COS 429: COMPUTER VISON

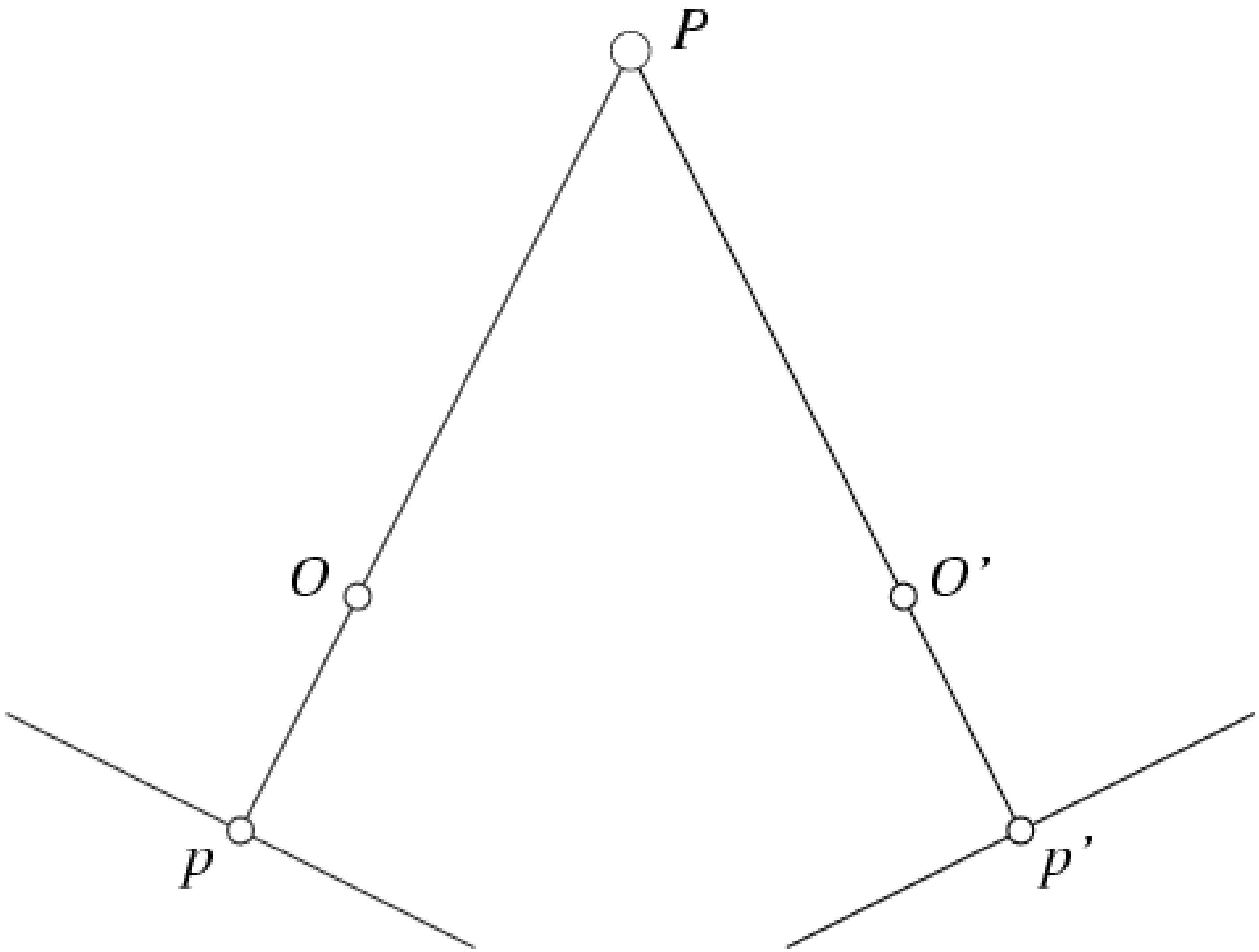
# MULTI-VIEW GEOMETRY (1 lecture)

- Epipolar Geometry
- The Essential and Fundamental Matrices
- The 8-Point Algorithm

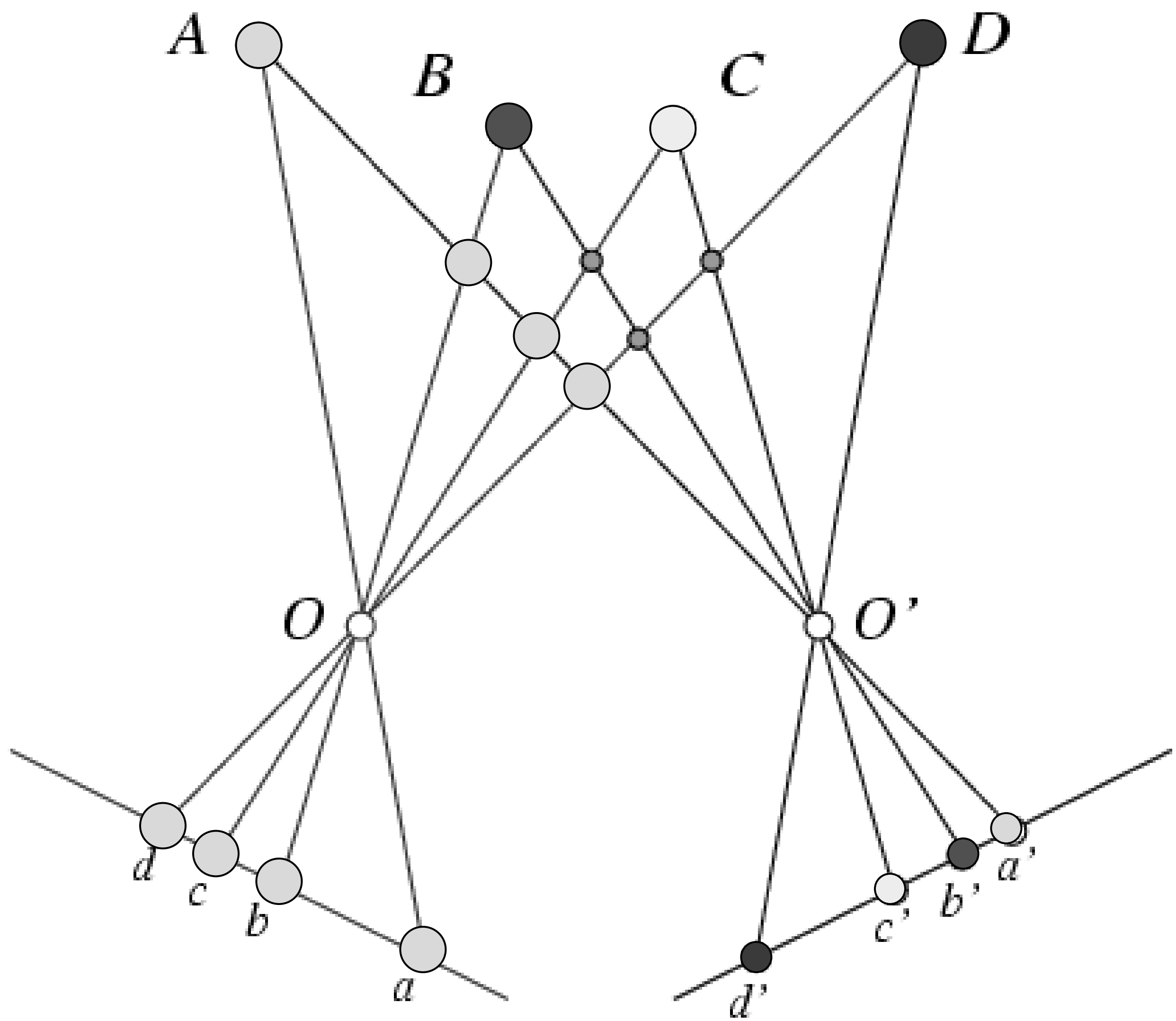
- Reading: Chapter 10

Many of the slides in this lecture are courtesy to Prof. J. Ponce

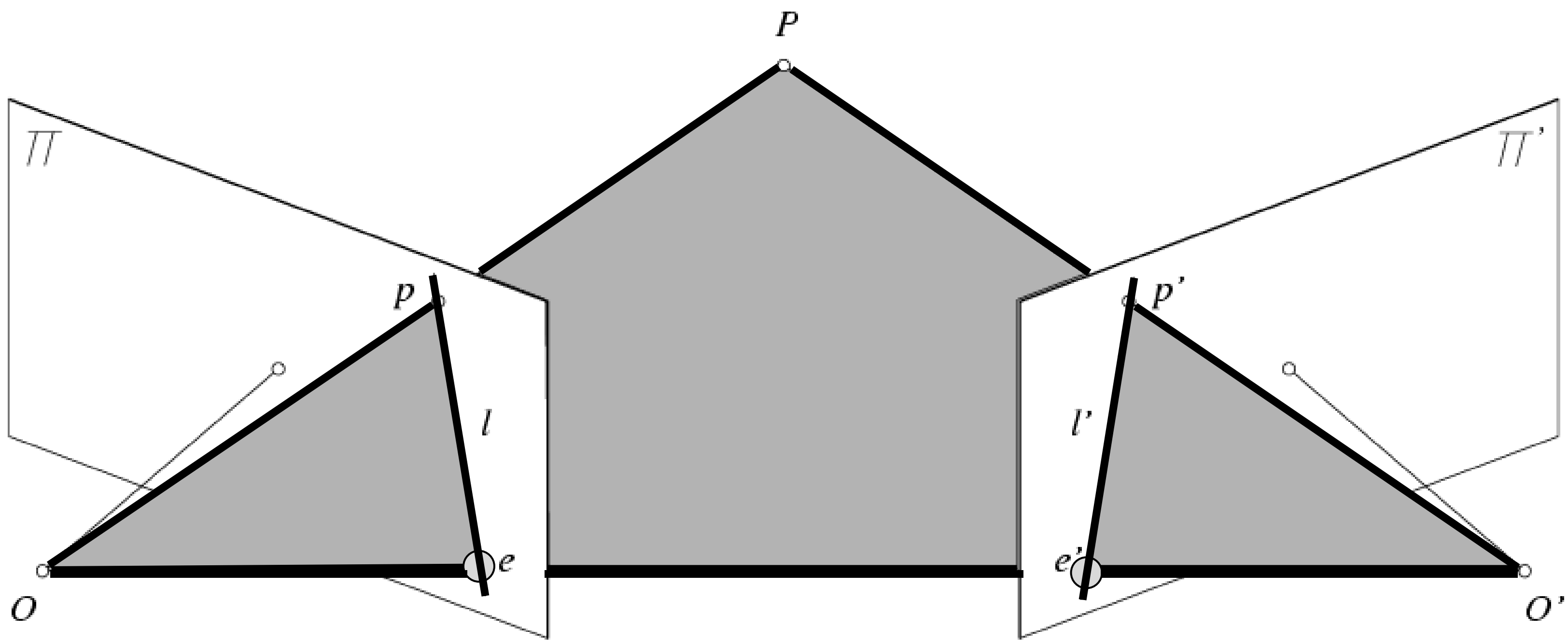
# Reconstruction / Triangulation



# (Binocular) Fusion

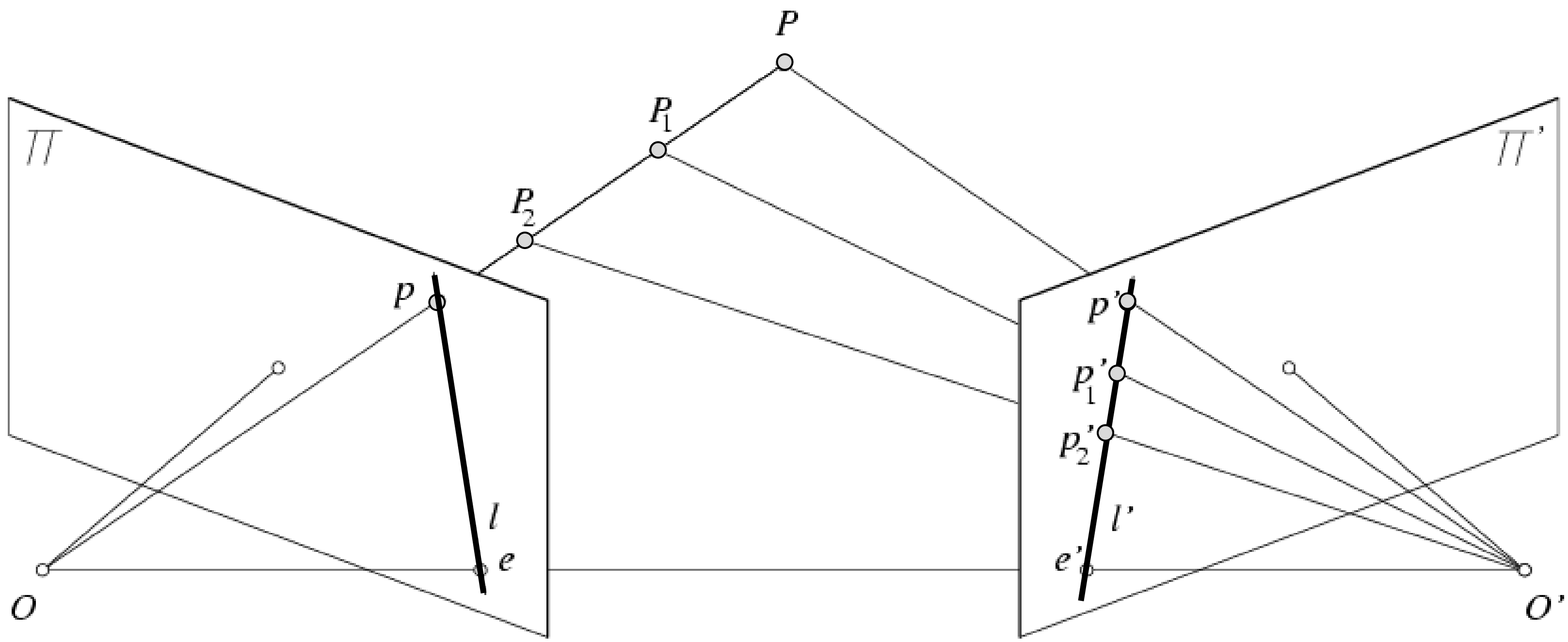


# Epipolar Geometry




- Epipolar Plane
- Baseline
- Epipoles
- Epipolar Lines

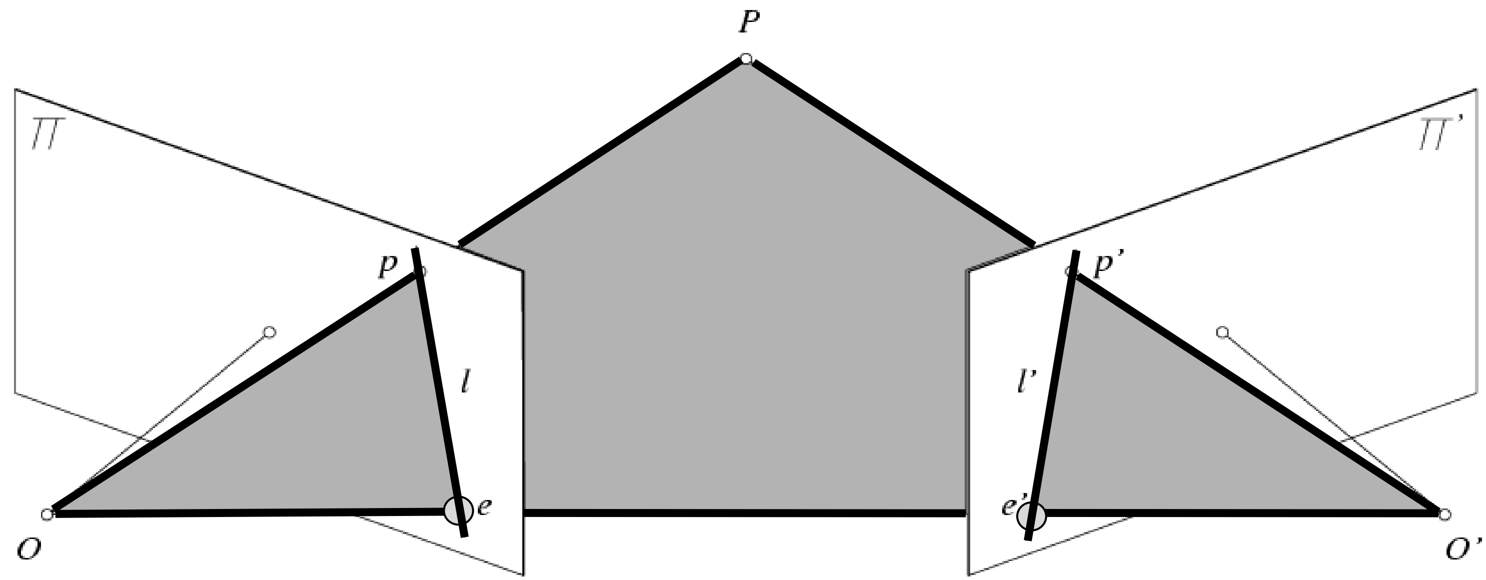
# Epipolar Constraint




• Potential matches for *p* have to lie on the corresponding epipolar line *l'*.

• Potential matches for *p'* have to lie on the corresponding epipolar line *l*.

# Epipolar Constraint: Calibrated Case




$$\overrightarrow{Op} \cdot [\overrightarrow{OO'} \times \overrightarrow{O'p'}] = 0$$

$$\boldsymbol{p} \cdot [\boldsymbol{t} \times (\mathcal{R}\boldsymbol{p}')] = 0 \quad \text{with} \begin{cases} \boldsymbol{p} = (u, v, 1)^T \\ \boldsymbol{p}' = (u', v', 1)^T \\ \mathcal{M} = (\,\mathrm{Id} \quad \boldsymbol{0}\,) \\ \mathcal{M}' = (\,\mathcal{R}^T, -\mathcal{R}^T \boldsymbol{t}\,) \end{cases}$$

$$\boldsymbol{p}^T \mathcal{E} \boldsymbol{p}' = 0 \quad \text{with} \quad \mathcal{E} = [\boldsymbol{t}_\times]\mathcal{R}$$

**Essential Matrix**
(Longuet-Higgins, 1981)

## Properties of the Essential Matrix

- $\mathcal{E}\, p'$ is the epipolar line associated with $p'$.
- $\mathcal{E}^T p$ is the epipolar line associated with $p$.
- $\mathcal{E}\, e'=0$ and $\mathcal{E}^T e=0$.
- $\mathcal{E}$ is singular.
- $\mathcal{E}$ has two equal non-zero singular values (Huang and Faugeras, 1989).

# Epipolar Constraint: Small Motions

$$\mathcal{R}(\boldsymbol{a},\theta) = e^{\theta[\boldsymbol{a}_\times]} \stackrel{\text{def}}{=} \sum_{i=0}^{+\infty} \frac{1}{i!}(\theta[\boldsymbol{a}_\times])^i$$

To First-Order:

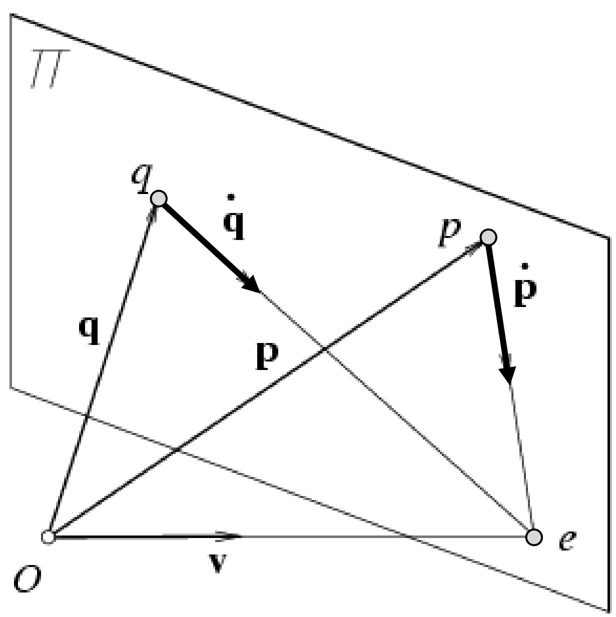
$$\begin{cases} \boldsymbol{t} = \delta t\, \boldsymbol{v} \\ \mathcal{R} = \mathrm{Id} + \delta t\, [\boldsymbol{\omega}_\times] \\ \boldsymbol{p}' = \boldsymbol{p} + \delta t\, \dot{\boldsymbol{p}} \end{cases}$$

$$\boldsymbol{p}^T \mathcal{E} \boldsymbol{p}' = 0 \quad \text{with} \quad \mathcal{E} = [\boldsymbol{t}_\times]\mathcal{R}$$

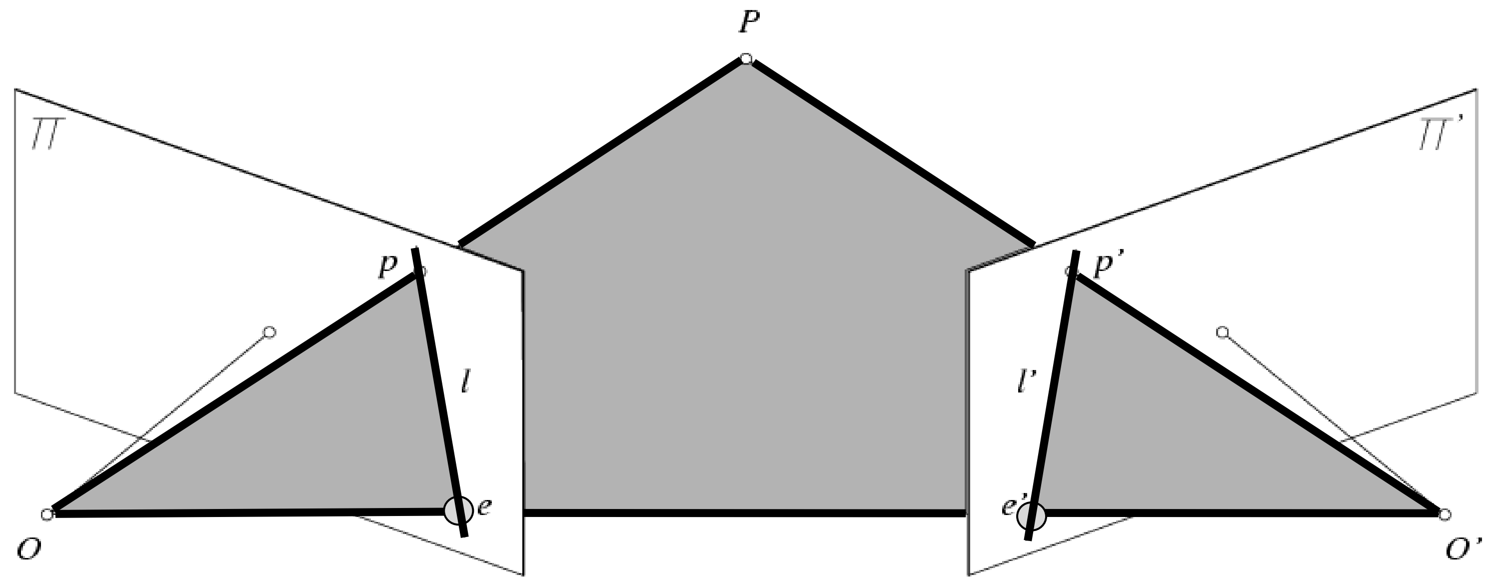
$$\boldsymbol{p}^T [\boldsymbol{v}_\times](\mathrm{Id} + \delta t\, [\boldsymbol{\omega}_\times])(\boldsymbol{p} + \delta t\, \dot{\boldsymbol{p}}) = 0$$

$$\boldsymbol{p}^T([\boldsymbol{v}_\times][\boldsymbol{\omega}_\times])\boldsymbol{p} - (\boldsymbol{p} \times \dot{\boldsymbol{p}}) \cdot \boldsymbol{v} = 0$$

Pure translation:
Focus of Expansion



# Epipolar Constraint: Uncalibrated Case




$$\hat{\boldsymbol{p}}^T \mathcal{E} \hat{\boldsymbol{p}}' = 0$$

$$\boldsymbol{p} = \mathcal{K} \hat{\boldsymbol{p}}$$

$$\boldsymbol{p}' = \mathcal{K}' \hat{\boldsymbol{p}}'$$

$$\Rightarrow \quad \boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0 \quad \text{with} \quad \mathcal{F} = \mathcal{K}^{-T} \mathcal{E} \mathcal{K}'^{-1}$$

$\Downarrow$

**Fundamental Matrix**
(Faugeras and Luong, 1992)

# Properties of the Fundamental Matrix

- $\mathcal{F}$ p' is the epipolar line associated with p'.
- $\mathcal{F}^T$p is the epipolar line associated with p.
- $\mathcal{F}$ e'=0 and $\mathcal{F}^T$ e=0.
- $\mathcal{F}$ is singular.

# The Eight-Point Algorithm (Longuet-Higgins, 1981)

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$(uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' \\ u_3u_3' & u_3v_3' & u_3 & v_3u_3' & v_3v_3' & v_3 & u_3' & v_3' \\ u_4u_4' & u_4v_4' & u_4 & v_4u_4' & v_4v_4' & v_4 & u_4' & v_4' \\ u_5u_5' & u_5v_5' & u_5 & v_5u_5' & v_5v_5' & v_5 & u_5' & v_5' \\ u_6u_6' & u_6v_6' & u_6 & v_6u_6' & v_6v_6' & v_6 & u_6' & v_6' \\ u_7u_7' & u_7v_7' & u_7 & v_7u_7' & v_7v_7' & v_7 & u_7' & v_7' \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' \end{pmatrix}\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Minimize:

$$\sum_{i=1}^{n} (\boldsymbol{p}_i^T \mathcal{F} \boldsymbol{p}_i')^2$$

under the constraint

$|F|^2=1$.

Non-Linear Least-Squares Approach (Luong et al., 1993)

Minimize

$$\sum_{i=1}^{n}[\mathrm{d}^2(\boldsymbol{p}_i, \mathcal{F}\boldsymbol{p}'_i) + \mathrm{d}^2(\boldsymbol{p}'_i, \mathcal{F}^T\boldsymbol{p}_i)]$$

with respect to the coefficients of F, using an appropriate rank-2 parameterization.

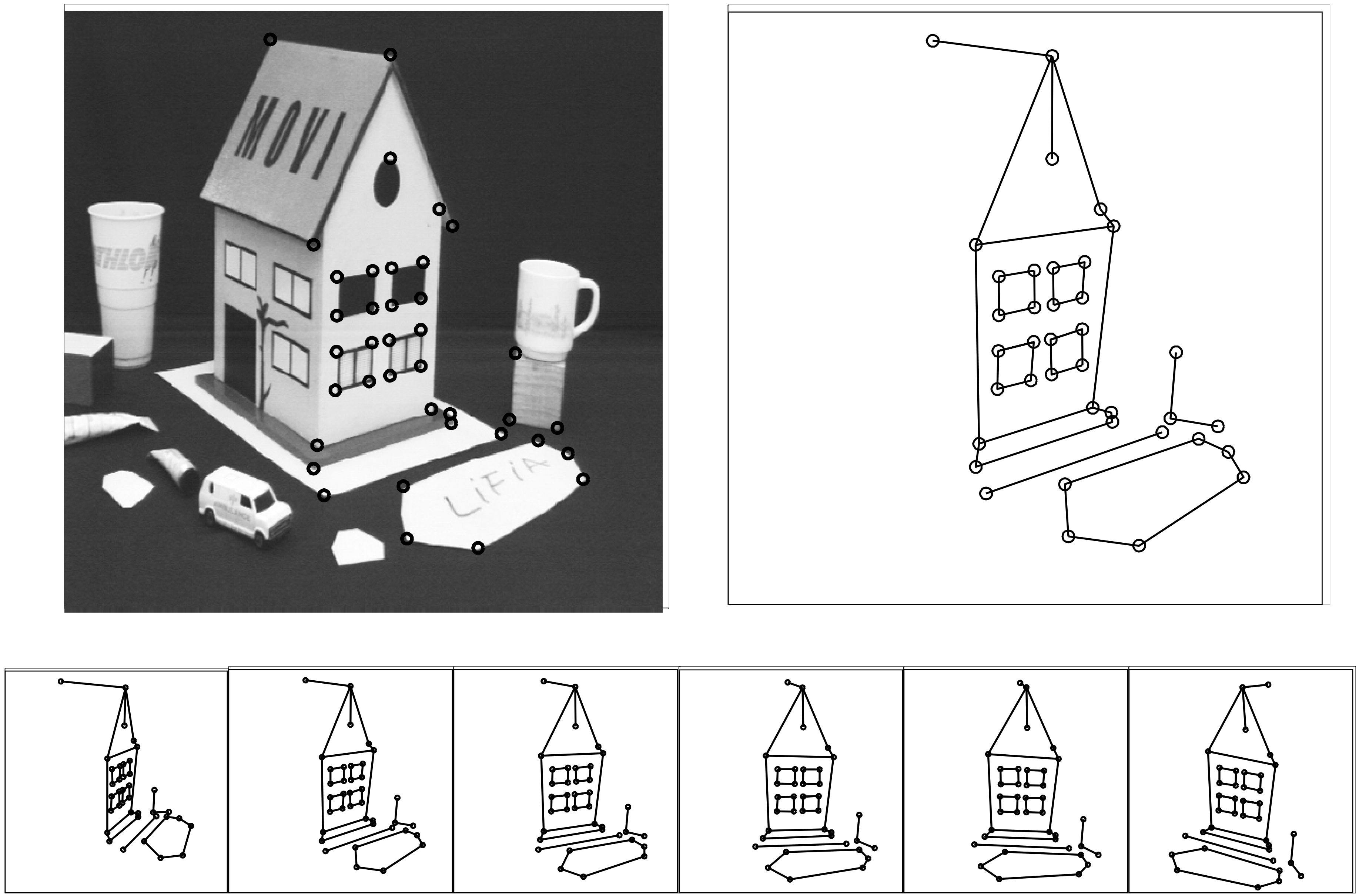
# The Normalized Eight-Point Algorithm (Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels: $q_i = T\,p_i$ , $q'_i = T'\,p'_i$.

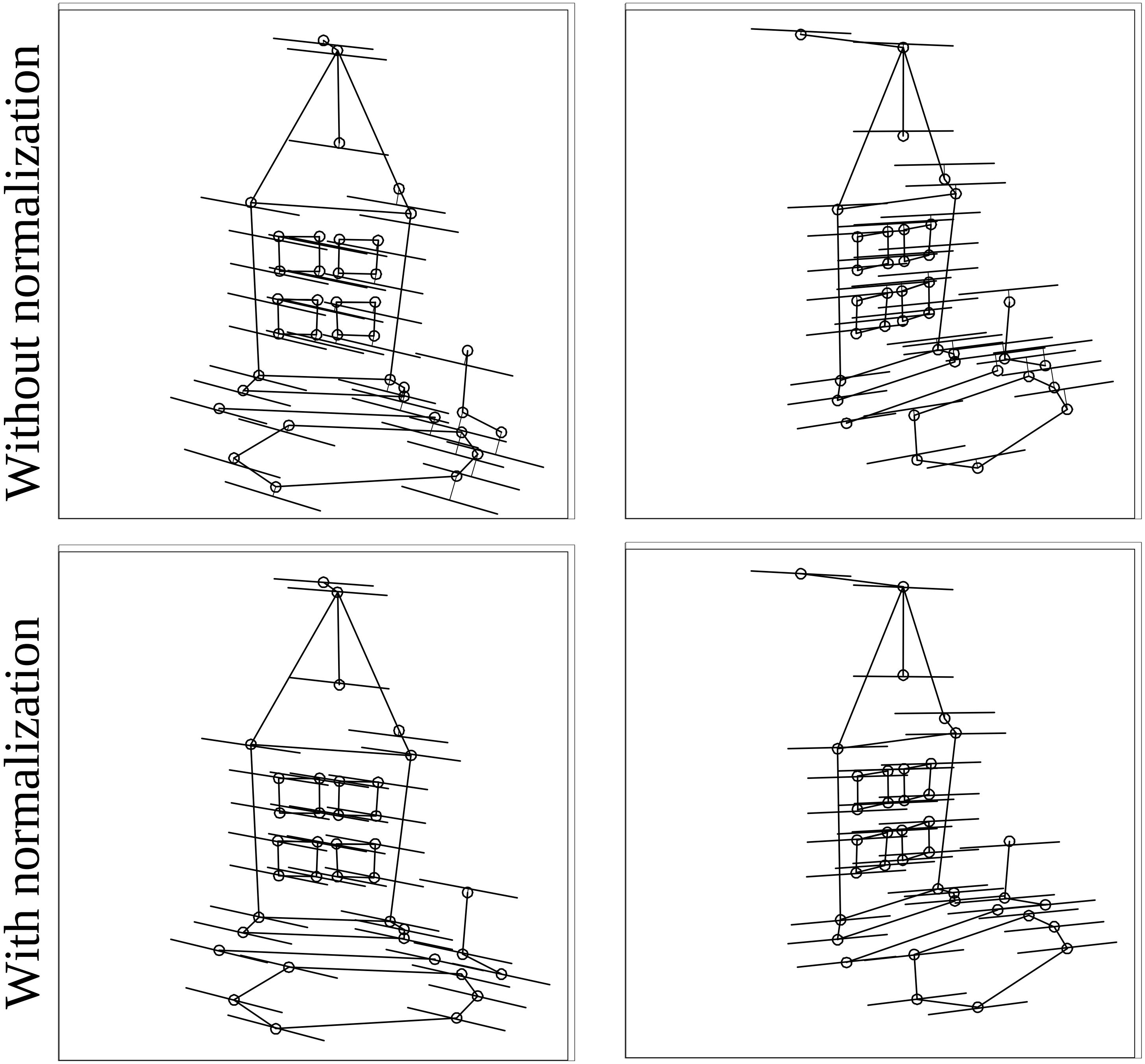
- Use the eight-point algorithm to compute $\mathcal{F}$ from the points $q_i$ and $q'_i$ .

- Enforce the rank-2 constraint.

- Output $T^T \mathcal{F}\, T'$.



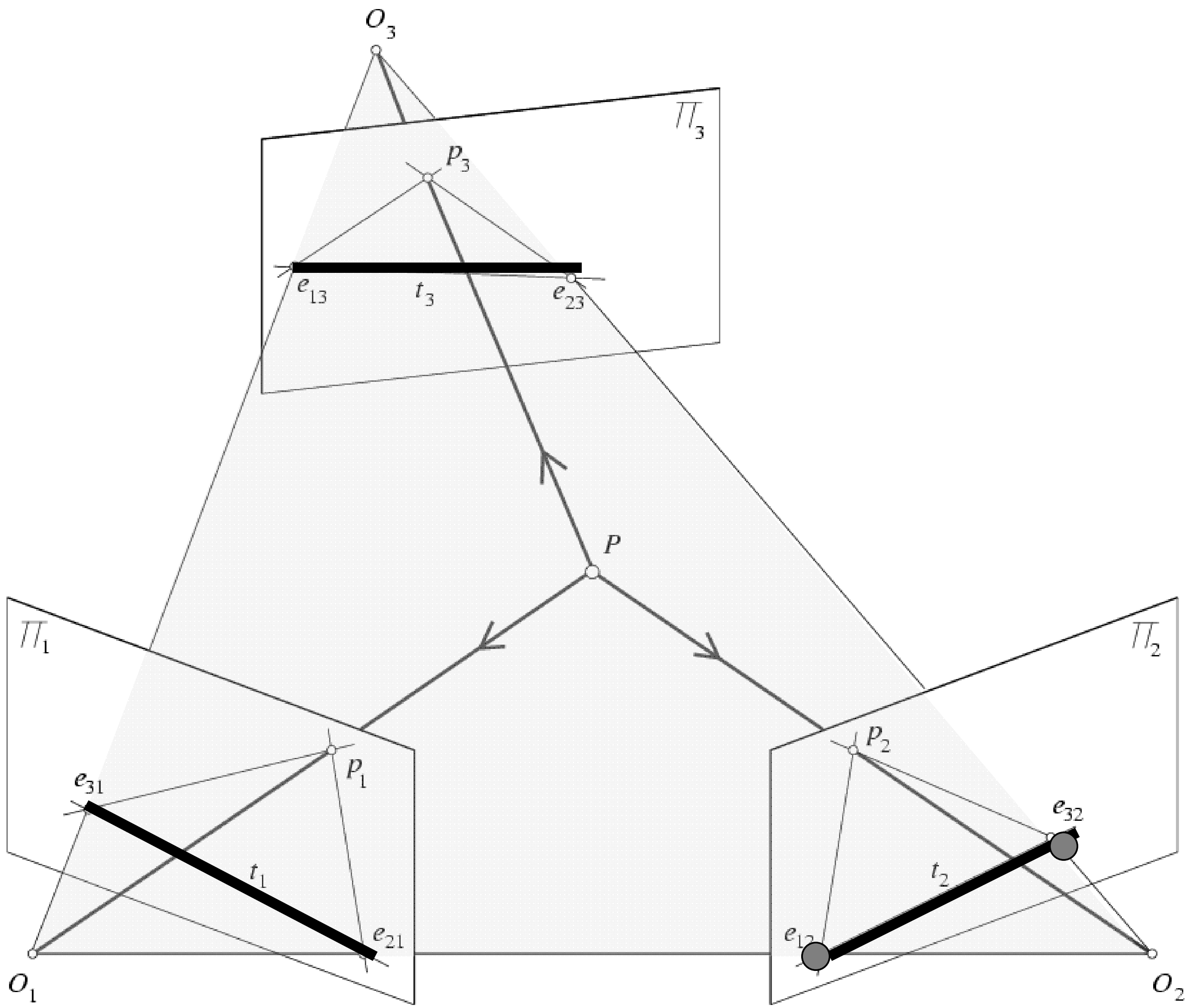

Data courtesy of R. Mohr and B. Boufama.



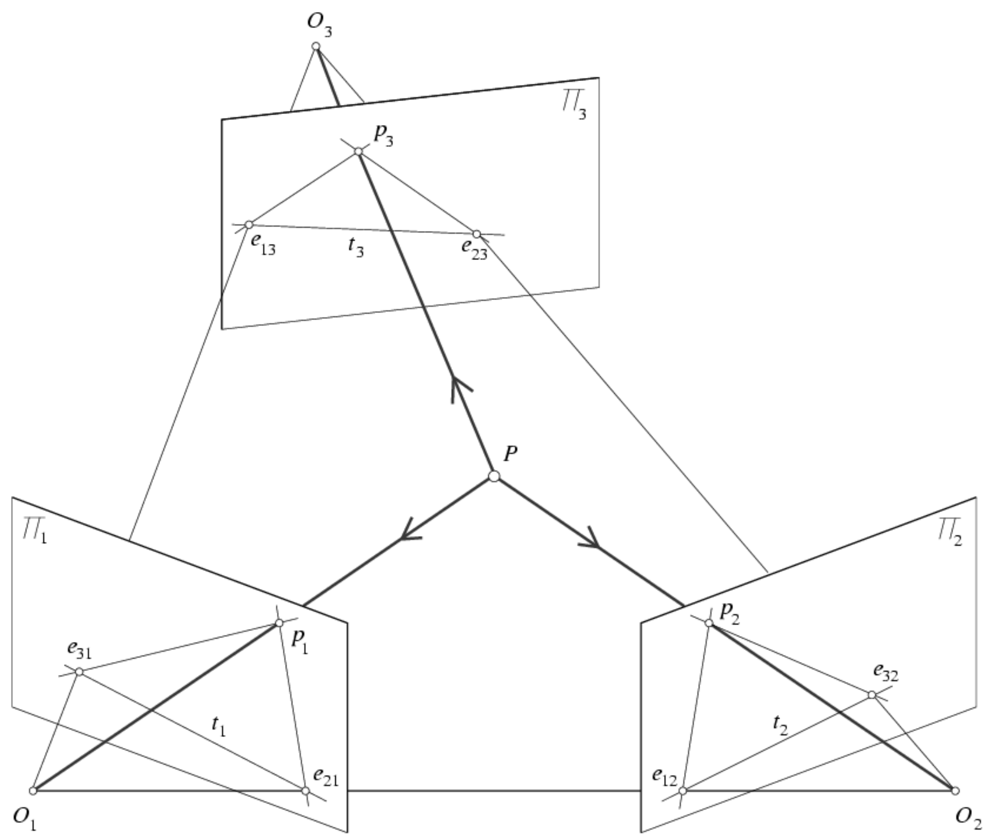

Mean errors:
10.0pixel
9.1pixel

Mean errors:
1.0pixel
0.9pixel

# Trinocular Epipolar Constraints



# Trinocular Epipolar Constraints

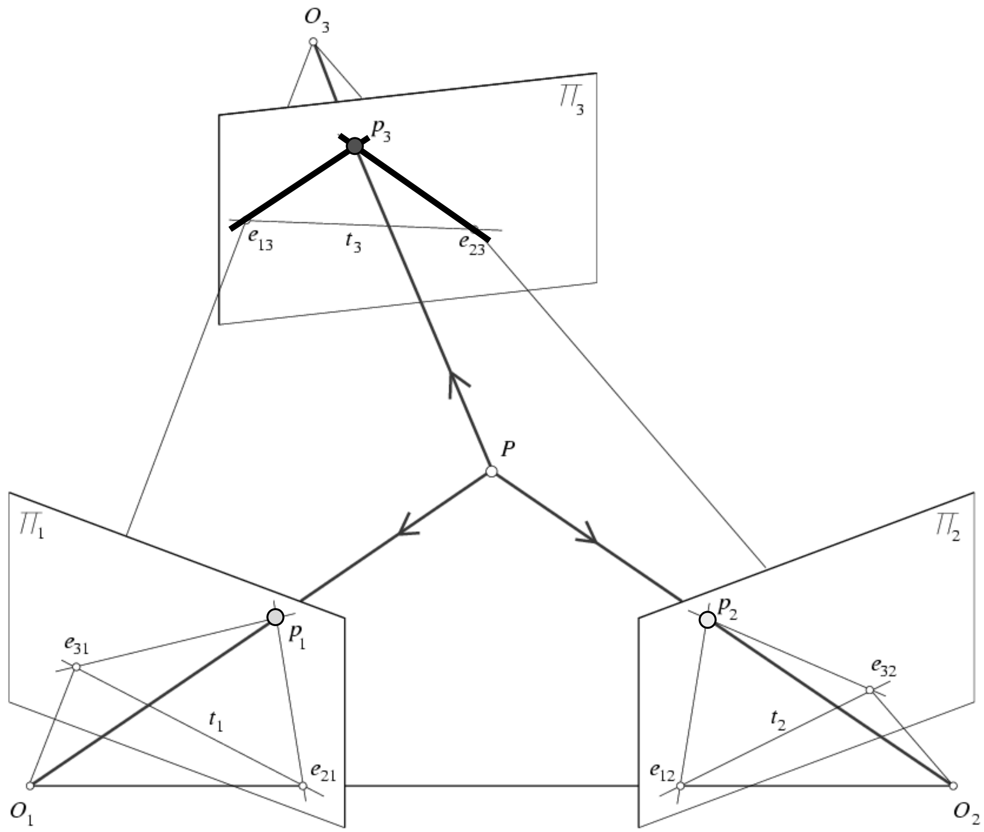



$$\begin{cases} \boldsymbol{p}_1^T \mathcal{E}_{12} \boldsymbol{p}_2 = 0 \\ \boldsymbol{p}_2^T \mathcal{E}_{23} \boldsymbol{p}_3 = 0 \\ \boldsymbol{p}_3^T \mathcal{E}_{31} \boldsymbol{p}_1 = 0 \end{cases}$$

⇒ These constraints are not independent!

$$\boldsymbol{e}_{31}^T \mathcal{E}_{12} \boldsymbol{e}_{32} = \boldsymbol{e}_{12}^T \mathcal{E}_{23} \boldsymbol{e}_{13} = \boldsymbol{e}_{23}^T \mathcal{E}_{31} \boldsymbol{e}_{21} = 0$$

# Trinocular Epipolar Constraints: Transfer

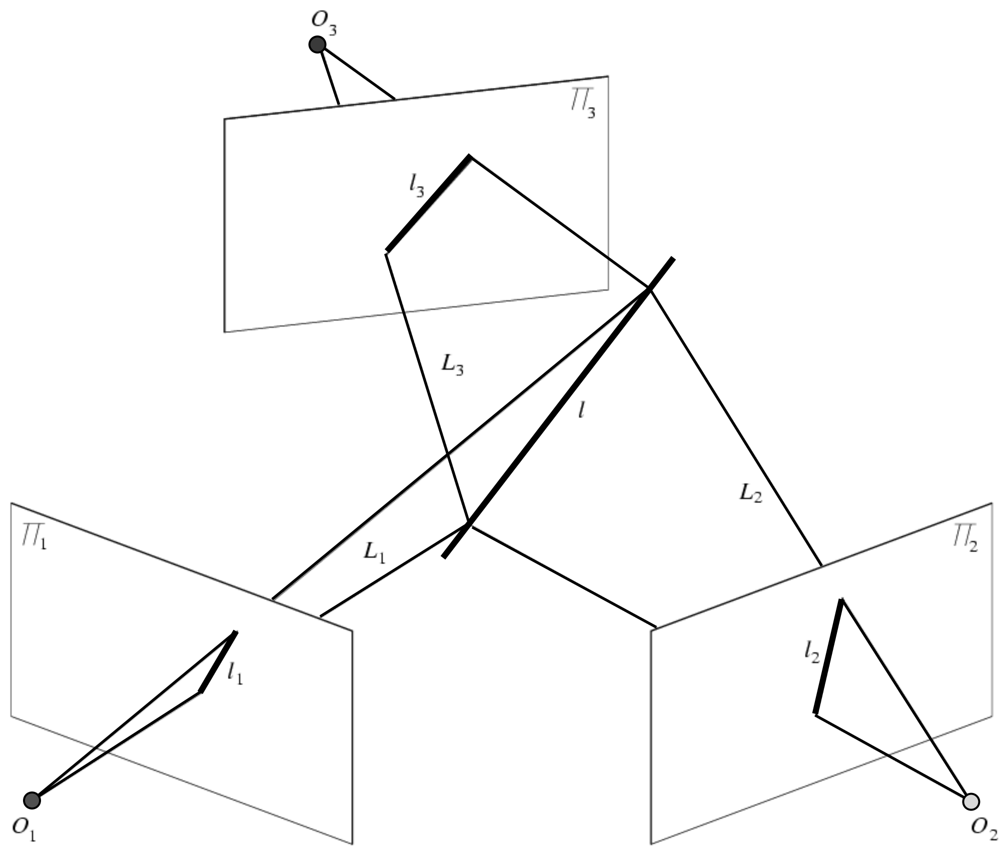



$$\begin{cases} \boldsymbol{p}_1^T \mathcal{E}_{12} \boldsymbol{p}_2 = 0 \\ \boldsymbol{p}_2^T \mathcal{E}_{23} \boldsymbol{p}_3 = 0 \\ \boldsymbol{p}_3^T \mathcal{E}_{31} \boldsymbol{p}_1 = 0 \end{cases} \Longrightarrow$$

Given $p_1$ and $p_2$, $p_3$ can be computed

as the solution of linear equations.

# Trifocal Constraints



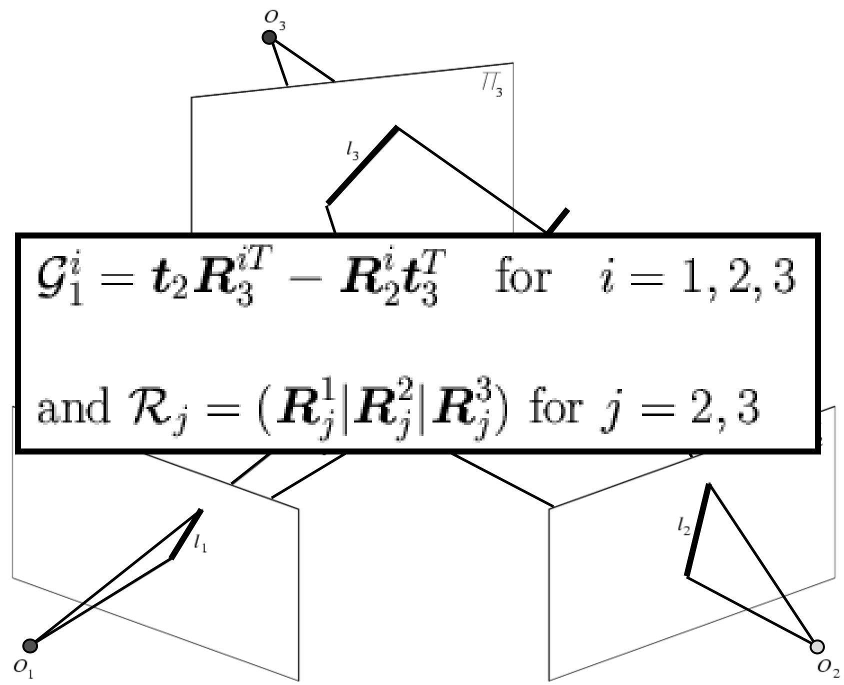

$$z\boldsymbol{p} = \mathcal{M}\boldsymbol{P} \iff \boldsymbol{l}^T\mathcal{M}\boldsymbol{P} = 0 \iff \boldsymbol{L}\cdot\boldsymbol{P} = 0 \text{ with } \boldsymbol{L} = \mathcal{M}^T\boldsymbol{l}$$

$$\begin{pmatrix} \boldsymbol{L}_1^T \\ \boldsymbol{L}_2^T \\ \boldsymbol{L}_3^T \end{pmatrix} \boldsymbol{P} = \boldsymbol{0}$$

$$\text{Rank}(\mathcal{L}) = 2 \quad \text{where} \quad \mathcal{L} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{l}_1^T\mathcal{M}_1 \\ \boldsymbol{l}_2^T\mathcal{M}_2 \\ \boldsymbol{l}_3^T\mathcal{M}_3 \end{pmatrix}$$

# Trifocal Constraints

## Calibrated Case




$$\mathcal{G}_1^i = \boldsymbol{t}_2 \boldsymbol{R}_3^{iT} - \boldsymbol{R}_2^i \boldsymbol{t}_3^T \quad \text{for} \quad i = 1, 2, 3$$

and $\mathcal{R}_j = (\boldsymbol{R}_j^1 | \boldsymbol{R}_j^2 | \boldsymbol{R}_j^3)$ for $j = 2, 3$

$$\mathrm{Rank}(\mathcal{L}) = 2 \quad \text{where} \quad \mathcal{L} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{l}_1^T \mathcal{M}_1 \\ \boldsymbol{l}_2^T \mathcal{M}_2 \\ \boldsymbol{l}_3^T \mathcal{M}_3 \end{pmatrix}$$

All 3x3 minors must be zero!

Pick $\mathcal{M}_1 = (\, \mathrm{Id} \quad \mathbf{0} \,)$, $\mathcal{M}_2 = (\, \mathcal{R}_2 \quad \boldsymbol{t}_2 \,)$ and $\mathcal{M}_3 = (\, \mathcal{R}_3 \quad \boldsymbol{t}_3 \,)$.

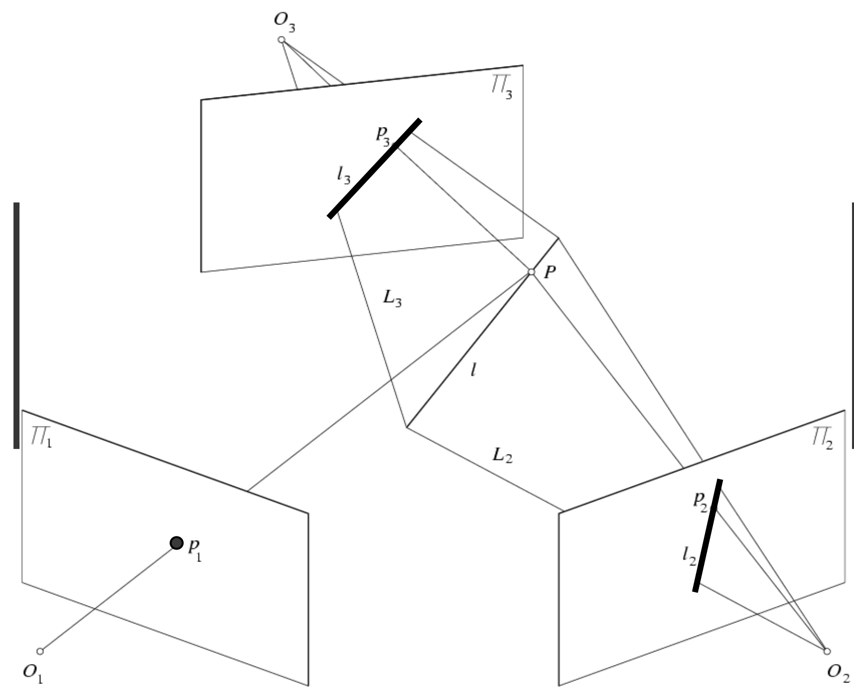
$$\mathcal{L} = \begin{pmatrix} \boldsymbol{l}_1^T & 0 \\ \boldsymbol{l}_2^T \mathcal{R}_2 & \boldsymbol{l}_2^T \boldsymbol{t}_2 \\ \boldsymbol{l}_3^T \mathcal{R}_3 & \boldsymbol{l}_3^T \boldsymbol{t}_3 \end{pmatrix} \Rightarrow \boldsymbol{p}_1^T \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix} = 0 \Rightarrow$$

$$\boxed{\begin{matrix} \mathcal{G}_1^1 \\ \mathcal{G}_1^2 \\ \mathcal{G}_1^3 \end{matrix}}$$

Trifocal Tensor

## Trifocal Constraints

## Uncalibrated Case




$$\mathrm{Rank}(\mathcal{L}) = 2 \quad \text{where} \quad \mathcal{L} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{l}_1^T \mathcal{M}_1 \\ \boldsymbol{l}_2^T \mathcal{M}_2 \\ \boldsymbol{l}_3^T \mathcal{M}_3 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} \boldsymbol{l}_1^T \mathcal{K}_1 & 0 \\ \boldsymbol{l}_2^T \mathcal{K}_2 \mathcal{R}_2 & \boldsymbol{l}_2^T \mathcal{K}_2 \boldsymbol{t}_2 \\ \boldsymbol{l}_3^T \mathcal{K}_3 \mathcal{R}_3 & \boldsymbol{l}_3^T \mathcal{K}_3 \boldsymbol{t}_3 \end{pmatrix}$$

Pick $\mathcal{M}_1 = (\,\mathcal{K}_1 \quad \mathbf{0}\,)$, $\mathcal{M}_2 = (\,\mathcal{A}_2\mathcal{K}_1 \quad \boldsymbol{a}_2\,)$ and $\mathcal{M}_3 = (\,\mathcal{A}_3\mathcal{K}_1 \quad \boldsymbol{a}_3\,)$.

$$\boldsymbol{l}_1 \propto \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix} \qquad \boldsymbol{p}_1^T \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix} = 0$$

$$\mathcal{G}_1^1 \quad \mathcal{G}_1^2 \quad \mathcal{G}_1^3$$

Trifocal Tensor

## Properties of the Trifocal Tensor

- For any matching epipolar lines, $\boldsymbol{l}_2^T \mathsf{G}_1^i \boldsymbol{l}_3 = 0$.

- The matrices $\mathsf{G}_1^i$ are singular.

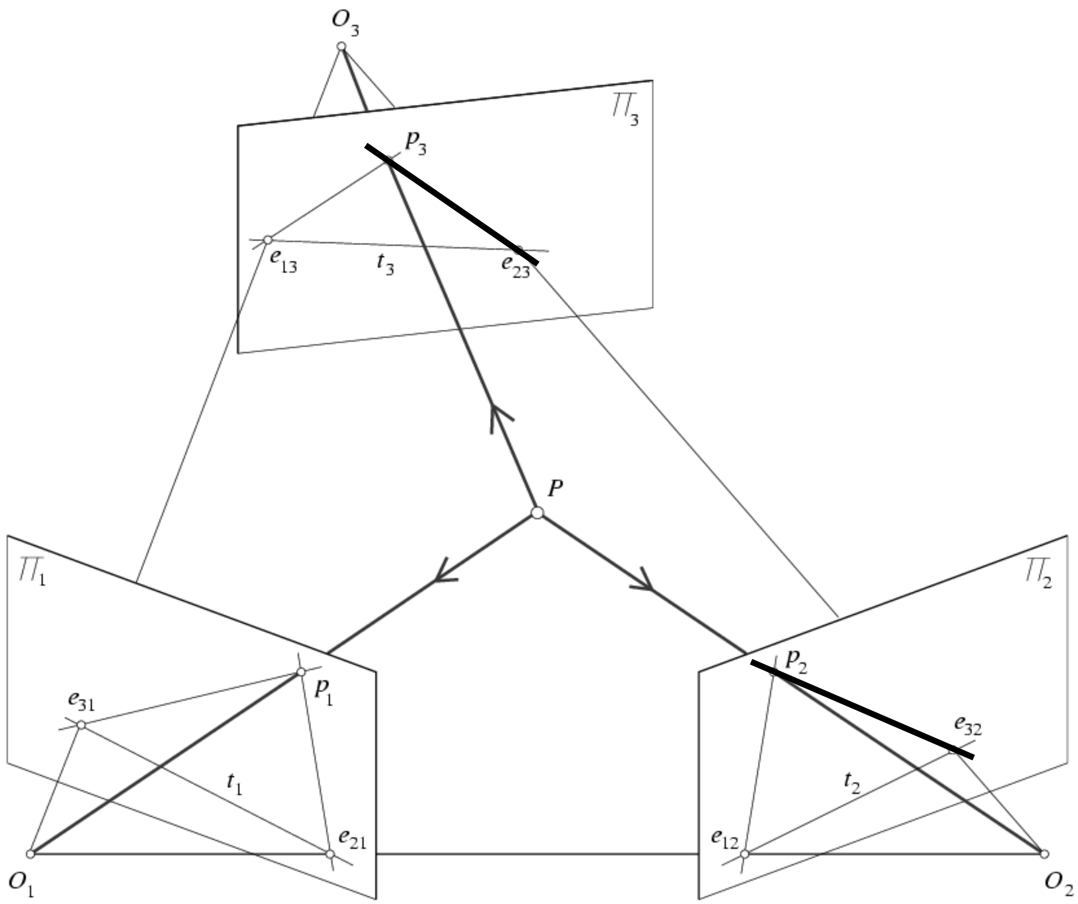
- They satisfy 8 independent constraints in the uncalibrated case (Faugeras and Mourrain, 1995).

## Estimating the Trifocal Tensor

- Ignore the non-linear constraints and use linear least-squares a posteriori.

- Impose the constraints a posteriori.

For any matching epipolar lines, $\boldsymbol{l}_2^{\mathrm{T}} \mathrm{G}_1^i \boldsymbol{l}_3 = 0.$




The backprojections of the two lines do not define a line!

# Multiple Views (Faugeras and Mourrain, 1995)

$$z\boldsymbol{p} = \mathcal{M}\boldsymbol{P} \Longleftrightarrow \boldsymbol{p} \times (\mathcal{M}\boldsymbol{P}) = ([\boldsymbol{p}_\times]\mathcal{M})\boldsymbol{P} = 0$$

$$\begin{pmatrix} u\mathcal{M}^3 - \mathcal{M}^1 \\ v\mathcal{M}^3 - \mathcal{M}^2 \end{pmatrix} \boldsymbol{P} = 0 \quad \text{where} \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}^1 \\ \mathcal{M}^2 \\ \mathcal{M}^3 \end{pmatrix}$$

$$\mathcal{Q}\boldsymbol{P} = 0 \quad \text{where} \quad \mathcal{Q} \stackrel{\text{def}}{=} \begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_2\mathcal{M}_2^3 - \mathcal{M}_2^2 \\ u_3\mathcal{M}_3^3 - \mathcal{M}_3^1 \\ v_3\mathcal{M}_3^3 - \mathcal{M}_3^2 \\ u_4\mathcal{M}_4^3 - \mathcal{M}_4^1 \\ v_4\mathcal{M}_4^3 - \mathcal{M}_4^2 \end{pmatrix} \Longrightarrow \boxed{\text{Rank}(\mathcal{Q}) \leq 3}$$

# Two Views

$$\mathcal{Q}\boldsymbol{P} = 0 \quad \text{where} \quad \mathcal{Q} \stackrel{\text{def}}{=} \begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_2\mathcal{M}_2^3 - \mathcal{M}_2^2 \\ u_3\mathcal{M}_3^3 - \mathcal{M}_3^1 \\ v_3\mathcal{M}_3^3 - \mathcal{M}_3^2 \\ u_4\mathcal{M}_4^3 - \mathcal{M}_4^1 \\ v_4\mathcal{M}_4^3 - \mathcal{M}_4^2 \end{pmatrix} \Longrightarrow \text{Rank}(\mathcal{Q}) \leq 3$$

$$\text{Det}\begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_2\mathcal{M}_2^3 - \mathcal{M}_2^2 \end{pmatrix} = 0 \quad \Longrightarrow \quad \text{Epipolar Constraint}$$

# Three Views

$$\boldsymbol{\mathcal{Q}}\boldsymbol{P} = 0 \quad \text{where} \quad \boldsymbol{\mathcal{Q}} \stackrel{\text{def}}{=} \begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_2\mathcal{M}_2^3 - \mathcal{M}_2^2 \\ u_3\mathcal{M}_3^3 - \mathcal{M}_3^1 \\ v_3\mathcal{M}_3^3 - \mathcal{M}_3^2 \\ u_4\mathcal{M}_4^3 - \mathcal{M}_4^1 \\ v_4\mathcal{M}_4^3 - \mathcal{M}_4^2 \end{pmatrix} \Longrightarrow \text{Rank}(\boldsymbol{\mathcal{Q}}) \leq 3$$

$$\text{Det}\begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_3\mathcal{M}_3^3 - \mathcal{M}_3^2 \end{pmatrix} = 0 \quad \Longrightarrow \quad \text{Trifocal Constraint}$$

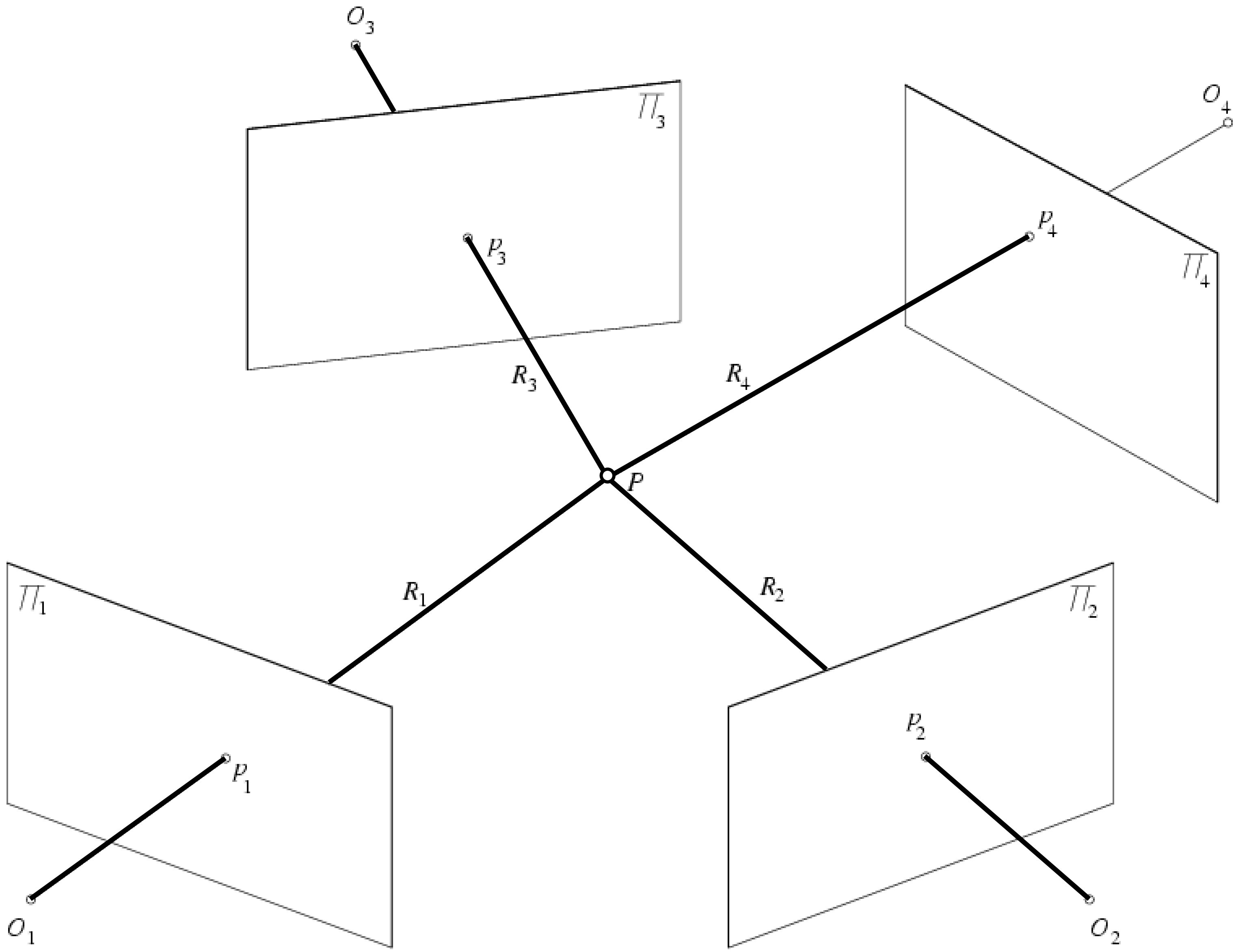
# Four Views

$$\boldsymbol{\mathcal{Q}}\boldsymbol{P} = 0 \quad \text{where} \quad \mathcal{Q} \stackrel{\text{def}}{=} \begin{pmatrix} u_1\mathcal{M}_1^3 - \mathcal{M}_1^1 \\ \boxed{v_1\mathcal{M}_1^3 - \mathcal{M}_1^2} \\ \boxed{u_2\mathcal{M}_2^3 - \mathcal{M}_2^1} \\ v_2\mathcal{M}_2^3 - \mathcal{M}_2^2 \\ u_3\mathcal{M}_3^3 - \mathcal{M}_3^1 \\ \boxed{v_3\mathcal{M}_3^3 - \mathcal{M}_3^2} \\ u_4\mathcal{M}_4^3 - \mathcal{M}_4^1 \\ \boxed{v_4\mathcal{M}_4^3 - \mathcal{M}_4^2} \end{pmatrix} \implies \text{Rank}(\mathcal{Q}) \leq 3$$

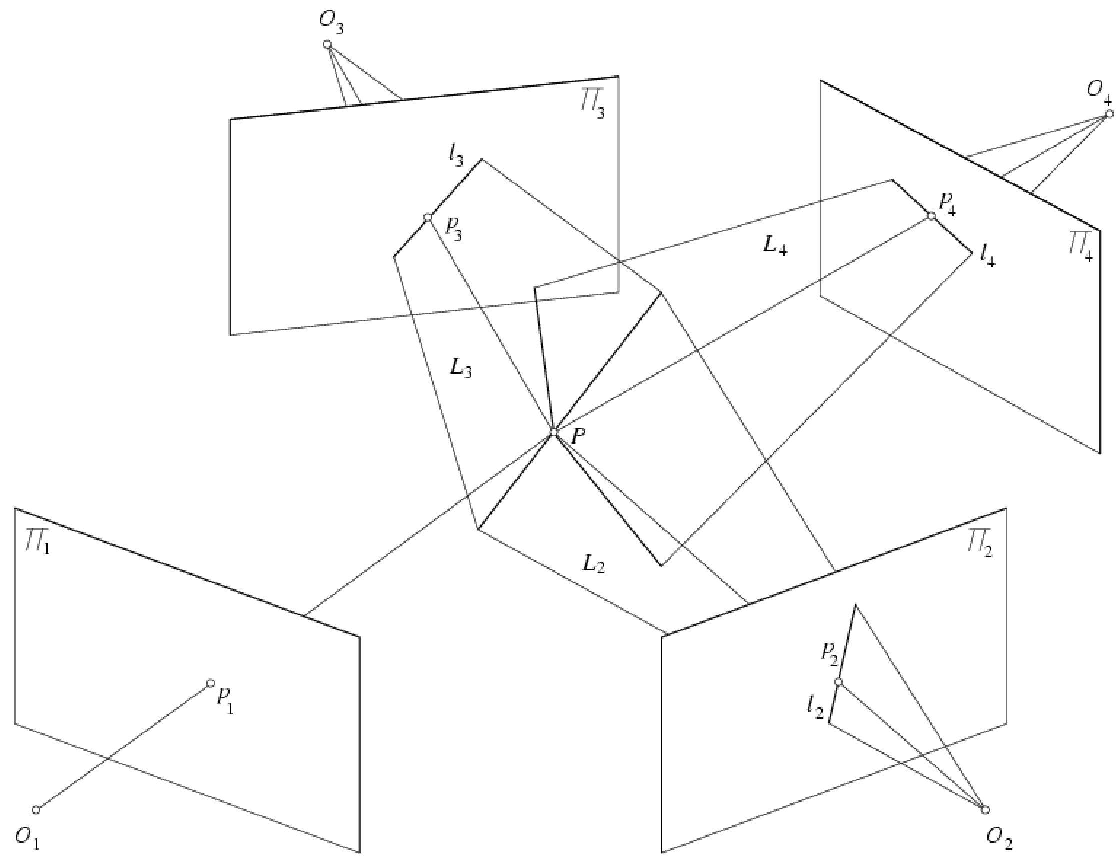
$$\text{Det}\begin{pmatrix} v_1\mathcal{M}_1^3 - \mathcal{M}_1^2 \\ u_2\mathcal{M}_2^3 - \mathcal{M}_2^1 \\ v_3\mathcal{M}_3^3 - \mathcal{M}_3^2 \\ v_4\mathcal{M}_4^3 - \mathcal{M}_4^2 \end{pmatrix} = 0$$

$\Rightarrow$ Quadrifocal Constraint
(Triggs, 1995)

Geometrically, the four rays must intersect in *P*..



# Quadrifocal Tensor and Lines




$$z\boldsymbol{p} = \mathcal{M}\boldsymbol{P} \Longleftrightarrow \boldsymbol{l}^T\mathcal{M}\boldsymbol{P} = 0 \Longleftrightarrow \boldsymbol{L}\cdot\boldsymbol{P} = 0 \text{ with } \boldsymbol{L} = \mathcal{M}^T\boldsymbol{l}$$

$$\begin{pmatrix} \boldsymbol{L}_1^T \\ \boldsymbol{L}_2^T \\ \boldsymbol{L}_3^T \\ \boldsymbol{L}_4^T \end{pmatrix} \boldsymbol{P} = \boldsymbol{0} \quad \Rightarrow \quad \mathrm{Rank}(\mathcal{L}) = 3 \quad \text{where} \quad \mathcal{L} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{l}_1^T\mathcal{M}_1 \\ \boldsymbol{l}_2^T\mathcal{M}_2 \\ \boldsymbol{l}_3^T\mathcal{M}_3 \\ \boldsymbol{l}_3^T\mathcal{M}_4 \end{pmatrix}$$

# Scale-Restraint Condition from Photogrammetry