

3. UNIQUENESS

When buyers are *ex ante* asymmetric, we do not generally obtain uniqueness of equilibrium bids that win zero probability. To see this, consider the following:

Example: Suppose that $n = 2$, that s_1 is distributed uniformly in the interval [0,1], and that s_2 is distributed uniformly in [3,4].[5] One equilibrium consists of buyer 2 bidding $b_2(s_2) = 1$ for all s_2 and $b_1(s_1) = s_1$ for all s_1. However, we can replace buyer 1's bid function with $\hat{b}_1(s_1) = s_1^2$ without destroying equilibrium. Indeed, there is a continuum of different possible equilibrium bids for buyer 1. Nevertheless, all this multiplicity occurs below $b_* = 1$, and thus pertains only to bids that have no chance of winning.

Such examples dictate that when we speak of "uniqueness of equilibrium" we will henceforth be referring only to the portions of the equilibrium bid functions *at* or *above* b_*.

Proposition 1: Suppose that $n = 2$. If Assumptions 1 and 2 hold, then equilibrium is unique.

Proof: Recall from Lemma 3 that $\underline{s}_2 \le b_* \le \underline{s}_1$. Suppose first that $b_* \ge \overline{s}_2$. But then, from Lemma 3, $b_* = \overline{s}_2$, and buyer 1 with reservation price $\underline{s}_1$ maximizes his payoff by bidding $\overline{s}_2$. Clearly the same is true for all other types of buyer 1, and so $b_1(s_1) = \overline{s}_2$ for all s_1, i.e., equilibrium is unique at or above b_*.

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[5] Strictly speaking, this example violates our assumption that $F_i(\underline{s}_i) > 0$, but we could modify it slightly to satisfy the assumption without changing our conclusion.

Thus, suppose that $b_* < \bar{s}_2$. Then, from Lemma 1, for any equilibrium there exists $b^* > b_*$ such that the distribution of winning bids has support $\left[b_*, b^*\right]$ with continuous c.d.f. $G_w(\cdot)$. From Lemma 4, both bidders bid with strictly positive probability in any subinterval of $\left(b_*, b^*\right]$. Hence, from Lemma 8, if $\left(\tilde{b}_1, \tilde{b}_2\right)$ is an equilibrium, the transforms $\left(p_1, p_2\right)$ of the inverse bid functions $\left(\phi_1, \phi_2\right)$ are differentiable everywhere and satisfy the differential equation system (2.9).

Now suppose that there exist equilibria $\left(\bar{p}_1, \bar{p}_2\right)$ and $\left(\hat{p}_1, \hat{p}_2\right)$ such that the support of the former is $\left[b_*, \bar{b}^*\right]$ and that of the latter is $\left[b_*, \hat{b}^*\right]$, where $\bar{b}^* > \hat{b}^*$. Then, for $i = 1, 2$,

$$1 = \bar{p}_i\left(\bar{b}^*\right) = \hat{p}_i\left(\hat{b}^*\right) > \bar{p}_i\left(\hat{b}^*\right). \tag{3.1}$$

Because both equilibria satisfy (2.9) on the interval $\left(b_*, \hat{b}^*\right]$, Lemma 9 and (3.1) imply that, for all $b \in \left(b_*, \hat{b}^*\right]$,

$$\sum_{j=1}^{2} \bar{p}_j'(b) > \sum_{j=1}^{2} \hat{p}_j'(b). \tag{3.2}$$

Integrating (3.2) and using the fact that $\bar{p}_j$ and $\hat{p}_j$ are continuous at b_*, we obtain

$$\sum_{j=1}^{2} \left(\bar{p}_j\left(\hat{b}^*\right) - \bar{p}_j\left(b_*\right)\right) \geq \sum_{j=1}^{2} \left(\hat{p}_j\left(\hat{b}^*\right) - \hat{p}_j\left(b_*\right)\right). \tag{3.3}$$

Hence, from (3.1) and (3.3), we have

$$\sum_{j=1}^{2} \hat{p}_j\left(b_*\right) > \sum_{j=1}^{2} \bar{p}_j\left(b_*\right). \tag{3.4}$$

But from Lemma 3, $\hat{p}_1(b_*) = \overline{p}_1(b_*) = \log F_1(\underline{s}_1)$ and $\hat{p}_2(b_*) = \overline{p}_2(b_*) = \log F_2(b_*)$, which contradicts (3.4). We conclude that $\overline{b}^* = \hat{b}^* = b^*$, and so uniqueness follows from FTODE with boundary condition $p_1(b^*) = p_2(b^*) = 1$.

Q.E.D.

The proof of Proposition 1 applies the FTODE to the upper endpoint of the distribution of winning bids. With two buyers, the upper endpoint is the same for both buyers, but with three or more buyers, not everyone need share the same maximum bid. To guarantee that they do, we shall impose two more fairly mild assumptions:

Assumption 3: Equal upper endpoints.

The upper endpoint of the support of the distribution of reservation prices is the same for all buyers, i.e., $\overline{s}_1 = \ldots = \overline{s}_n = \overline{s}$ [6]

We also assume that when bidders have the same reservation price, then they have the same preferences. Formally, we have:

Assumption 4: Identical reservation prices imply identical preferences

For all *i* and *j*, if $s_i = s_j$, then $U_i(\cdot, s_i) = U_j(\cdot, s_j)$.

Note that Assumption 4 is satisfied if buyers are risk neutral, as is often assumed in the auctions literature. We can now state:

[6] Assumption 3 is weak in the sense that, for any vector of distributions $(F_1, \ldots, F_n)$, there exists another vector $(\hat{F}_1, \ldots, \hat{F}_n)$ that is arbitrarily close to $(F_1, \ldots, F_n)$ and satisfies the assumption. Moreover our method of proof can be extended readily to the case of different upper endpoints.

Lemma 10: If Assumptions 3 and 4 hold, then the upper endpoints in the supports of all buyers' equilibrium bid distributions are the same.

Proof: Suppose that we index the buyers according to the upper endpoints of their equilibrium bid distributions: $b_1^* \geq \ldots \geq b_n^*$. Since equilibrium bidding is monotonic, b_1^* is a best reply for bidder 1 when his type is $\bar{s}$ (by leaving the subscript off $\bar{s}$, we are invoking Assumption 3). Using the logarithmic transformation of buyer 1's expected utility, it follows that

$$e_1(b_n^*, \bar{s}) = \sum_{j=2}^{n} p_j(b_n^*) - c(b_n^*, \bar{s}) \leq \sum_{j=2}^{n} p_j(b_1^*) - c(b_1^*, \bar{s}) = -c(b_1^*, \bar{s}) = e_1(b_1^*, \bar{s}),$$

where we have used the fact that $p_j\left(b_1^*\right) = \log F_j\left(\bar{s}_j\right) = 0$, and we have invoked Assumption 4 by leaving the subscript off c_1. Suppose that $b_n^* < b_1^*$. Since b_1^* is in the support of buyer 1's distribution of winning bids, $p_1(b_n^*) < 0 = p_n(b_n^*)$. Substituting for $p_n(b_n^*)$, we have, from the above inequality,

$$e_n(b_n^*, \bar{s}) = \sum_{j=1}^{n-1} p_j(b_n^*) - c(b_n^*, \bar{s}) < -c(b_1^*, \bar{s}) = e_n(b_1^*, \bar{s}).$$

Thus b_n^* is not a best response for buyer *n* after all, a contradiction. We conclude that $b_n^* = b_1^*$.

Q.E.D.

The proof of Proposition 1 also relies on the property that, with just two buyers, equilibrium bid functions are continuous above b_*. But with three or more buyers, our assumptions so far do not suffice to rule out the possibility that some buyer *i* has a "gap" $[b', b'']$

in the support of his equilibrium bid distribution. Still, we require only one additional weak condition to rule out such gaps:

Assumption 5: Nonincreasing absolute risk-aversion

For all *i*, the coefficient of absolute risk aversion, $A_i\left(b, s_i\right) = \frac{\partial^2 U_i}{\partial b^2} \Big/ \frac{\partial U_i}{\partial b}$, is nonnegative and nonincreasing in s_i.

We can now establish our final preliminary result:

Lemma 11: If Assumptions 1, 2, 4, and 5 hold, the support of each buyer *i*'s equilibrium bid distribution is an interval $[b_*, b_1^*]$.

Remark: We ignore bids that have no chance of winning for the reasons illustrated by the example at the beginning of the section.

Proof: Suppose, to the contrary, that some buyer *i*'s equilibrium bid distribution has a "gap" $[b^\circ, b^{\circ\circ}]$. That is, there exists some reservation price $s_i^\circ = \phi_i(b^\circ)$ for which both b^o and $b^{\circ\circ}\left(> b^\circ\right)$ are best replies, and $\phi_i\left(b\right) = s_i^\circ$ for all $b \in \left[b^\circ, b^{\circ\circ}\right]$. Buyer *i* with reservation price s_i° chooses *b* to maximize

$$e_i(b, s_i^\circ) = \sum_{j \neq i} p_j(b) - c_i(b, s_i^\circ). \tag{3.5}$$

Thus, at b°,

$$\frac{\partial e_i}{\partial b} = \sum_{j \neq i} p_j'(b^o) - \frac{\partial c_i}{\partial b}\left(b^\circ, s_i^\circ\right) \leq 0.$$

Let $\hat{b}$ be the biggest bid in $[b^\circ, b^{\circ\circ}]$ such that

$$\frac{\partial e_i}{\partial b} = \sum_{j \neq i} p_j'(b) - \frac{\partial c_i}{\partial b}\left(b, s_i^\circ\right) \leq 0 \tag{3.6}$$

for all $b \in \left[b^\circ, \hat{b}\right]$. Suppose that m of the equilibrium bid functions are strictly increasing at b°. Without loss of generality, let these be the bid functions of bidders 1 to m and suppose that they are increasing throughout the interval $\left[b^\circ, \hat{b}\right]$ (if not, we can conduct the following argument on each subinterval of strictly increasing bid functions). Then, from (3.6),

$$\frac{\partial e_i}{\partial b} = \sum_{j=1}^{m} p_j'(b) - \frac{\partial c_i}{\partial b}\left(b, \phi_i(b)\right) \leq 0, \tag{3.7}$$

and from (2.6),

$$\sum_{\substack{j=1 \\ j \neq k}}^{m} p_j'(b) - \frac{\partial c_k}{\partial b}\left(b, \phi_k(b)\right) = 0, \quad k = 1, \ldots, m. \tag{3.8}$$

Comparing (3.7) with (3.8), we obtain

$$\frac{\partial c_i}{\partial b} > \frac{\partial c_k}{\partial b} \text{ for all } k = 1, \ldots, m. \tag{3.9}$$

Hence, from Assumptions 2 and 4,

$$\phi_i(b) < \phi_k(b), \quad k = 1, \ldots, m \text{ for all } b \in \left[b^\circ, \hat{b}\right]. \tag{3.10}$$

Summing (3.8) over k, we have

$$(m-1)\sum_{j=1}^{m} p_j'(b) = \sum_{j=1}^{m} \frac{\partial c_j}{\partial b}\left(b, \phi_j(b)\right). \tag{3.11}$$

Differentiating (3.11) by b, we obtain, using Assumption 2 and (1.3),

$$(m-1)\sum_{j=1}^{m} p_j''(b) < \sum_{j=1}^{m} \frac{\partial^2 c_j}{\partial b^2} = \sum_{j=1}^{m} A_j \frac{\partial c_j}{\partial b} + \sum_{j=1}^{m} (\frac{\partial c_j}{\partial b})^2$$

$$< A_i \sum_{j=1}^{m} \frac{\partial c_j}{\partial b} + \sum_{j=1}^{m} (\frac{\partial c_j}{\partial b})^2, \tag{3.12}$$

where the last inequality follows from Assumptions 4 and 5 and (3.10), and where the fact that $p_j(b)$ is twice differentiable at b follows from our assumptions about F_j and the FTODE.

From (3.7) and (3.11),

$$\frac{\partial e_i}{\partial b}\left(b, \phi_i(b)\right) = \frac{1}{m-1}[\sum_{j=1}^{m} \frac{\partial c_j}{\partial b}\left(b, \phi_j(b)\right) - (m-1)\frac{\partial c_i}{\partial b}\left(b, \phi_i(b)\right)]. \tag{3.13}$$

Also from (3.7),

$$\frac{\partial^2 e_i}{\partial b^2} = \sum_{j=1}^{m} p_j'' - \frac{\partial^2 c_i}{\partial b^2} = \sum_{j=1}^{m} p_j'' - A_i \frac{\partial c_i}{\partial b} - (\frac{\partial c_i}{\partial b})^2$$

$$< \frac{A_i}{m-1} \sum_{j=1}^{m} \frac{\partial c_j}{\partial b} + \frac{1}{(m-1)} \sum_{j=1}^{m} (\frac{\partial c_j}{\partial b})^2 - A_i \frac{\partial c_i}{\partial b} - (\frac{\partial c_i}{\partial b})^2 \quad \text{(from (3.12))}$$

$$< \frac{A_i}{(m-1)}[\sum_{j=1}^{m} \frac{\partial c_j}{\partial b} - (m-1)\frac{\partial c_i}{\partial b}] + \frac{1}{(m-1)}[\sum_{j=1}^{m} (\frac{\partial c_j}{\partial b})^2 - (m-1)(\frac{\partial c_i}{\partial b})^2]$$

$$< A_i \frac{\partial e_i}{\partial b} + \frac{1}{(m-1)}[\sum_{j=1}^{m} (\frac{\partial c_j}{\partial b})^2 - (m-1)(\frac{\partial c_i}{\partial b})^2] \text{ (from (3.13).} \tag{3.14}$$

If $\frac{\partial e_i}{\partial b} \leq 0$ it follows from (3.13) that $\sum_{j=1}^{m} \frac{\partial c_j}{\partial b} - (m-1)\frac{\partial c_i}{\partial b} \leq 0$ Hence,

$$\frac{\partial c_i}{\partial b}\left(\sum_{j=1}^{m}\frac{\partial c_j}{\partial b}\right)-(m-1)\left(\frac{\partial c_i}{\partial b}\right)^2 \leq 0,$$

and so, from (3.9), the bracketed expression on the right-hand side of (3.14) is negative. Thus, for all $b \in [b^\circ, b^{\circ\circ})$,

$$\frac{\partial e_i}{\partial b} \leq 0 \Rightarrow \frac{\partial^2 e_i}{\partial b^2} < 0.$$

It follows that $\hat{b} = b^{\circ\circ}$, and so $e_i(b, s_i^\circ)$ is strictly decreasing over $[b^\circ, b^{\circ\circ}]$, a contradiction of our hypothesis that bidder *i* with reservation price s_i° is indifferent between bidding b° and $b^{\circ\circ}$. Thus there can be no such "gap" after all.

Q.E.D.

Proposition 2: **Uniqueness with n buyers**

If Assumptions 1-5 hold, equilibrium is unique.

Proof: Lemmas 8 and 11 imply that equilibrium inverse bid functions are differentiable, and Lemma 10 implies that, in equilibrium, each buyer makes the same maximum bid. Hence, we can apply Lemma 9, as in the proof of Proposition 1, to show that the maximum bid b^* is the same in any equilibrium. Uniqueness then follows from FTODE.

Q.E.D.

4. **Concluding Remarks**

We have limited our attention to the case of "independent private values," in which a buyer's reservation price does not depend on other buyers' private information, and reservation prices are independently distributed. Note that, for this case, our arguments also establish equilibrium existence without the need to invoke existence theorems for discontinuous games such as Dasgupta and Maskin (1986), Simon and Zame (1990), and Reny (1999) (existence results for high-bid auctions that *do* use these theorems include Lebrun (1996), Maskin and Riley (2000b), Bresky (1999), Jackson and Swinkels (2001), and Reny and Zamir (2002)).

When there are only two buyers, Lizzeri and Persico (2000) relax the independence and private-values assumptions and establish uniqueness (and existence) under affiliation and certain forms of interdependent values. We believe that our methods can be adapted to accommodate such relaxations when there are more than two buyers, but this avenue remains to be explored (Bajari, 2001, establishes uniqueness when there are more than two buyers under the assumption the inverse bid functions are everywhere differentiable).

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APPENDIX

Lemma 5: **Strict monotonicity property of bid distributions.**

For any $b > b_*$ and any *i*, $\sum_{j \neq i} p_j(b)$ is strictly increasing at *b*.

Proof: Choose $\varepsilon > 0$. From Lemma 4, there must be at least one buyer $k \neq i$ who bids in $[b-\varepsilon, b]$ with positive probability. Hence $p_k(b-\varepsilon) < p_k(b)$ and so $\sum_{j \neq i} p_j(b-\varepsilon) < \sum_{j \neq i} p_j(b)$.

Q.E.D.

Lemma 6: If $\phi_i(b)$ is strictly increasing to the right (or left) at $b = \hat{b} \geq b_*$, then $\hat{b}$ is a best response for buyer *i* with reservation price $\hat{s}_i = \phi_i(\hat{b})$.

Proof: Since both cases are handled in the same way, we consider only the case in which $\phi_i(b)$ is strictly increasing to the right. If $\phi_i(b)$ is also strictly increasing to the left, then $\phi_i(\hat{b}) = y_i(\hat{b})$, and so the Lemma follows. Thus for some $\delta > 0$, suppose that $\phi_i(b) = s_i^*$ for all $b \in [\hat{b}-\delta, \hat{b}]$. That is, for some $b^* \in [\hat{b}-\delta, \hat{b}]$, $y_i(b^*) = s_i^*$. Because $\phi_i(b)$ is strictly increasing to the right at $\hat{b}$, there exists a decreasing sequence $\{b^1,...,b^t,...\}$ converging to $\hat{b}$ such that sequence $\{y_i(b^1),...,y_i(b^t),...\}$ converges to s_i^*.

Since b^t is optimal for reservation price $y_i(b^t)$, we have

$$e_i(b^t, y_i(b^t)) = \sum_{j \neq i} p_j(b^t) - c_i(b^t, y_i(b^t)) \geq \sum_{j \neq i} p_j(b^*) - c_i(b^*, y_i(b^*)), \text{ for all } t. \qquad \text{(A.1)}$$

From Lemma 5, it follows that $\sum_{j \neq i} p_j(b) = \log G_i(b)$ is continuous. Also $c_i(b, s_i)$ is continuous.

Therefore we have, in the limit,

$$\sum_{j \neq i} p_j(\hat{b}) - c_i(\hat{b}, s_i^*) \geq \sum_{j \neq i} p_j(b^*) - c_i(b^*, s_i^*) . \qquad \text{(A.2)}$$

From (A.2) it follows that buyer *i* with reservation price s_i^* is at least as well off choosing $\hat{b}$ as b^*.

Q.E.D.

Lemma 7: If $\phi_i(b)$ is strictly increasing to the right (or to the left) at $b = \hat{b} > b_*$, then $\sum_{j \neq i} p_j(b)$ is correspondingly right (or left) continuously differentiable at $\hat{b}$. Moreover, the right (left) derivative satisfies

$$\sum_{j \neq i} p_j(\hat{b}) = \frac{\partial c_i}{\partial b}(\hat{b}, \phi_i(\hat{b})) . \qquad \text{(A.3)}$$

<u>Proof</u>: Since the two cases are handled in the same way, we consider only the case in which $\phi_i(b)$ is strictly increasing to the right. We know that $\phi_i(b)$ is continuous. Thus at $\hat{b}$ there

exists a decreasing sequence $\{b^1,...,b^t,...\}$ converging to $\hat{b}$ such that $y_i(b^t)$ converges to $s_i^* = \phi_i(\hat{b})$ monotonically from above. Because b^t is optimal for buyer *i* with reservation price $s_i^t = y_i(b^t)$ we have

$$\sum_{j\neq i} p_j(\hat{b}) - c_i(\hat{b}, y_i(b^t)) \leq \sum_{j\neq i} p_j(b^t) - c_i(b^t, y_i(b^t)).$$

Rearranging, we obtain

$$\sum_{j\neq i} \frac{p_j(b^t) - p_j(\hat{b})}{b^t - \hat{b}} \geq \frac{c_i(b^t, y_i(b^t)) - c_i(\hat{b}, y_i(b^t))}{b^t - \hat{b}}. \tag{A.4}$$

By Lemma 6, $\hat{b}$ is optimal for buyer *i* with reservation price $\phi_i(\hat{b})$. Thus,

$$\sum_{j\neq i} p_j(\hat{b}) - c_i(\hat{b}, \phi_i(\hat{b})) \geq \sum_{j\neq i} p_j(b^t) - c_i(b^t, \phi_i(\hat{b})) \text{ for all } t.$$

Rearranging, we obtain

$$\sum_{j\neq i} \frac{p_j(b^t) - p_j(\hat{b})}{b^t - \hat{b}} \leq \frac{c_i(b^t, \phi_i(\hat{b})) - c_i(\hat{b}, \phi_i(\hat{b}))}{b^t - \hat{b}}. \tag{A.5}$$

In the limit as $b^t \to \hat{b}$, the right-hand sides of (A.4) and (A.5) equal $\frac{\partial}{\partial b} c_i\left(\hat{b}, \phi_i\left(\hat{b}\right)\right)$, which is continuous in $\hat{b}$. Thus $\sum_{j\neq i} p_j(b)$ is right continuously differentiable at $\hat{b}$, and its right derivative satisfies (A.3).

Q.E.D.

Lemma 8: $\phi_i(b)$ is right (left) continuously differentiable at all $b > b_*$.

<u>Proof</u>: Suppose $\phi_1(b),...,\phi_k(b)$ are strictly increasing to the right at $\hat{b}$ and that $\phi_{k+1}(b),...,\phi_n(b)$ are constant to the right at $\hat{b}$. By assumption, $i \leq k$. By Lemma 7, $\sum_{j \neq i}^{k} p_j(b)$ is right differentiable at $\hat{b}$, $i = 1,..,k$. Summing over *i* and dividing by *k*-1, we conclude that

$$\sum_{j=1}^{k} p_j(b) = \frac{1}{k-1} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} p_j(b)$$

is also right differentiable at $\hat{b}$. Since the difference between these last two expressions is just $p_i(b)$, $i = 1,...,k$, this too is right differentiable at $\hat{b}$. But $p_i(b) = F_i(\phi_i(b))$. Thus $\phi_i(b)$ is right differentiable at $\hat{b}$.

Q.E.D.

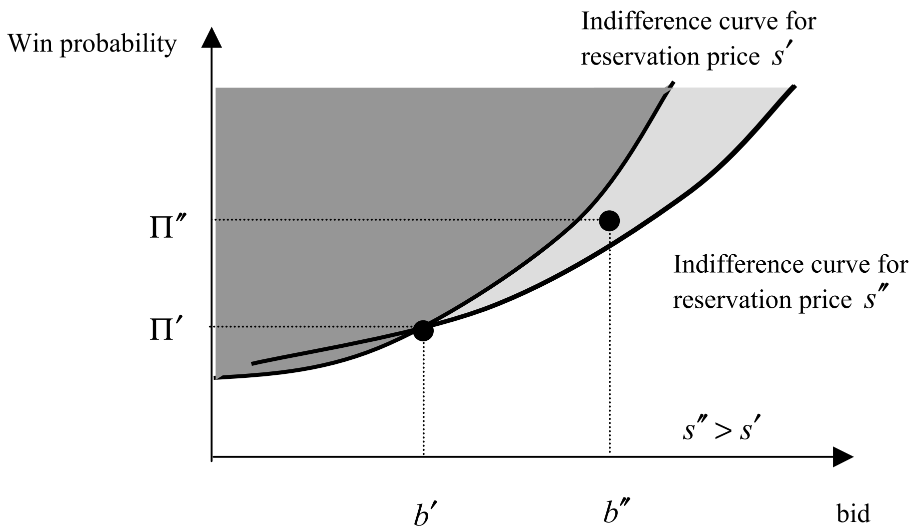

Figure 1.1: Single-crossing property

Footnotes