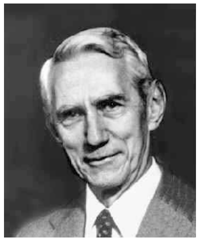
6. Combinational Circuits

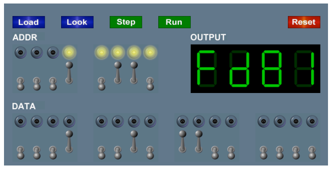
George Boole (1815 - 1864)



Claude Shannon (1916 - 2001)

Computer Architecture

TOY lectures. von Neumann machine.



This lecture. Boolean circuits.

Ahead. Putting it all together and building a TOY machine.

Digital Circuits

What is a digital system?
- Digital: signals are 0 or 1.
- Analog: signals vary continuously.

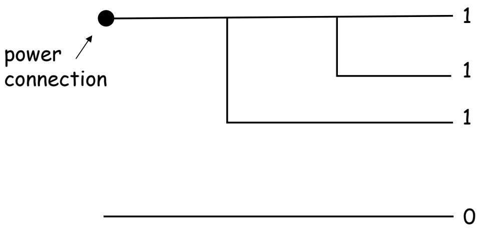
Why digital systems?
- Accuracy and reliability.
- Staggeringly fast and cheap.

Basic abstractions.
- On, off.
- Wire: propagates on/off value.
- Switch: controls propagation of on/off values through wires.

Digital circuits and you.
- Computer microprocessors.
- Antilock brakes, cell phones, iPods, etc.

Wires

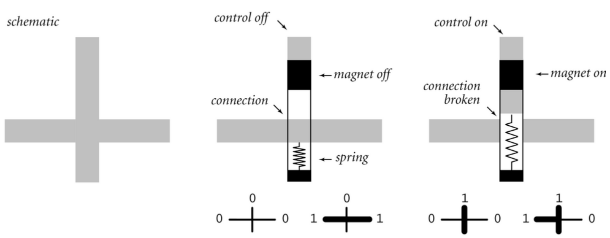
Wires.
- On (1): connected to power.
- Off (0): not connected to power.
- If a wire is connected to a wire that is on, that wire is also on.
- Typical drawing convention: "flow" from top, left to bottom, right.



Controlled Switch

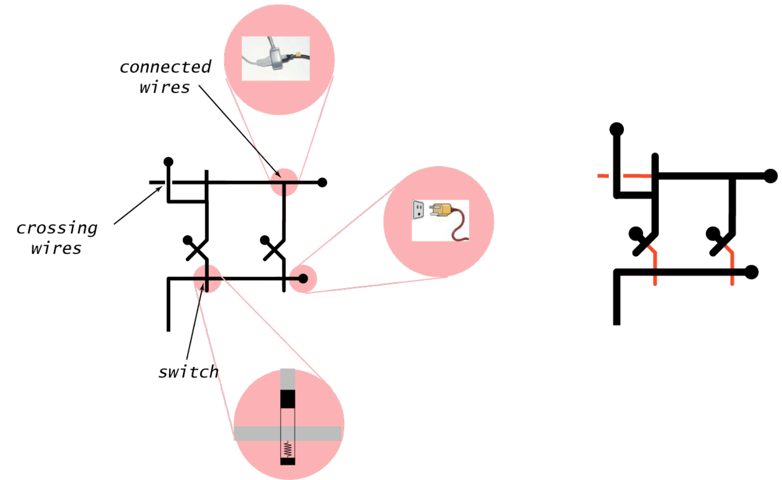
Controlled switch. [relay implementation]

- 3 connections: input, output, control.
- Magnetic force pulls on a contact that cuts electrical flow.
- Control wire affects output wire, but output does not affect control; establishes forward flow of information over time.



Anatomy of a relay (controlled switch)

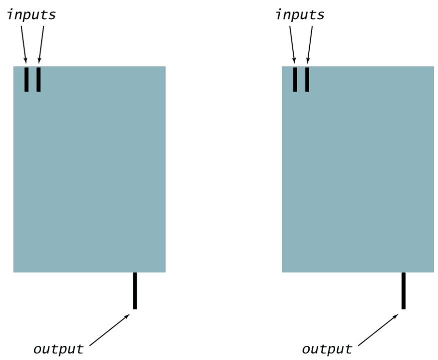
Circuit Anatomy



Layers of Abstraction

Layers of abstraction.

- Circuits are built from wires and switches. (implementation)
- A circuit is defined by its inputs and outputs. (interface)
- To control complexity, we encapsulate circuits. (ADT)



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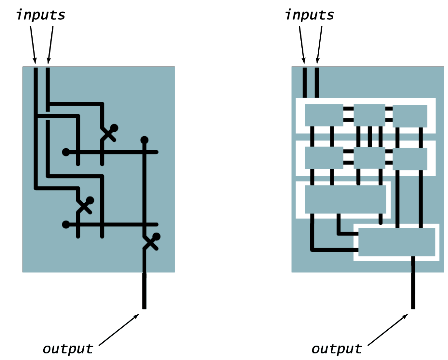


Logic Gates: Fundamental Building Blocks

NOT = x'

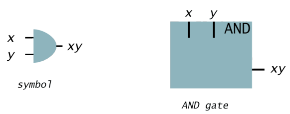
x	NOT
0	1
1	0

OR = x+y

x y	OR
0 0	0
0 1	1
1 0	1
1 1	1

AND = xy

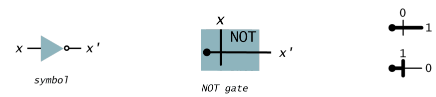
x y	AND
0 0	0
0 1	0
1 0	0
1 1	1



Logic Gates: Fundamental Building Blocks

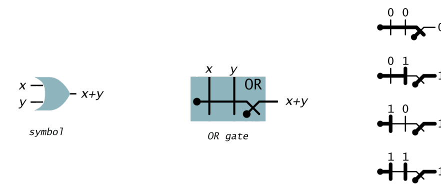
NOT = x'

x	NOT
0	1
1	0



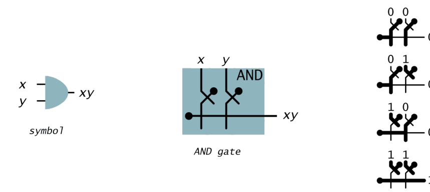
OR = x+y

x y	OR
0 0	0
0 1	1
1 0	1
1 1	1



AND = xy

x y	AND
0 0	0
0 1	0
1 0	0
1 1	1

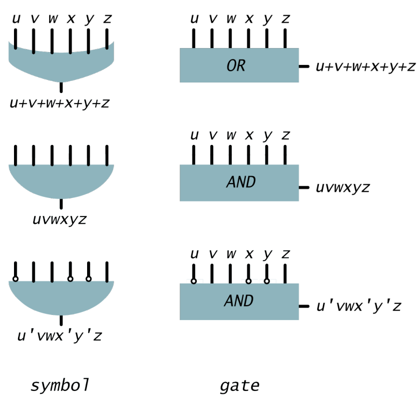


implementations with switches

Multiway Gates

Multiway gates.

- OR: 1 if any input is 1; 0 otherwise.
- AND: 1 if all inputs are 1; 0 otherwise.
- Generalized: negate some inputs.



Multiway Gates

Multiway gates.

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- AND: 1 if all inputs are 1; 0 otherwise.
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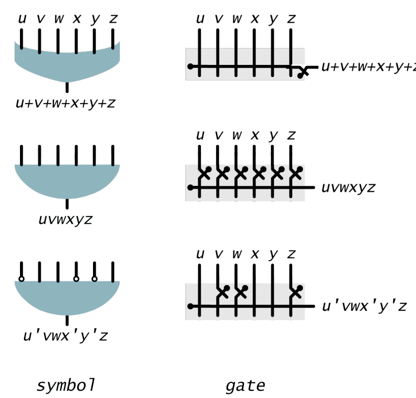
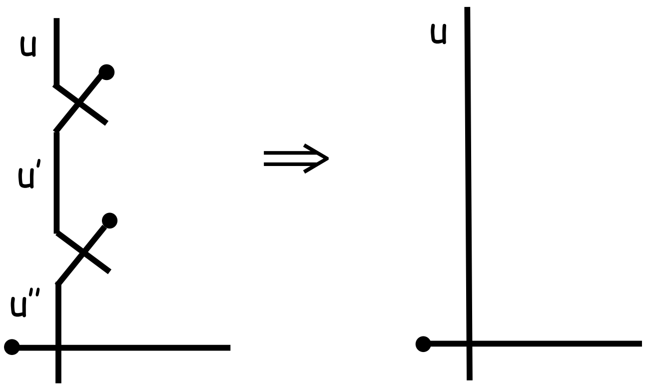


Cancelling inverters



Boolean Algebra

History.

- Developed by Boole to solve mathematical logic problems (1847).
- Shannon master's thesis applied it to digital circuits (1937).

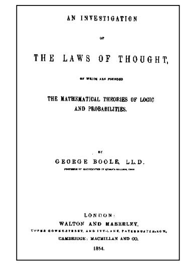
"possibly the most important, and also the most famous, master's thesis of the [20th] century" --Howard Gardner

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

- Boolean variables: signals.
- Boolean functions: circuits.



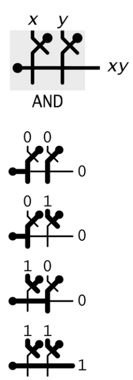
Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs ⇒ 2^N rows.

x	y	x y
0	0	0
0	1	0
1	0	0
1	1	1

AND Truth Table



Truth Table for Functions of 2 Variables

Truth table.

- 16 Boolean functions of 2 variables. ← every 4-bit value represents one

x	y	ZERO	AND		x		y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth table for all Boolean functions of 2 variables

x	y	NOR	EQ	y'		x'		NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth table for all Boolean functions of 2 variables

Truth Table for Functions of 3 Variables

Truth table.

- 16 Boolean functions of 2 variables. ← every 4-bit value represents one
- 256 Boolean functions of 3 variables. ← every 8-bit value represents one
- 2^(2^n) Boolean functions of n variables! ← every 2^n-bit value represents one

x	y	z	AND	OR	MAJ	ODD
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Some Functions of 3 Variables

Universality of AND, OR, NOT

Fact. Any Boolean function can be expressed using AND, OR, NOT.

- { AND, OR, NOT } are universal.
- Ex: XOR(x,y) = xy' + x'y.

Notation	Meaning
x'	NOT x
x y	x AND y
x + y	x OR y

Expressing XOR Using AND, OR, NOT

x	y	x'	y'	x'y	xy'	x'y + xy'	x XOR y
0	0	1	1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	1	1	1
1	1	0	0	0	0	0	0

Exercise. Show {AND, NOT}, {OR, NOT}, {NAND}, {AND, XOR} are universal.
Hint. DeMorgan's law: (x'y')' = x + y.

Sum-of-Products

Sum-of-products. Systematic procedure for representing a Boolean function using AND, OR, NOT.

- Form AND term for each 1 in Boolean function.
- OR terms together.

↖ proves that { AND, OR, NOT } are universal

x	y	z	MAJ	x'yz	xy'z	xyz'	xyz	x'yz + xy'z + xyz' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

expressing MAJ using sum-of-products

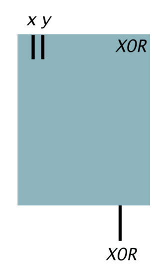
Translate Boolean Formula to Boolean Circuit

Sum-of-products. XOR.

XOR = x'y + xy'

x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Truth table



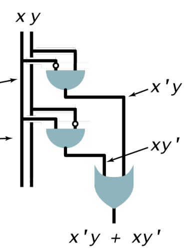
Circuit

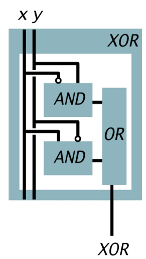
Translate Boolean Formula to Boolean Circuit

Sum-of-products. XOR.

XOR = x'y + xy'

x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0





Truth table *Abstract circuit* *Circuit*

Translate Boolean Formula to Boolean Circuit

Sum-of-products. XOR.

XOR = x'y + xy'

x	y	XOR
0	0	0
0	1	1
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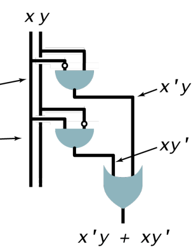
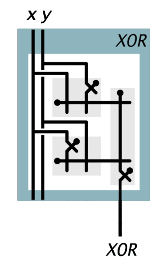
1	1	0





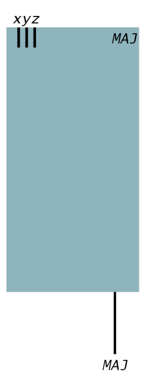
Truth table *Abstract circuit* *Circuit*

Translate Boolean Formula to Boolean Circuit

Sum-of-products. Majority.

MAJ = x'yz + xy'z + xyz'+ xyz

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



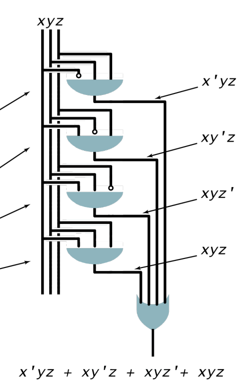
Truth table *Circuit*

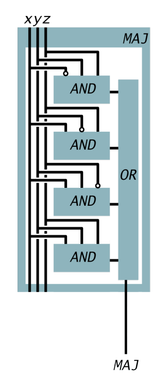
Translate Boolean Formula to Boolean Circuit

Sum-of-products. Majority.

MAJ = x'yz + xy'z + xyz'+ xyz

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1





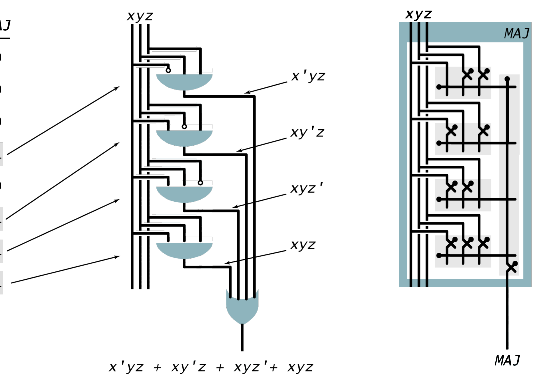
Truth table *Abstract circuit* *Circuit*

Translate Boolean Formula to Boolean Circuit

Sum-of-products. Majority.

MAJ = x'yz + xy'z + xyz'+ xyz

x y z	MAJ
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1



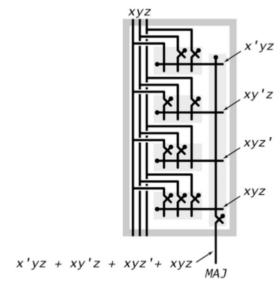
Truth table *Abstract circuit* *Circuit*

Simplification Using Boolean Algebra

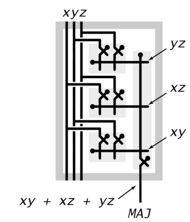
Many possible circuits for each Boolean function.

- Sum-of-products not necessarily optimal in:
  - number of switches (space)
  - depth of circuit (time)

Ex. MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.



size = 10, depth = 2



size = 7, depth = 2

Expressing a Boolean Function Using AND, OR, NOT

Ingredients.

- AND gates.
- OR gates.
- NOT gates.
- Wire.

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

ODD Parity Circuit

ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.

x	y	z	ODD	x'y'z	x'yz'	xy'z'	xyz	x'y'z + x'yz' + xy'z' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	1	0	0	0	1	1

Expressing ODD using sum-of-products

ODD Parity Circuit

ODD(x, y, z).

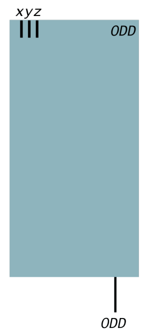
- 1 if odd number of inputs are 1.
- 0 otherwise.

MAJ = x'yz + xy'z + xyz'+ xyz

ODD = x'y'z + x'yz'+ xy'z'+ xyz

x y z	MAJ
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

x y z	ODD
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1



ODD Parity Circuit

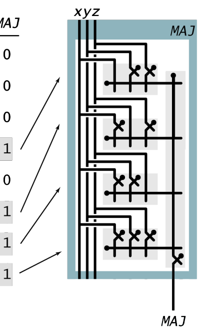
ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.

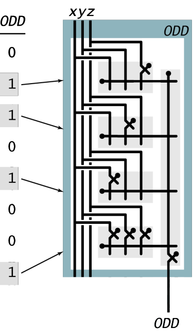
MAJ = x'yz + xy'z + xyz'+ xyz

ODD = x'y'z + x'yz'+ xy'z'+ xyz

x y z	MAJ
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1



x y z	ODD
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1



Let's Make an Adder Circuit

Goal. x + y = z for 4-bit integers.

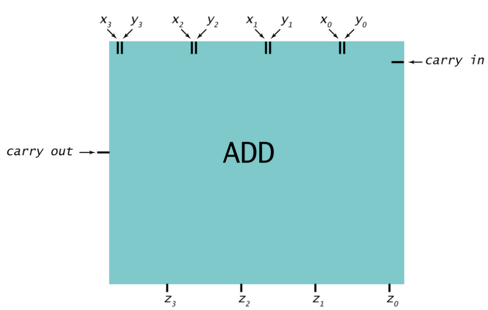
- We build 4-bit adder: 9 inputs, 4 outputs.
- Same idea scales to 128-bit adder.
- Key computer component.

	1	1	1	0
	2	4	8	7
+	3	5	7	9
	6	0	6	6

Step 1. Represent input and output in binary.

	1	1	0	0
	0	0	1	0
+	0	1	1	1
	1	0	0	1

	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0



Let's Make an Adder Circuit

Goal. x + y = z for 4-bit integers.

c_{out}				c_{in}
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 2. (first attempt)

- Build truth table.
- Why is this a bad idea?
  - 128-bit adder: 2^{256+1} rows > # electrons in universe!

c_0	x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1
.	.	.	.	.	.	.	.	.	.	.	.	.
1	1	1	1	1	1	1	1	1	1	1	1	1

} 2^{8+1} = 512 rows!

4-Bit Adder Truth Table

Let's Make an Adder Circuit

Goal. x + y = z for 4-bit integers.

Step 2. (do one bit at a time)
- Build truth table for carry bit.
- Build truth table for summand bit.

c_{out}	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Carry Bit

x_i	y_i	c_i	c_{i+1}
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Summand Bit

x_i	y_i	c_i	z_i
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Let's Make an Adder Circuit

Goal. x + y = z for 4-bit integers.

Step 3.
- Derive (simplified) Boolean expression.

c_{out}	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Carry Bit

x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

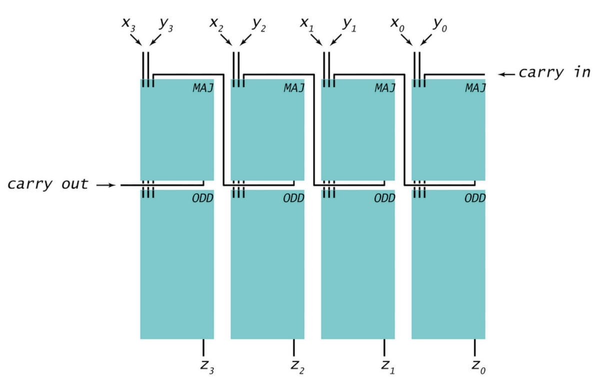
Summand Bit

x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

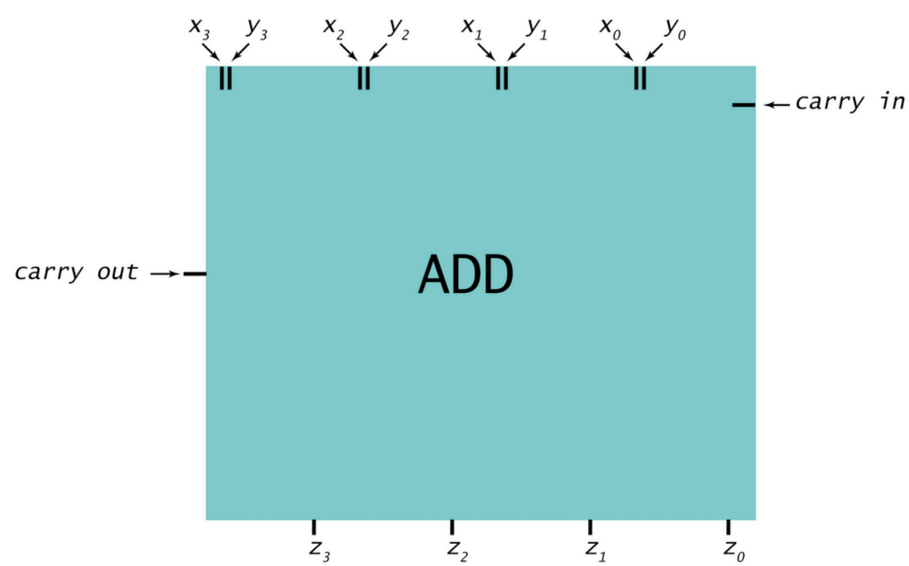
Let's Make an Adder Circuit

Goal. x + y = z for 4-bit integers.

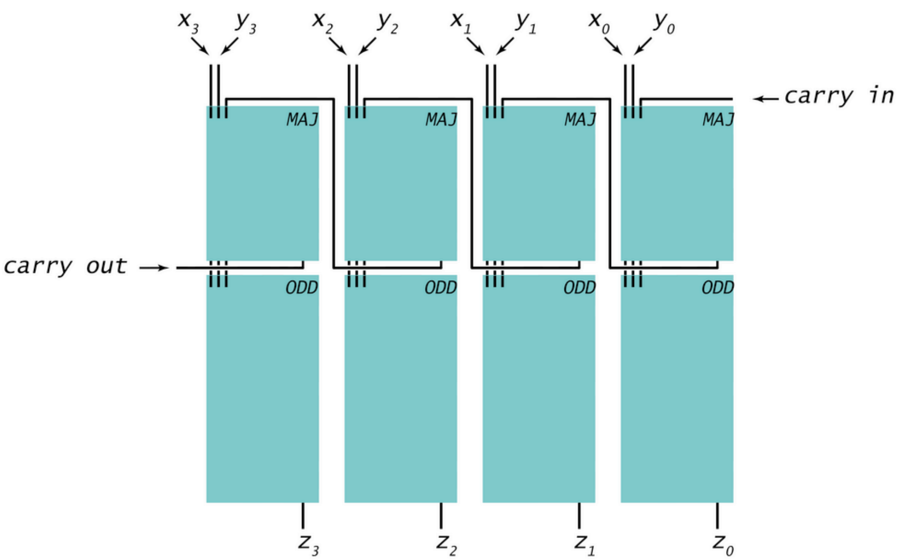
Step 4.
- Transform Boolean expression into circuit.
- Chain together 1-bit adders.



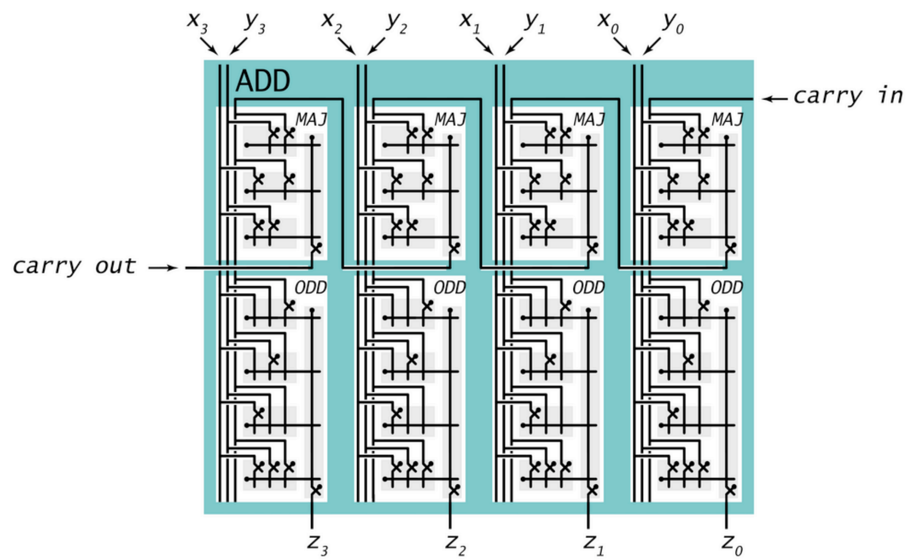
Adder: Interface



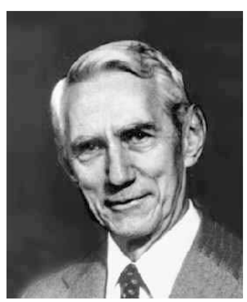
Adder: Component Level View



Adder: Switch Level View



George Boole (1815 – 1864)



Claude Shannon (1916 – 2001)